Finite flavour groups of fermions

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Seminar DESY Zeuthen
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Review:

W. Grimus, P.O. Ludl
Finite flavour groups of fermions
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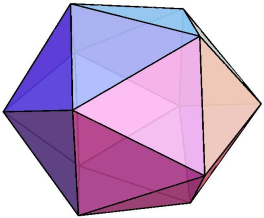


Groups

Definition: group $(G, \circ)$

$$\circ : \begin{array}{ccc} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_1 \circ g_2 \end{array}$$

1. Associative law:
   $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
2. Neutral element:
   $\exists\, e \in G$ with $e \circ g = g \;\forall\, g \in G$
3. Inverse element:
   $\forall\, g \in G \;\exists\, g^{-1} \in G$ with $g^{-1} \circ g = e$

"Any set of $n \times n$ matrices, closed under multiplication and formation the inverse matrix, is a group."

Motivation and application

- Gauge group:
  - completely fixes gauge interactions
  - flavour-blind (Yukawa couplings free)
- Flavour group: determines flavour sector?
  - Lie group? Gauged?
  - Spontaneous symmetry breaking: Goldstone bosons?

Flavour group:

Discrete $\hookrightarrow$ avoid Goldstone bosons

There are no compelling argument in favour of discrete flavour groups!

Motivation and application

Lepton mixing: until 2011 tri-bimaximal mixing and group A_4
TBM: Harrison, Perkins, Scott (2002)

$$U_{\mathrm{PMNS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

θ_{13} **revolution:**
Daya Bay, RENO exps. (2012): $\theta_{13} \neq 0$
Gonzalez-Garcia et al.: $\sin^2\theta_{13} = 0.023 \pm 0.0023$ or $\theta_{13} \simeq 9^\circ \pm 0.5^\circ$

Tri-bimaximal not a good approximation anymore!
TBM no good guiding principle anymore for discrete flavour group!

Motivation and application

New approach for lepton mixing:

Majorana neutrinos, flavour group G_f

$$\text{Symmetry breaking:} \quad G_f \begin{array}{l} \nearrow \quad G_\ell \subset U(1) \times U(1) \times U(1) \\ \searrow \quad G_\nu \subset \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \end{array}$$

$G_\ell =$ residual symmetry group in charged-lepton sector
$G_\nu =$ residual symmetry group neutrino sector

Idea: purely group-theoretical approach

- $G_\ell \cup G_\nu$ determines G_f
- Mismatch $G_\ell \neq G_\nu$ determines U_{PMNS}
- Lam; Adelhart Toorop, Feruglio, Hagedorn; Hernandez, Smirnov; Holthausen, Lim, Lindner (2012): $G_e = \mathbb{Z}_3$, $|G_f| \leq 1536$, group scan using GAP $\Rightarrow$ $\Delta(6 \times 10^2)$, $(\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes S_3$, $\Delta(6 \times 16^2)$

Plan of the talk

1. General properties of discrete groups and their representations
    - Subgroups and normal subgroups
    - Semidirect products
    - Characters and character tables
2. Symmetric and alternating groups
    - A_4, S_4
3. The finite subgroups of $SU(3)$

General properties: generators

Basic notions

Generators

Subset S of G such that every element of G can be written as a finite product of elements of S and their inverses

Presentation of a group

Set S of generators and a set R of relations among the generators

Examples:

- Cyclic group $\mathbb{Z}_n$ with one generator a and $a^n = e$
- Two generators a, b with $a^3 = b^2 = (ab)^2 = e \Rightarrow$ group with 6 elements (S_3) due to $ba = a^2 b$

General properties: subgroup

Subgroup

Subset H of G which is closed under under multiplication and inverse
Proper subgroup: $H \neq \{e\}$ or G

Cosets

H subgroup
Left coset: $aH := \{ah|h \in H\}, \quad a \in G$
Right coset: $Hb := \{hb|h \in H\}, \quad b \in G$

