R-operation, anomalous dimensions and all that

Konstantin Chetyrkin

Main topic of the talk:

discussion of the current status of the multiloop RG calculations (β-functions and anomalous dimensions) as for their *algebraic* aspects in connection to the R-operation and its generalizations (R^*-operaion)

DESY, Zeuthen 17.01.2013

”Es gibt nichts Praktischeres als eine gute Theorie”

“Нет ничего более практического, чем хорошая теория”

“Nothing is more practical than a good theory”

(A popular saying of another great physicist **Ludwig Boltzmann** addressed to all his students and friends)

One of Main Aims of the present talk:
to demonstrate that the **Bogolyubov-Parasiyuk-Hepp- Zimmerman R-operation*** is the **GOOD one** in the Boltzmann's sense:

combined with Dim. Regularization and the Minimal Subtraction Scheme the R-operation has developed in the powerful and versatile tool for multiloop RG calculations

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* **N.N. Bogoliubov , Izwestija Akad. Nauk SSSR, 19 (1955) 237; N.N. Bogoliubov and O.S. Parasyuk, Doklady Akad. Nauk SSSR, 100 (1955) 25-28, 429-432; also Acta Mathematica, 97 (1957), 227-266.**

Important qualification:

- There will be (almost) no discussion of the tremendous (and still ongoing!) progress of last 30 $+$ years in computing multiloop Feynman Integrals (FI's) per se (see best-seller books by Volodja Smirnov).

  Instead, I will concentrate on the various methods (all related to R-operation) of massaging of FI's before real calculations are even started to make the final evaluatiuon of Z-constants (read UV-counterterms) as simple as possible.

- There will be no discussion of very interesting and promising but rather theoretical recent developments of the theory of the R-operation related to Hopf algebra, knot theory, etc. /D. Kreimer, D. Broadhurst, A. Connes, . . . /

  The reason: the developments have not yet (IMHO) fully demonstrated their potential in practice of multiloop calculations..

Consider a simple theory, the ϕ^4-model with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4$$

and let

$$\Gamma[\mathcal{L}, \phi]$$

is the the generating functional of all 1PI Green functions corresponding to the Lagrangian $\mathcal{L}$ (some UV regularization is assumed). Renormalizability of the model means that there exits such a choice of the *renormalization constants* Z_2, Z_m and Z_4 (in the form of the *formal* series in the cc g, starting from 1) that the *renormalized* generating functional

$$\Gamma_r[\phi] \equiv \Gamma[\mathcal{L}^c, \phi]$$

with

$$\mathcal{L}^c = \frac{1}{2}Z_2(\partial\phi)^2 - \frac{1}{2}Z_m m^2\phi^2 - \frac{g}{4!}Z_4\phi^4$$

produces finite (after regularization is removed) Green functions in every order of PT in the coupling g.

To prove the renormalzability Bogolyubov and Parasyuk invented the R-operation. Let'remind some definitions.

Let $< \Gamma >$ be a Feynman integral (FI) corresponding to a diagram Γ, then R-operation is defined as

$$R < \Gamma >= \sum_{\gamma_1,\dots\gamma_j} \prod_i \Delta(\gamma_i) < \Gamma >= R' < \Gamma > + \Delta(\Gamma) < \Gamma >$$

1. sum goes over all sets $\{\gamma_1, \dots \gamma_j\}$ of (pairwise) disjoint 1PI subgraphs, with $\Delta(\emptyset) = 1$

2. $\Delta(\gamma)$ is a counterterm (c-) operation which acts as follows:

$$\Delta(\gamma) < \Gamma >= P_\gamma * < \Gamma/\gamma >$$

3. $P_\gamma = \Delta < \gamma >$ is a polynomial in external momenta (mandatory) and masses (desirable) of FI $< \gamma >$ which is inserted into the vertex v_γ inside of the reduced graph Γ/γ

4. a specific choice of the c-operation $\longleftrightarrow$ choice of a renormalization scheme

fine print for experts in R-operation:

I have modernized a little bit the definitions, original ones were somewhat different as they corresponded to the use of normal ordering in the Lagrangian. The difference between both formulations is well-understood and, as a metter of principle, is not essential . . .

Main (analytical) theorem of the R-operation:

If FI $< \Gamma >$ does not contain IR divergences (which is certanly true if all lines are massive and extrenal momenta are off-shell), then the $renormalized$ FI $R < \Gamma >$ is finite in the limit of removed $UV\ regularization$ if the c-operation is choosen as follows:

$$\Delta(\gamma) < \gamma >= -T_{\omega_\gamma} R' < \gamma >$$

where the operator T_n select n first terms of the Taylor expansion of the FI $R' < \gamma >$ in the corresponding external momenta and ω_γ is the UV index of divergency of the FI $< \gamma >$.

In terms of the R-operation the generating functional of the renormalized Green function is written as:

$$\Gamma_r[\phi] = R\,\Gamma[\mathcal{L}, \phi]$$

Important: new version does not contain any divergent counterterms and could be even formulated without any use of any regularization.

Still, in real life a good regularization is extremely usefull.

Connection to the multiplicative regularization and the Lagrangian with counter-terms is given by the following

Main (combinatorial) theorem of the R-operation:

$$\mathcal{L}^c \equiv \Delta\,\Gamma[\mathcal{L}, \phi]$$

The theorem provides us with a very convenient and flexible way of computing of contributions to Z-factors from *separate diagrams*.