$*$ Cosets aH, $bH \Rightarrow$ either $aH = bH$ or $aH \cap bH = \emptyset$ $*$

General properties: subgroup

Order of a finite group

ord G = number of elements of G

Order of an element of G

Order of $a \in G$ is the smallest power ν such that $a^\nu = e$

Theorem (Lagrange)

* The number of elements of a subgroup is a divisor of ord G
* The order of an element of G is divisor of ord G

Proof:

a) Consider cosets $a_1 H, \ldots, a_k H \Rightarrow k \times \text{ord } H = \text{ord } G$

b) a generates $\mathbb{Z}_\nu$ Q.E.D.

General properties: invariant subgroup

Normal or invariant subgroup

N is a proper normal subgroup of G ($N \lhd G$) if $gNg^{-1} = N$ for all $g \in G$

Factor group

The cosets $gN = Ng$ of $N \lhd G$ with the multiplication rule $(aN)(bN) = (ab)N$ form a group called factor group G/N

General properties: classes

Conjugate elements

* $a, b \in G$ are called conjugate ($a \sim b$) if there exists an element $g \in G$ such that $gag^{-1} = b$
* The equivalence relation $a \sim b$ allows to divide G into distinct "classes" C_k ($C_1 \cup \cdots \cup C_{n_c} = G$ with $C_k \cap C_l = \emptyset \, \forall \, k \neq l$)
* The class of an element $a \in G$ is defined as $C_a = \{gag^{-1} \,|\, g \in G\}$

Remarks: $C_e \equiv C_1 = \{e\}$

G Abelian $\Rightarrow$ every element is its own class

General properties: direct product, Abelian groups

Direct product

The set $G \times H$ with the multiplication law
$(g_1, h_1)(g_2, h_2) := (g_1 g_2, h_1 h_2) \quad g_1, g_2 \in G,\ h_1, h_2 \in H$ is a group which is called the *direct product* of G and H.

Theorem (structure of Abelian groups)

* A Abelian, ord $A = p_1^{a_1} \cdots p_n^{a_n}$ (p_i distinct primes) $\Rightarrow$
  $A \cong A_1 \times \cdots \times A_n$ with ord $A_i = p_i^{a_i}$
* A' Abelian, ord $A' = p^b$ (p prime) $\Rightarrow$
  $\exists\ b_1 + \cdots + b_m = b$ with $A' \cong \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_m}$

Examples:
A Abelian with four elements $\Rightarrow$
$A = \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein four-group) ($\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$)
ord $A = 512 = 2^9$: 30 Abelian groups

General properties: semidirect product

Semidirect product

Automorphisms

- Aut(G) = group of isomorphisms $f : G \to G$
- Inner automorphisms: for every $g \in G$ $f_g(a) = gag^{-1}$

Semidirect product $G \rtimes_\phi H$

Homomorphism $\phi : H \to \mathrm{Aut}(G)$ ("H acts on G")
$G \times H$ with the multiplication law
$(g_1, h_1)(g_2, h_2) := (g_1\phi(h_1)g_2, h_1 h_2)$
forms a group, the semidirect product

Note: generalization of direct product with $\phi(h) = \mathrm{id}\ \forall\, h \in H$
* $G \times \{e'\}$ is a normal subgroup, $\{e\} \times H$ a subgroup of $G \rtimes_\phi H$ *

General properties: semidirect product

Decomposition of a group into a semidirect product

Group S, G normal subgroup of S, H subgroup of S with following properties:

1. $G \cap H = \{e\}$,
2. every element $s \in S$ can be written as $s = gh$ with $g \in G$, $h \in H$.

Then the following holds:

- $S \cong G \rtimes_\phi H$ with $\phi(h)g = hgh^{-1}$,
- decomposition $s = gh$ is unique,
- $S/G \cong H$.