Few words about proofs. The original proof of the main analytical theorem is notoriously difficult. Later it was somewhat simplified in works by S. Anikin, O. Zavialov, B. Stepanov, K. Hepp, W. Zimmerman and others.

Significant simplifications have beem made by E. Speer in late 60-ties and early 70-th by introducing a dedicated system of sectors (in alpha-representation). **Fortunately, for practice is enough to know that the proof exists.**

The combinatorial theorem is significantly simpler to prove and the proof is much more instructive and usefull for applications. For instance, the combinatoric of heavy mass expansions and Wilson's OPE at short distances is precisely the same as that of R-operation.

In early seventies t' Hooft and M. Veltman discovered and elaborated the Dimensional Regularization (DR) by formally considering* FI's as analytically continued in the space time dimension $D = 4 - 2\epsilon$. The DR and related Minimal Subtraction (MS) happened to be extremely convenient for all kinds of perturbative calculations and, *especially* for the RG calculations (that is for calculations of the β-functions and anomalous dimensions).

The R-operation for MS-scheme is easily formulated by defining the c-operation as follows

$$\Delta(\gamma) < \gamma >= -K\, R' < \gamma >$$

where K picks up the pole part in $\epsilon = 2 - D/2$. An important property of such formulated R-operation is its commutativity with differentiations wrt masses and external momenta. This commutativity naturally leads to the following remarkable statement

Theorem 1. (J. Collins, 75) *Any UV counterterm for any Feynman integral and, consequently, any RG function in arbitrary minimally renormalized model is a polynomial in momenta and masses.*

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* Mathematically correct definition of DR requires the use of α-parameters and has been done for massive case by Breitelohner and Maison in 1977 and by V. Smirnov and K.Ch. in 1984 for a general case of FI with UV **and** IR divergences

Starting point of multiloop RG business

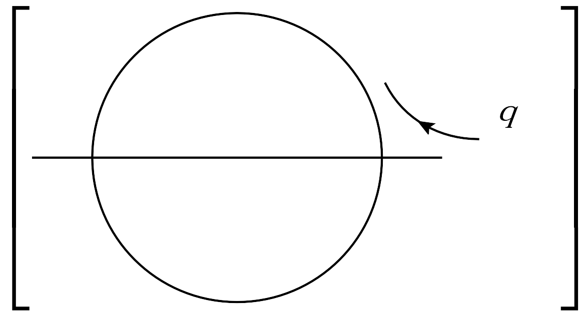
Since the early 70-th in Dubna the strong group of young LTF theorists led by D.V Shirkov (V. Belokurov, D. Kazakov, O. Tarasov and A. Vladimirov) began to investigate the RG properties of various QFT models including the scalar fields . . . .

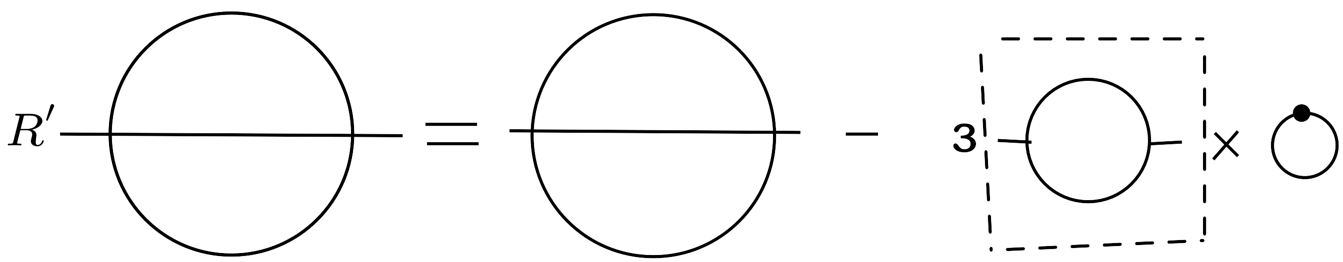
That was the beginning of the multiloop RG. In particular, such important results as the first calculation of the 3-loop β-function in non-abelian theories and 4-loop one for the ϕ^4 model were performed in the group in in the course of time (in fact, in the end of 70-th and the beginng of 80-th).

Such complicated calculations would be impossible without creating new methods. The first such method is based on the R-operation and the Theorem 1. It is now called Infrared ReaRrangement (IRR) /A. Vladimirov 1978/. Let me explain the essence of the method on the simplest example.

Suppose we want to compute contributions to Z_2 and Z_m from the diagram

$$q^2 \delta Z_2 + m^2 \delta Z_m = K R'$$





δZ_2 is simple: one just set $m = 0$ and the result directly comes from double application of the textbook 1-loop formula for *massless propagators* (called *p-integrals* later on):

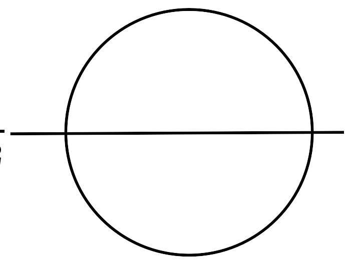
$$\boldsymbol{\frac{1}{i}\int \frac{d^D l}{(-l^{2\alpha})(-(q-l)^{2\beta})} = \pi^{D/2}(-q^2)^{2-\epsilon-\alpha-\beta} G(\alpha, \beta)}$$

with $G(\alpha, \beta)$ being just a simple combination of 6 Γ-functions.

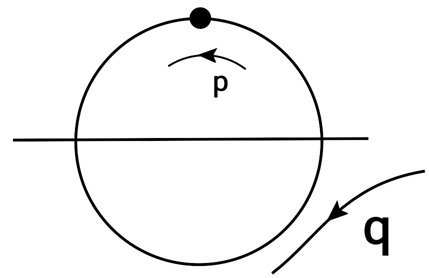
To compute δZ_m one could, of course, set $q = 0$ but resulting 2-loop massive vacuum graph is certainly more complicated than 1-loop p-integral. Anticipating even more complicated cases in future, let us try to stay with massless integrals . . . .

Let us first perform a derivative wrt m^2 of the initial integral:

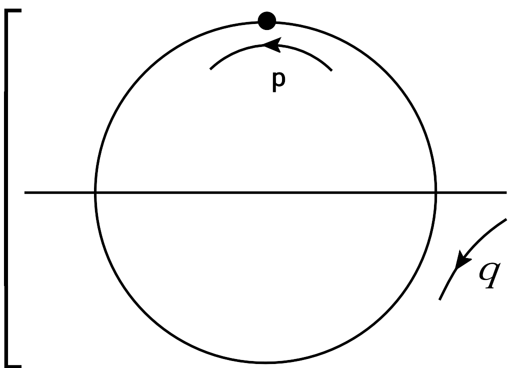
$$-\frac{\partial}{\partial m^2}$$



$$\Longrightarrow \quad \mathbf{3}$$



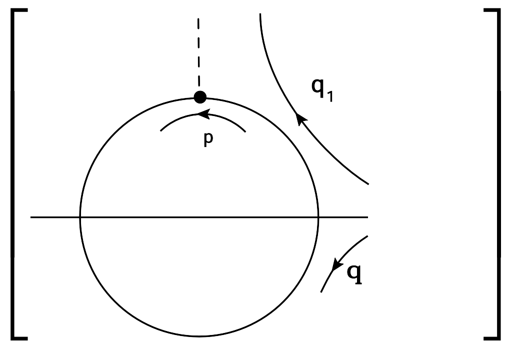
$$-\frac{1}{3}\,\delta Z_m = KR' \Bigg[\qquad \Bigg]$$



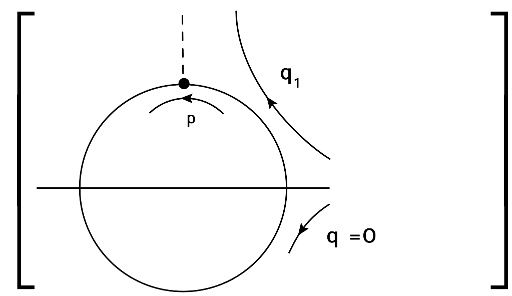
Now would be nice to set m=0 but we can not as the dotted line corresponds to $\frac{1}{p^4}$ and leads to an IR divergency! (which certainly spoil the result for δZ_m)

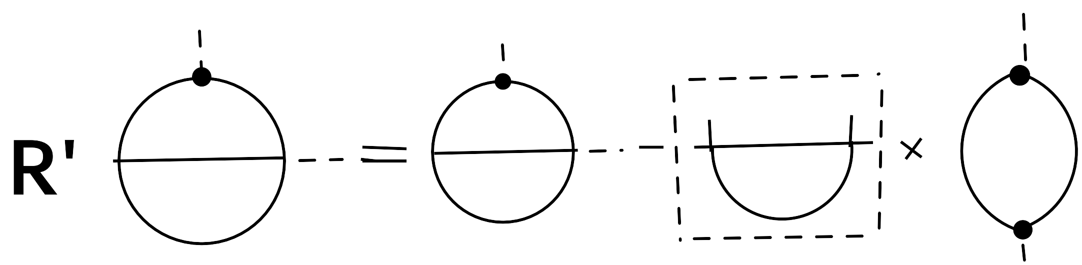
But as we are dealing with log-divergent integral, its UV counter-term is just a pole without *any* dependence on external momenta. So one could freely *change* external momenta without touching δZ_m!

$$-\frac{1}{3}\,\delta Z_m = KR'$$



$$= KR'$$



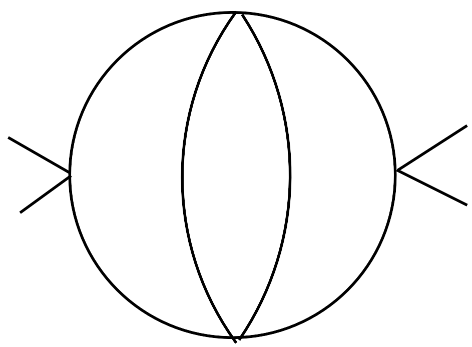


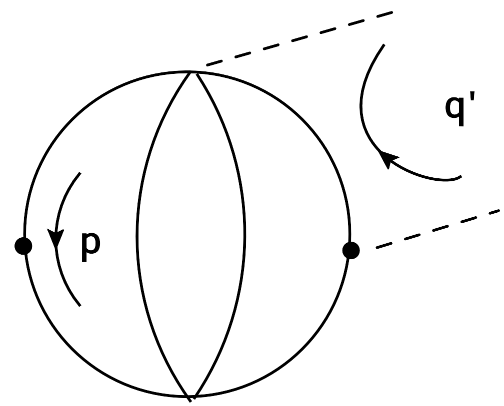
With $q = m == 0$ and $q' \neq 0$ there is no IR divergences and we could easily perform integrations using massless formulas.