Semidirect product structure of S simply given by

$*$ $(g_1 h_1)(g_1 h_1) = (g_1 h_1 g_2 h_1^{-1})(h_1 h_2)$ $*$

General properties: semidirect product

Note: $G \triangleleft S$ alone does in general *not* result in a semidirect product on S.

Semidirect products are ubiquitous!
$S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, $A_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$, $S_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3$, etc.

General properties: simple groups

Are there groups without normal subgroups?

Simple group

A group is called simple if it has no non-trivial normal subgroups

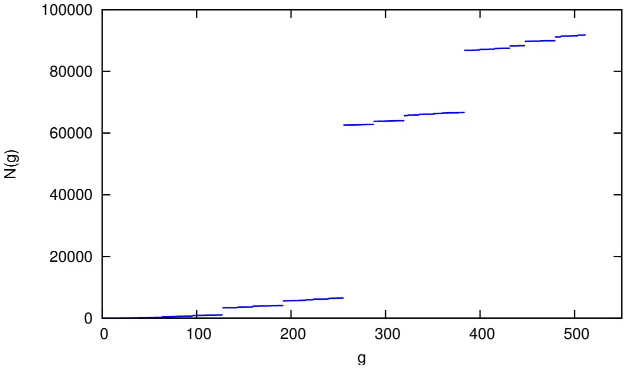
Finite Abelian simple groups: $\mathbb{Z}_p$ with p prime

Finite non-Abelian simple groups: *All* groups have been classified!
Though infinitely many, they are "rare":
Orders below 1000 are 60 (A_5), 168, 360, 504, 660.

1. Alternating groups A_n with $n \geq 5$
2. 16 series of Lie type
3. 26 sporadic groups

Order of largest sporadic group $\simeq 8 \times 10^{53}$

General properties: number of finite groups



P.O. Ludl (2010)

$g \equiv \text{ord}\, G$, $N(g)$ = number of non-Abelian groups with ord $G \leq g$

Remarks: jumps in $N(g)$ at $g = 2^8 = 256$ and $3 \times 2^7 = 384$

Jump at $g = 512$: $N(511) = 91774 \rightarrow N(512) = 10494193$

General properties: existence of subgroups

There are groups without normal subgroups, however, every group except $\mathbb{Z}_p$ with p prime has subgroups!

First theorem of Sylow

ord $G = p_1^{a_1} \cdots p_n^{a_n}$ (prime factor decomposition) $\Rightarrow$ G possesses subgroups of all orders $p_i^{s_i}$ with $0 \leq s_i \leq a_i$ $(i = 1, \ldots, n)$

General properties: representations

Representation of group G

$\mathcal{V}$ vector space over $\mathbb{C}$
Homorphism $D : G \to \mathrm{Lin}(\mathcal{V})$ with $D(e) = \mathbb{1}$

All representations of finite groups are equivalent to unitary representations $\Rightarrow$ If D is reducible, there exists a basis such that

$$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}$$

Irreducible representations (irreps) are basic building blocks of representations.

General properties: characters

Character of a representation

The character $\chi_D : G \to \mathbb{C}$ is defined by
$\chi_D(a) := \mathrm{Tr}\, D(a), \quad a \in G.$

Properties:

- Equivalent representations have the same character
- $a \sim b$, i.e. a, b in same class $\Rightarrow \chi_D(a) = \chi_D(b)$
- $\chi_D(a^{-1}) = \chi_D^*(a)$
- $\chi_{D \oplus D'}(a) = \chi_D(a) + \chi_{D'}(a)$
- $\chi_{D \otimes D'}(a) = \chi_D(a)\, \chi_{D'}(a).$

General properties: orthogonality theorem

Bilinear form on space of functions $G \to \mathbb{C}$:

$$(f|g) = \frac{1}{\mathrm{ord}\, G} \sum_{a \in G} f(a^{-1}) g(a)$$

Real subspace of functions with property $f(a^{-1}) = f^*(a)$
$\Rightarrow$ $(\cdot|\cdot)$ scalar product on this space.

Notation: $D^{(\alpha)}$ with $\dim D^{(\alpha)} = d_\alpha$ denotes all inequivalent irreps
Schur's Lemmata $\Rightarrow$

Orthogonality theorem

$$((D^{(\alpha)})_{ij} | (D^{(\beta)})_{kl}) = \frac{1}{d_\alpha} \delta_{\alpha\beta} \delta_{il} \delta_{jk}$$

General properties: orthogonality of characters

Orthogonality of characters

$$(\chi^{(\alpha)}|\chi^{(\beta)}) = \delta_{\alpha\beta} \quad \Leftrightarrow \quad \sum_{k=1}^{n_c} c_k \left(\chi_k^{(\alpha)}\right)^* \chi_k^{(\beta)} = \text{ord}\, G\, \delta_{\alpha\beta}$$

C_k . . . . class of G
n_c . . . . number of classes
c_k . . . . number of elements in C_k
$\chi_k^{(\alpha)}$ . . . . value of $\chi^{(\alpha)}$ on C_k

General properties: "number theorems"

From orthogonality of characters it follows:

$*$ number of inequivalent irreps $\leq n_c =$ number of classes $*$

However on can show that equality holds:

Theorem

number of inequivalent irreps = number of classes

Two more very important theorems:

Theorems on the dimensions of irreps

$$\sum_\alpha d_\alpha^2 = \text{ord } G$$

All d_α are divisors of ord G

General properties: character table

G $(\#\ C)$ $\mathrm{ord}\,(C)$	C_1 (c_1) ν_1	C_2 (c_2) ν_2	$\cdots$ $\cdots$ $\cdots$	C_{n_c} (c_{n_c}) ν_{n_c}
$D^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$	$\cdots$	$\chi_{n_c}^{(1)}$
$D^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_{n_c}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D^{(n_c)}$	$\chi_1^{(n_c)}$	$\chi_2^{(n_c)}$	$\cdots$	$\chi_{n_c}^{(n_c)}$

Lines $\Rightarrow$ ON system $\left(\sqrt{\frac{c_1}{\mathrm{ord}\, G}}\, \chi_1^{(\alpha)}, \ldots, \sqrt{\frac{c_{n_c}}{\mathrm{ord}\, G}}\, \chi_{n_c}^{(\alpha)} \right)$

Columns $\Rightarrow$ ON system $\sqrt{\frac{c_k}{\mathrm{ord}\, G}} \begin{pmatrix} \chi_k^{(1)} \\ \vdots \\ \chi_k^{(n_c)} \end{pmatrix} \quad (k = 1, \ldots, n_c)$

General properties: reduction of representations

Characters and character tables: means of finding irreducible components of a representation D

$$D = \bigoplus_\alpha m_\alpha D^{(\alpha)} \quad \Rightarrow \quad \chi_D = \sum_\alpha m_\alpha \chi^{(\alpha)}$$

Theorem

Let D be a representation of the group G $\Rightarrow$
The multiplicity m_α with which an irrep $D^{(\alpha)}$ occurs in D is given by

$$m_\alpha = (\chi^{(\alpha)}|\chi_D)$$

* $(\chi_D|\chi_D) = \sum_\alpha m_\alpha^2 \Rightarrow D$ irreducible iff $(\chi_D|\chi_D) = 1$ *

Application to tensor products of irreps:

$$\chi^{(\alpha\otimes\beta)}(a) = \chi^{(\alpha)}(a) \times \chi^{(\beta)}(a) \quad \Rightarrow \quad m_\gamma = (\chi^{(\gamma)}|\chi^{(\alpha)} \times \chi^{(\beta)})$$

Symmetric and alternating groups

Symmetric group S_n: Group of all permutations of n objects

$$p = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}, \quad \text{ord}\, S_n = n!$$