The IRR in many cases is able to reduce the problem of evaluation of (L+1)-loop UV countertem to evaluation of some L-loop p-integrals (the latter is necessary to know up to and including the constant ϵ^0 part in the corresponding ϵ-expansion.

But *there are cases when it does not work*: no **simple** (read: flowing through exactly one line) choice of the external momentum in a massless FI can kill all IR problems:

An example:





Here the IR divergency in p-integration makes problems. One, of course, could regulate it with a small "auxiliary" mass:

$$\frac{1}{p^4} \to \frac{1}{(p^2+m^2)^2}$$

but that will complicates integration, leading to a 2-scale integral.

The idea how to overcome the problem (in fact, it came from the Bogolyubov's distributional approach to QFT) is very simple: to subtract the unwanted IR divergency with the help of an IR counterterm but now local in *position space*:

$$\frac{\mathbf{1}}{\boldsymbol{p}^4} \;\to\; \frac{\mathbf{1}}{(\boldsymbol{p}^4)} \;-\; \frac{\boldsymbol{c}}{\boldsymbol{\epsilon}}\,\boldsymbol{\delta}^{\boldsymbol{D}}(\boldsymbol{p})$$

with the constant $\boldsymbol{c}$ choosen such that there would be no IR poles coming from the integration region of small momentum p.

After such a replacement no IR poles survive and integrations are made easily.

This simple observation led to the so-called R^*-operation[1]:

a generalization of the R-operation which recursively subtracts all UV and IR divergences from any (Euclidean!) Feynman integral:

$$R^* = R\,\tilde{R} = \tilde{R}\,R$$

where $\tilde{R}$ is so-called **Infrared** R-operation which removes all IR poles from any (Euclidean!) FI.

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[1] K.Ch., F. Tkachov (1982); K.Ch., V. Smirnov (1984-1991)

The main use of the R^* -operation is in proof of the following statement

Theorem 2. *Any (L+1)-loop UV counterterm for any Feynman integral may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

Theorem 2 is a key tool for multiloop RG calculations as it reduces the general task of evaluation of (L+1)-loop UV counterterms to a well-defined and clearly posed purely mathematical problem: the calculation of L-loop p-integrals (that is massless propagator-type FI's).

In the following we shall refer to the latter as the L-loop Problem.

1. 1-loop Problem is trivial

2. the 2-loop Problem was solved after inventing and developing the Gegenbauer polynomial technique in x-space (GPTX) (K.Ch.,F. Tkachov (1980)). *In principle* GTPX is applicable to analytically compute some quite non-trivial three and even higher loop p-integrals. However, in practice calculations quickly get clumsy, especially for diagrams with nominators.

The main breakthrough at the three loop level happened with elaborating the method of integration by parts of DR integrals

. . .

. . .

. . .

Historical references:

At one loop, IBP (for DR integrals) was used in *, a crucial step — an appropriate modification of the *integrand* before differentiation was undertaken in ** (in momentum space, 2 and 3 loops) and in *** (in position space, 2 loops)

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* G. 't Hooft and M. Veltman (1979)
*** A. Vasiliev, Yu. Pis'mak and Yu. Khonkonen (1981)
** F. Tkachov (1981); K. Ch. and F. Tkachov (1981)

4-loop Problem has been under study in the Karlsruhe-Moscow group (P. Baikov, K.Ch., J. Kühn) since late 90th. It is solved by now. Master integrals are all computed and published. The (idea of) coresponding reduction algorithm (based on the Baikov's method) is published but its computer implementation and details of the algorithm are not publicly available.

As a result during last 10 years in our group the the results for $R^{VV}(s)$ and a closely related quantity – Z-decay rate into hadrons have been extended by one more loop (that is to order α_s^4).

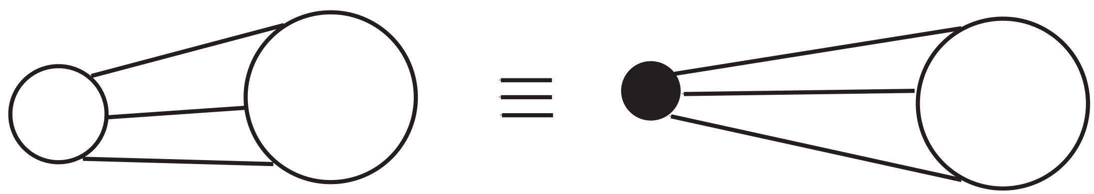
These results +some others related to 5 and 4-loop correlators (Higgs decays into hadrons, etc.) can be found in:

Phys.Rev.Lett. 88 (2002) 01200
Phys.Rev.Lett. 95 (2005) 012003
Phys.Rev.Lett. 96 (2006) 012003
Phys.Rev.Lett. 97 (2006) 061803
Phys.Rev.Lett.101:012002,2008
Phys.Rev.Lett. 102 (2009) 212002
Phys.Rev.Lett.104:132004,2010
Phys.Rev.Lett. 108 (2012) 222003
JHEP 1207 (2012) 017
Phys.Lett. B714 (2012) 62-65

$\tilde{R}$-operation: some technical details in modern exposition

Let us consider a typical situation: we are given a log-divergent (and Euclidean!) 1PI FI $< \Gamma >$ without external momenta and with only one massive line ℓ_h. How to identify and remove all possible IR poles from it? What is an IR analog of a 1PI UV divergent subgraphs?