Cycle of length r: $(n_1 \to n_2 \to n_3 \to \cdots n_r \to n_1) \equiv (n_1 n_2 n_3 \cdots n_r)$
All numbers $n_1, \ldots, n_r$ are different

Theorem

Every permutation is a unique product of cycles which have no common elements

Example: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix} = (145)(3)(26)$

Remarks:
Cycles which have no common element commute
A cycle which consists of only one element is identical with the unit element of S_n

Symmetric and alternating groups

Even and odd permutations:

Every permutation of S_n is associated with an $n \times n$ permutation matrix $M(p)$

Even and odd permutations

Sign of a permutation: $\mathrm{sgn}(p) = \det M(p)$

A permutation p is called even (odd) if $\mathrm{sgn}(p) = +1$ (-1)

For instance: $(1)(23) \in S_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Alternating group A_n:

Group of all *even* permutations of n objects

Theorem

A_n normal subgroup of S_n with $n!/2$ elements, $S_n \cong A_n \rtimes \mathbb{Z}_2$

Symmetric and alternating groups

Classes of S_n and A_n

The classes of S_n:

Consist of the permutations with the same cycle structure

The classes of A_n:

Obtained from those of S_n in the following way:

- All classes of S_n with even permutations are also classes of A_n,
- except those which consist exclusively of cycles of unequal odd length.
- Each of the latter classes of S_n refines in A_n into two classes of equal size.

Examples:

S_4: $(a)(b)(c)(d), (a)(b)(cd), (a)(bcd), (abcd), (ab)(cd) \Rightarrow n_c = 5$

A_4: $(a)(b)(c)(d)$, $(a)(bcd) \rightarrow 2$ classes, $(ab)(cd) \Rightarrow n_c = 4$

Symmetric and alternating groups

One-dimensional irreps of S_n

S_n has exactly two 1-dimensional irreps:
$p \mapsto 1$ and $p \mapsto \mathrm{sgn}(p)$

Discussion of S_4 and A_4

Dimensions if irreps of S_4: $1^2 + 1^2 + d_3^2 + d_4^2 + d_5^2 = 24 \Rightarrow$
$d_3 = 2,\ d_4 = d_5 = 3$

Dimensions if irreps of A_4: $1^2 + d_2^2 + d_3^2 + d_4^2 = 12 \Rightarrow$
$d_2 = d_3 = 1,\ d_4 = 3$

Symmetric and alternating groups

Structure of A_4 and S_4:

Klein's four-group: $k_1 k_2 = k_2 k_1 = k_3$ plus permutations of indices

$$K = \{e,\ (12)(34),\ (14)(23),\ (13)(24)\} \equiv \{e,\ k_1,\ k_2,\ k_3\}$$

$*$ K is a normal subgroup of A_4 and S_4 $*$

K and $s \equiv (123)$ generate A_4:

$$k_1^2 = k_2^2 = k_3^2 = e, \quad s^3 = e, \quad s k_1 s^{-1} = k_2, \quad s k_2 s^{-1} = k_3$$

K, s and $t \equiv (12)$ generate S_4:

$$t^2 = e, \quad t k_1 t^{-1} = k_1, \quad t k_2 t^{-1} = k_3, \quad t s t^{-1} = s^2$$

Symmetric and alternating groups

Theorem

Every element of $p \in S_4$ can be uniquely decomposed into $p = kq$ with $k \in K$ and q being a permutation of the numbers 1,2,3.

$$A_4 \cong K \rtimes \mathbb{Z}_3 \quad \text{and} \quad S_4 \cong K \rtimes S_3$$

Note: $\{e\} \lhd K \lhd A_4 \lhd S_4$
Every kernel of a non-faithful irrep is a normal subgroup $\Rightarrow$
In non-faithful irrep of A_4 and S_4 always $K \mapsto \mathbb{1}$

Side remark: * Simple groups have only faithful non-trivial irreps *

Irreps of A_4:

One-dimensional irreps:

$\mathbf{1}^{(p)}: \quad k_i \mapsto 1,\ s \mapsto \omega^p\ (p = 0, 1, 2)$ with $\omega = e^{2\pi i/3}$

Three-dimensional irrep: K represented as diagonal matrices

$$\mathbf{3}: \quad k_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: A, \quad s \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} =: E$$

$\Rightarrow$

$k_2 = s k_1 s^{-1} \mapsto \mathrm{diag}\,(-1, -1, 1),\ k_3 = s k_2 s^{-1} \mapsto \mathrm{diag}\,(-1, 1, -1)$

Irreps of S_4:

$$
\begin{aligned}
\mathbf{1}: &\quad p \mapsto 1 \\
\mathbf{1}': &\quad p \mapsto \mathrm{sign}(p) \\
\mathbf{3}: &\quad k_1 \mapsto A, \quad s \mapsto E, \quad t \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} =: R_t \\
\mathbf{3}': &\quad k_1 \mapsto A, \quad s \mapsto E, \quad t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\mathbf{2}: &\quad k_i \mapsto 1, \quad s \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

Remarks: $\mathbf{2}$ is irrep of $S_3 \cong S_4/K$, $\mathbf{3}' = \mathbf{1}' \otimes \mathbf{3}$

Character table of A_4:

$T \cong A_4$	$C_1(e)$	$C_2(s)$	$C_3(s^2)$	$C_4(k_1)$
$(\# \ C)$	(1)	(4)	(4)	(3)
ord (C)	1	3	3	2
$\mathbf{1}^{(0)}$	1	1	1	1
$\mathbf{1}^{(1)}$	1	ω	ω^2	1
$\mathbf{1}^{(2)}$	1	ω^2	ω	1
$\mathbf{3}$	3	0	0	-1

Character table of S_4: $r := s^{-1} k_1 s t = (1423)$

$O \cong S_4$	$C_1(e)$	$C_2(t)$	$C_3(k_1)$	$C_4(s)$	$C_5(r)$
$(\#\ C)$	(1)	(6)	(3)	(8)	(6)
ord (C)	1	2	2	3	4
$\mathbf{1}$	1	1	1	1	1
$\mathbf{1'}$	1	-1	1	1	-1
$\mathbf{2}$	2	0	2	-1	0
$\mathbf{3}$	3	-1	-1	0	1
$\mathbf{3'}$	3	1	-1	0	-1

Symmetric and alternating groups

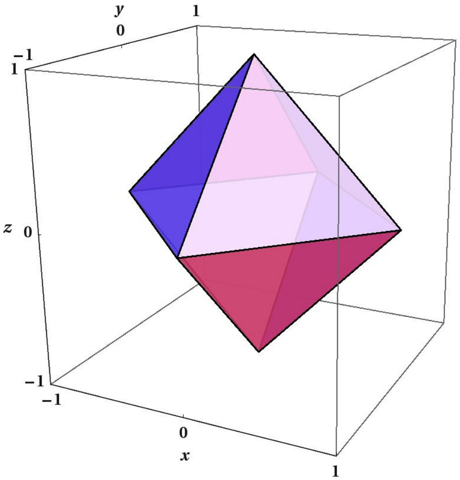
Remark:
Of the non-trivial symmetric and alternating groups, only
$S_3,\ A_4,\ S_4,\ A_5$
can be considered as finite subgroups of $SO(3)$,
i.e. possess a faithful representation by 3×3 rotation matrices.