Let $\gamma_h \subset \Gamma$ be a hard subgraph of Γ: that is $\ell_h \in \gamma$ and the graph γ/ℓ_h is 1PI. It is usefull to consider two decompositions:



Γ $\qquad\qquad$ Γ/γ_h

graph decomposition: $\Gamma \equiv \gamma_h \star \Gamma/\gamma_h$ and FI decomposition: $< \Gamma > \equiv < \gamma_h > \star < \Gamma/\gamma_h >$

A *co-subgraph* $\tilde{\gamma}_h$ is defined as Γ/γ_h; the corresponding FI $< \tilde{\gamma}_h >$ is, obviously, 1PI massless tadpole. It is a full IR analog of a UV divergent 1PI subgraph with IR index of divergence $\tilde{\omega}(< \tilde{\gamma}_h >) = -\omega_{UV}(< \tilde{\gamma}_h >)$.

IR c-operation is defined as (assuming that $\tilde{\omega}(< \tilde{\gamma}_h >) = 0$, which is essentially a general case, see below)

$$\tilde{\Delta} < \tilde{\gamma}_h > = \tilde{Z}_{\gamma_h},$$

with $\tilde{Z}_{\gamma_h}$ being some pure polynomial in $1/\epsilon$.

Now we define the the IR $\tilde{R}$ operation as follows:

$$\tilde{R} < \Gamma >= \sum_{\gamma_h} T_{\tilde{\omega}} < \gamma_h > \star \tilde{\Delta} < \tilde{\gamma}_h >$$

where the operator T_n select n first terms of the Taylor expansion of the FI $R' < \gamma >$ in the corresponding external momenta.

Comments:

1. the main theorem of $\tilde{R}$-operation is that a proper choice of IR countertems makes $\tilde{R} < \Gamma >$ completely IR finite and, correspondingly, the combination $\tilde{R}R < \Gamma >$ **completely** finite.

2. For any massless tadpole $< \gamma_0 >$ $\tilde{\Delta} < \gamma_0 >$ could be found in terms of UV counterterms for $< \gamma_0 >$ in its UV subgraphs.

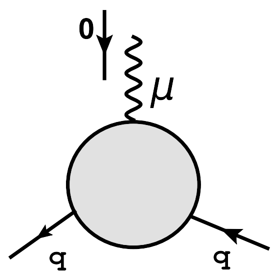
3. main formula for UV Z-factors:

$$Z_\Gamma = -K\tilde{R} \cdot R' < \Gamma > \quad (\star)$$

If $< \Gamma >$ has at least 1 massive line then all IR Z-factors appearing on the rhs of $(\star)$ will come from co-subgraphs with loop number *strictly* less than that of original FI $< \Gamma >$. This nicely explains why we can not start from a purely massless tadpole.

Example of global calculation of an IR Z-factor

Let us consider vertex function $\Gamma_B(a_s^0, q^2) = 1 + \delta\Gamma_B(a_s^0, q^2)$ of a vector quark curent:



$$= \gamma_\mu \cdot \Gamma_B(a_s^0, q^2)$$

Its renormalized (and, hence, *finite*) version reads:

$$\Gamma_R(a_s, q^2) = Z_2\,\Gamma_B(a_s^0, q^2) = Z_2 + Z_2\,\delta\Gamma_B(a_s^0, q^2)$$

In fact, the combination $Z_2\,\delta\Gamma_B(a_s^0, q^2) = -K\,R'\,\Gamma(a_s, q^2)$. Now we set $q \equiv 0$, then the vertex function $\Gamma_R(a_s, q = 0)$ will be *not* zero but Z_2!

Now if we apply, in addition, the IR $\tilde{R}$ operation everything should be finite. Natural normalization of R^* operation is $R^* <$ massless tadpole $>\equiv 0$.
Thus, we have:

$$Z_2 + Z_2\,\tilde{\Delta}\,\delta\Gamma_B(a_s^0, q = 0) = 1$$

and, consequently,

$$\tilde{\Delta}\,\delta\Gamma_B(a_s^0, q = 0) = -1 + \frac{1}{Z_2}$$

By defining that $\tilde{\Delta} 1 = 1$ we arrive to a beautiful formula

$$\tilde{\Delta}\, \Gamma_B(a_s^0, q=0) = \frac{1}{Z_2}$$

Note that the appearance of an inverted Z-factor in the above formula is not by chance: in fact, the $\tilde{R}$ operation is intimately related to so-called R^{-1}-operation (F. Tkachov, G. Pivovarov (1986); K. Ch. (1989)) defined by the equation:

$$R^{-1} \cdot R = 1$$

β-function in 6 loop in ϕ^4-model

First 5-loop RG calculation in a 4-dim model (ϕ^4-model) was done *long before* the 4-loop Problem was solved:

/K.Ch, Gorishnii, Larin, and Tkachov, Phys.Lett. B132 (1983) 351/; D.I. Kazakov,Phys.Lett. B133 (1983) 406; Kleinert, Neu, Schulte-Frohlinde , K.Ch., and Larin, Phys.Lett. B272 (1991) 39/

The reason: relative simplicity of the corresponding Feynman amplitudes (only limited number of topologies, no nominators).