- $S_3 \cong S_\triangle =$ symmetry group of unilateral triangle
- $A_4 \cong T =$ symmetry group of tetrahedron
- $S_4 \cong O =$ symmetry group of octahedron
- $A_5 \cong I =$ symmetry group of icosahedron

$A_4,\ S_4,\ A_5$ are the symmetry groups of the Platonic solids

Classes of rotation groups: $R_2\, R(\alpha, \vec{n})\, R_2^{-1} = R(\alpha, R_2 \vec{n})$

Symmetric and alternating groups



The finite subgroups of $SU(3)$

H.F. Blichfeldt (1916)[1]:

Classification of the finite subgroups of SU(3) into five types:

(A) Abelian groups.

(B) Finite subgroups of $U(2)$

(C) The groups $C(n, a, b)$ generated by the matrices

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n, a, b) = \mathrm{diag}(\eta^a, \eta^b, \eta^{-a-b}),$$

where $\eta = \exp(2\pi i/n)$.

[1] G.A. Miller, H.F. Blichfeldt and L.E. Dickson: Theory and applications of finite groups, New York (1916)

(D) The groups $D(n,a,b;d,r,s)$ generated by E, $F(n,a,b)$ and

$$\widetilde{G}(d,r,s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix},$$

where $\delta = \exp(2\pi i/d)$.

(E) Six exceptional finite subgroups of $SU(3)$:

- $\Sigma(60) \cong A_5$, $\Sigma(168) \cong PSL(2,7)$
- $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$ and $\Sigma(360 \times 3)$,

as well as the direct products $\Sigma(60) \times \mathbb{Z}_3$ and $\Sigma(168) \times \mathbb{Z}_3$.

The finite subgroups of $SU(3)$: (A) Abelian groups

Simple (but powerful) theorem: P.O. Ludl (2011)

Abelian finite subgroups of SU(3)

Every finite Abelian subgroup A of SU(3) is isomorphic to

$$\mathbb{Z}_m \times \mathbb{Z}_p,$$

where m is the maximum of the orders of the elements of A and n is a divisor of p.

The finite subgroups of $SU(3)$: type (C) and (D)

$$\begin{aligned} C(n,a,b) &\cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3 \\ D(n,a,b;d,r,s) &\cong (\mathbb{Z}_{m'} \times \mathbb{Z}_{p'}) \rtimes S_3 \end{aligned}$$

Note: $n, a, b \Rightarrow m, p$ in a complicated way
n divisor of m, p' divisor of m'

For which (m,n), (m',n') do such groups exist?
No complete solution known

Dimensions of irreps (Grimus, Ludl (2012)):
Type (C): 1 and 3
Type (D): 1, 2, 3 and 6

The finite subgroups of $SU(3)$: type (C)

Special cases of type (C):

- $p = m$: Such groups exist $\forall$ m
  $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3 \equiv \Delta(3m^2)$
- $p = 1$: $T_m \cong \mathbb{Z}_m \rtimes \mathbb{Z}_3$ where m is a product of powers of primes of the form $6k + 1$.
- $A_4 \cong \Delta(12)$
- $C(9, 1, 1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$

Note: $\Delta(3m^2)$ generated by E, $\mathrm{diag}\left(\eta, 1, \eta^{-1}\right)$, $\mathrm{diag}\left(\eta^{-1}, \eta, 1\right)$ with $\eta = \exp(2i\pi/m)$

Special cases of type (D):

- $p = m$: Such groups exist $\forall\ m$
  $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes S_3 \equiv \Delta(6m^2)$
- $S_3 \cong \Delta(6),\ S_4 \cong \Delta(24)$
- $D(9,1,1;2,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes S_3$

Summary: “number theorems” of finite groups

Divisors of ord G:

- order of a subgroup
- order of an element
- number of elements in a class
- dimension of an irrep

$$\text{number if inequivalent irreps} = \text{number of classes}$$

$$\text{Irreps } D^{(\alpha)} \text{ with } \dim D^{(\alpha)} = d_\alpha \quad \Rightarrow \quad \sum_\alpha d_\alpha^2 = \text{ord } G$$

Thank you for your attention!