What about 6 loops? (Would be of some use for the the statistical physics /critical indexes/

Analytically: no hope at present (imho)

Numerically: no hope at present (imho)

Mixed way: YES! (imho)

WHY?

Lets neglect all IR singularities (having in mind the possibility of their complete removal with $\tilde{R}$-operation). Then the IR reduction by one loop could be easily understood as follows (for a log-divergent (L+1)loop FI $< \Gamma >$):

1. set zero all (except for one) massess and all external momenta
2. cut the massive line with internal momentum ℓ, then there is an obvious formal representatin:

$$< \Gamma >= \int < \Gamma' > (\ell) \, \frac{d\ell}{m^2 + \ell^2}$$

Now in order to find the UV div. of $< \Gamma >$ one should, obviously, compute the p-integral $< \Gamma' > (\ell)$ including its constant (ϵ^0 part) + some some "easy" FI's with less # of loops then L. (Note that since $< \Gamma' > (\ell) \approx 1/\ell^{2\alpha}$ the last integral over $d\ell$ is trivial!) That is esentially the statement of Theorem 2!

3. for the ϕ^4 model in many cases the FI $< \Gamma' > (\ell)$ could be chosen to be a product of 2 FI's with loop numbers less then 5 $\rightarrow$ representable in terms of 4-loop p-ntegrals

4. All *primitive* (that is without UV subdivergences) 6-loop contributions to the ϕ^4 β-function are known with high accuracy since long (via GPTx, D. Broadhurst, D. Kreimer (1995))

4 Γ_4

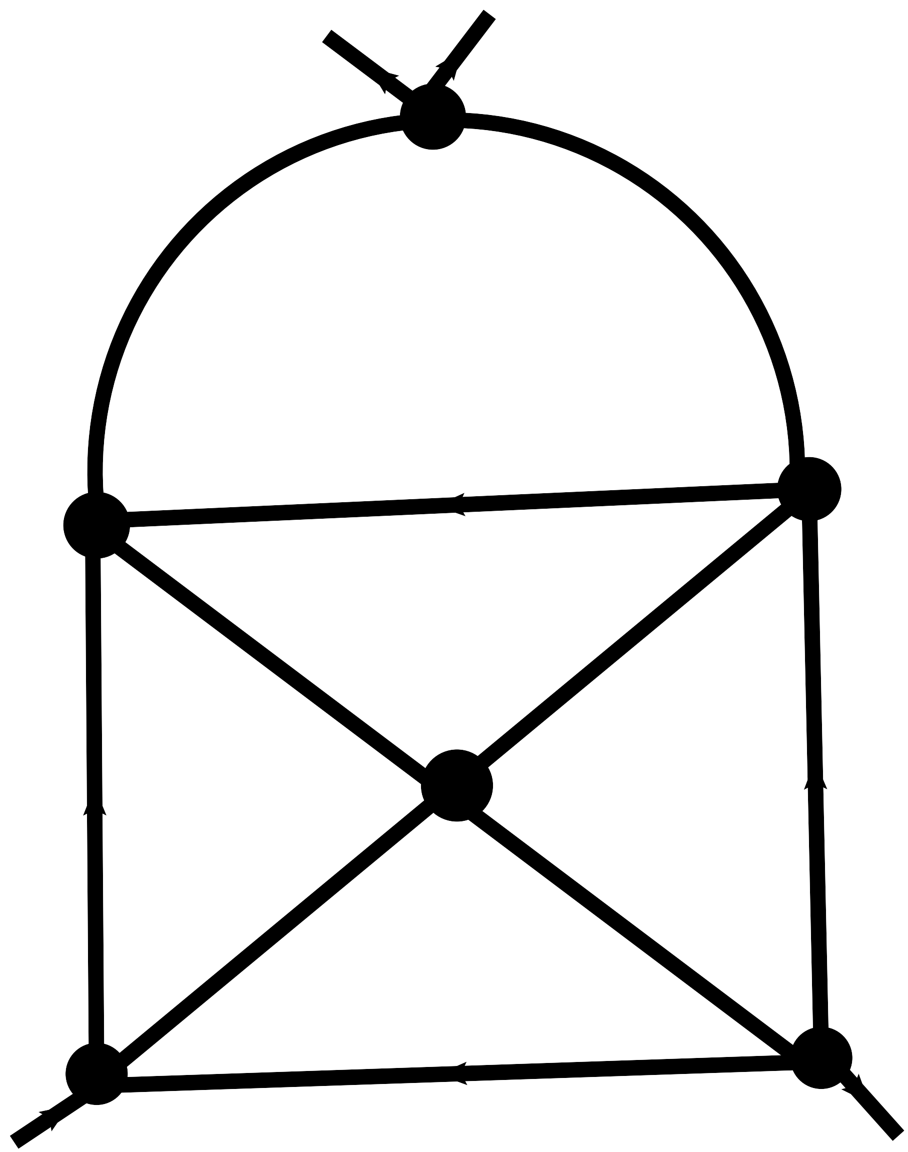
loops	total	factorized	primitive	(1-2)-loop irreducible	(1-2)-loop reducible
1	1	0	1	0	0
2	2	1	0	0	1
3	8	2	1	0	5
4	26	7	1	0	18
5	124	28	3	2	91
6	627	125	10	12	480

decomposition of the vertex diagrams of the ϕ^4 model (due to Misha Kompaniets):

The 12 really difficult diagrams could be certainly computed numerically (via some kind of sector technique. Recently a few complicated scalar 6-loop integrals have been evaluated in this way in "$D = 5$ maximally supersymmetric Yang-Mills theory diverges at six loops" Z. Bern, J. Joseph Carrasco, L. Dixon, M. Douglas, Matt von Hippel, H. Johansson, e-Print: arXiv:1210.7709.

The citation from above:

"Thus the computation must be split into smaller parts to make it feasible. To do so, we only need a small portion of the FIESTA 2 software, and we have extracted this portion and adapted it to our purpose by hand. We did the sector decomposition on a small (10 node) cluster, and then performed the numerical integrations on a large (1000 node) cluster, using the adaptive quasi-Monte Carlo integrator Vegas [54]."



Quark Mass Anomalous Dimension $\gamma_m = \sum_{i\geq 0} \gamma_i \, a_s^i$: history

3-loops: /O, Tarasov (82, with IRR reduced to 2-loop p-integrals);

3-loops: /S. Larin/ (92; direct evaluation of 3-loop p-integrals with MINCER)

4-loops: /K. Chetyrkin/ (97; with R^*-operation all FI's were reduced to 3-loop p-integrals; the latter were performed with MINCER)

4-loops: /J.A.M. Vermaseren, S.A. Larin, T. van Ritbergen/ (97; direct evaluation of the **completely massive 4-loop tadpoles** /via a kind of Laporta machine (?)/)

$$\gamma_0 = 1 \quad \gamma_1 = \frac{1}{16}\left\{ \frac{202}{3} + n_f \left[-\frac{20}{9} \right] \right\}, \gamma_2 = \frac{1}{64}\left\{ 1249 + n_f \left[-\frac{2216}{27} - \frac{160}{3}\zeta(3) \right] + n_f^2 \left[-\frac{140}{81} \right] \right\}$$

$$\begin{aligned}
\gamma_3 = {} & \frac{1}{256}\left\{ \frac{4603055}{162} + \frac{135680}{27}\zeta(3) - 8800\,\zeta(5) \right. \\
+ & \quad n_f \left[-\frac{91723}{27} - \frac{34192}{9}\zeta(3) + 880\,\zeta(4) + \frac{18400}{9}\zeta(5) \right] \\
+ & \quad n_f^2 \left[\frac{5242}{243} + \frac{800}{9}\zeta(3) - \frac{160}{3}\zeta(4) \right] + n_f^3 \left[-\frac{332}{243} + \frac{64}{27}\zeta(3) \right] \left. \right\}.
\end{aligned}$$

Quark Mass Anomalous Dimension $\gamma_m = \sum_{i\geq 0} \gamma_i \, a_s^i$: today

New result (preliminary) /P. Baikov, J. Künh, K. Ch./ (2013; with R^*-operation all FI's were reduced to 4-loop p-integrals; the latter were performed with BAICER)

$$\gamma_4^{n_f=3} = -\frac{156509815}{497664} + \frac{23663747}{124416}\,\zeta_3 - 85\zeta_3^2 - \frac{23765}{256}\,\zeta_4 + \frac{22625465}{62208}\,\zeta_5 - \frac{1875}{32}\,\zeta_6 - \frac{118405}{576}\,\zeta_7$$

Numerically:

$$\gamma_m^{n_f=3} = -\left\{a_s + 3.792\,a_s^2 + 12.420\,a_s^3 + 44.263\,a_s^4 + 198.906\,a_s^5\right\}$$

To construct scale-inavariant mass (or to run the quark mass) one needs also β-function at 5-loop (not yet available)

$$\beta(n_f = 3) = -\left(\beta_0 = \frac{4}{9}\right) \cdot \left\{a_s + 1.777\,a_s^2 + 4.4711\,a_s^3 + 20.990\,a_s^4 + \overline{\beta}_4\,a_s^5\right\}$$

It is natural to estimate $\overline{\beta}_4$ as sitting in the interval 50 –100

The mass evolution is described by equation $\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}$ where

$$
\begin{aligned}
c(x) &= \exp\left\{\int \frac{dx'}{x'}\frac{\gamma_m(x')}{\beta(x')}\right\}(x)^{\bar{\gamma}_0}\left\{1 + (\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)x\right. \\
&+ \frac{1}{2}\left[(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0\right]x^2 \\
&+ \left[\frac{1}{6}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^3 + \frac{1}{2}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)(\bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0)\right. \\
&\left.\left. + \frac{1}{3}\left(\bar{\gamma}_3 - \bar{\beta}_1{}^3\bar{\gamma}_0 + 2\bar{\beta}_1\bar{\beta}_2\bar{\gamma}_0 - \bar{\beta}_3\bar{\gamma}_0 + \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_2\right)\right]x^3 + \mathcal{O}(x^4)\right\}
\end{aligned} \tag{1}
$$

$\bar{\gamma}_i = \gamma_i/\beta_0$, $\bar{\beta}_i = \beta_i/\beta_0$, (i=1,2,3) and β_i are the coefficients of the QCD beta-function

Running (strange quark) mass from the RGI mass $\hat{m} \equiv m(\mu_0)/c(a_s(\mu_0))$:

$$m_s(\mu) = c(a_s(\mu))\hat{m}_s$$

with ($c_s(x) \equiv c(x)$ in QCD with $n_f = 3$)

$$c_s(x) = x^{4/9}(1 + 0.895062\,x + 1.37143\,x^2 + 1.95168\,x^3 + (15.6982 - 0.1111\overline{\beta}_4)\,x^4)$$

The function $c(x)$ is used, e.g, by the **ALPHA** collaboration to find the ($\overline{\mathsf{MS}}$) mass of the strange quark at a lower scale from the RGI mass determned from lattice simulations

Example (setting $a_s(\mu = 2\,\mathsf{GeV}) = \frac{\alpha_s(\mu)}{\pi} = .1$; h counts loops)

$$m_s(2\,\mathsf{GeV}) = \hat{m}_s \cdot (a_s(2\,\mathsf{GeV}))^{\frac{4}{9}} \cdot$$

$$\left(1 + 0.0895\, h^2 + 0.0137\, h^3 + 0.00195\, h^4 + (0.00157 - .000011\, \overline{\beta}_4)\, h^5\right)$$

Note that for any reasonable value of $\overline{\beta}_4$ (positive and ≤ 100) the (apparent) convergency of the above series is quite good even at rather small energy scale

Concluding Notes I:

- IRR based on R^* operation significantly simplifies RG calculations. It reduces (L+1)-loop RG function in any model to a combination of properly consructed p-integrals; the latter include not only standard UV- but also **IR** subtractions. It is always possible to do at the level of *separate* diagrams. IR counterterms are expressible diagramwise through UV-ones.
- IRR + R^* + Baikov Algorithm to reduce 4-loop p-integrals + paralell Form (J. Vermaseren, M. Tentyukov + ...) + known 4-loop masters (P. Baikov, K.Ch.) $\Longrightarrow$ the 5-loop RG functions are *in principle* doable in *any* model.
- But: global representation of neccessary IR subtractions (that is on the level of Green functions) strongly depends on the problem and not always easy.
- For example: the notorious problem of evalutaion of the 4-loop anomalous dimensions of spin n DIS operators of spin even for relatively moderate n (say, around five) is difficult. The problem is in global representation of corresponding IR subtractions. It boils down to our inability to globally renormalize a specific *gauge non-invariant operator*, namely:

$$\mathcal{D}_{\mu_1}\mathcal{D}_{\mu_2}\ldots\mathcal{D}_{\mu_n}\psi$$

Concluding Notes II:

- The 5-loop quark anomalous dimension γ_m is done in $n_f = 3$ QCD. The full n_f dependence will come soon (SFB in March?). The phenomenological implications seem to be not very dramatic.

- The 5-loop QCD β-function is significantly more complicated; first results are expected in a year or so. Phenomenogically result would be of some importance for the analysis of the τ-decay rate within so-called CIPT and for various QCD "optimization" schemes like PMS and PMC (the Principles of Maximal Sensitivity P. Stevenson, 1981) and of Maximal Conformality (S. Brodsky, X. G. Wu,L. Di Giustino,M. Mojaza, 2012).

  (The full QED β function in 5-loops is available since recently: P. Baikov, K. Ch., JH. Kühn, J. Rittinger, JHEP 1207:017,2012.).

- Truly remarkable fact: N=4 SYM theory seems to be simpler than QCD: "Konishi" (anomalous dimension of a specific operator in N=4 SYM) in 5-loop has been recenltly computed with a via IRR + p-intergrals + Laporta machine + a lot of ingenuity; the result confirms the prediction from non-perturbative methods (" Five-loop Konishi in N=4 SYM", B. Eden, P. Heslop, G. Korchemsky, V. Smirnov, E. Sokatchev, arXiv:1202.5733)

Concluding Notes III:

- There are some theoretical problems requiring analytical evaluation of 6-loop anomalous dimensions: e.g. "Konishi" (anomalous dimension of a specific operator in N=4 SYM) in 6-loop is already available from non-perturbative methods:

  Six and seven loop Konishi from Luscher corrections. Z. Bajnok, R. Janik e-Print: arXiv:1209.0791

  Here the main problem is the very reduction to masters (the way to compute the resulting masters is known /K.Ch. and Baikov, 2010). BUT: shear # of contributing diagrams in "normal" gauge theories would presumably be prohibitively large for, say, QCD 6-loop β-function. Probably the situation should be better for N=4 SYM and such quantities as $R(s)$ and DIS sum rules (here the next loop number is 5 *not* 6!)

- The 6-loop β-function in the ϕ^4-model is certainly doable in the "mixed" way. But a diagram-wise computer algebra implementation of the R^* operation is requied; it is certainly doable /the scalar theories are much simpler to deal with than the gauge ones, but not completely trivial/.