# High Order Corrections in Heavy to Light Decays

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# Introduction

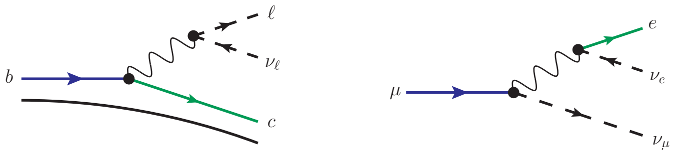
The Standard Model has been extremely successful.

This success relies on precision measurements of the free parameters.

Two values of interest are the Fermi coupling constant $G_F$ and the CKM matrix element $V_{cb}$.

# Introduction

Both parameters are measured using heavy to light decays.




In the limit $M_W \to \infty$ they are equivalent at LO and NLO.

At higher orders 3- and 4-gluon vertices play a role and introduce extra diagrams in QCD.

# Semileptonic B decay

The parton decay rate contributes to the inclusive decay rate $\Gamma(\overline{B} \to X_c \ell \overline{\nu}_\ell)$.

This is used to determine the most precise value of $V_{cb}$.

There is currently some small amount of tension between the inclusive and exclusive determinations.

$$
\begin{aligned}
|V_{cb}| &= (39.6 \pm 0.9) \times 10^{-3} \qquad \text{(exclusive)} \\
|V_{cb}| &= (41.96 \pm 0.45 \pm 0.07) \times 10^{-3} \qquad \text{(inclusive)}
\end{aligned}
$$

# From Meson to Parton

The total decay rate is composed from a few different factors.

$$\Gamma(\overline{B} \rightarrow X_c \ell \overline{\nu}_\ell) = \frac{G_F^2 m_b^5(\mu)}{192\pi^3} |V_{cb}| (1 + A_{ew}) A^{\mathrm{pert}}(\rho, \mu) A^{\mathrm{non\text{-}pert}}$$

$$\Gamma(b \rightarrow c \ell \overline{\nu}_\ell) = \frac{G_F^2 m_b^5(\mu)}{192\pi^3} |V_{cb}| A^{\mathrm{pert}}$$

The corrections to $A^{\mathrm{pert}}$ are known to $\mathcal{O}(\alpha_S^2)$.

# History of $\alpha_S^2$

The lepton pair can be replaced with a virtual boson using

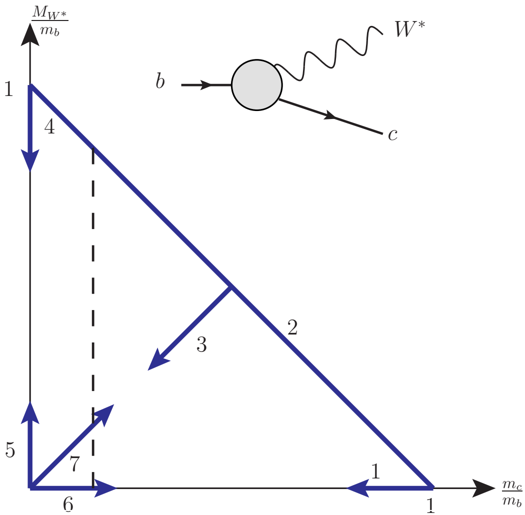
$$\Gamma(b \to c\ell\overline{\nu}_\ell) = \frac{g_W^2}{96\pi^2 M_{W*}^4} \int dq^2 q^2 \Gamma(b \to cW^*)\Big|_{M_{W*}=\sqrt{q^2}}$$

This simplifies the calculation but only gives the rate for a specific value of $M_{W*}$.

Computing the decay rate at a number of points allows an estimate of the full partonic rate.

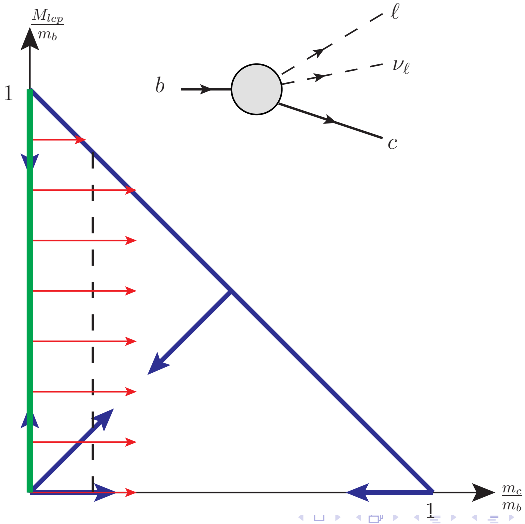
# The Different $\alpha_S^2$ Calculations

1. Czarnecki, Melnikov 97
2. Czarnecki, Melnikov 97
3. Czarnecki, Melnikov 99
4. Czarnecki, Melnikov 02
5. Blokland et al 05
6. Pak et al 06
7. M.D., Pak, Czarnecki 08



# The Different $\alpha_S^2$ Calculations

- van Ritbergen, Stuart 00
- Pak, Czarnecki 08
- Melnikov 08



# Muon Decay

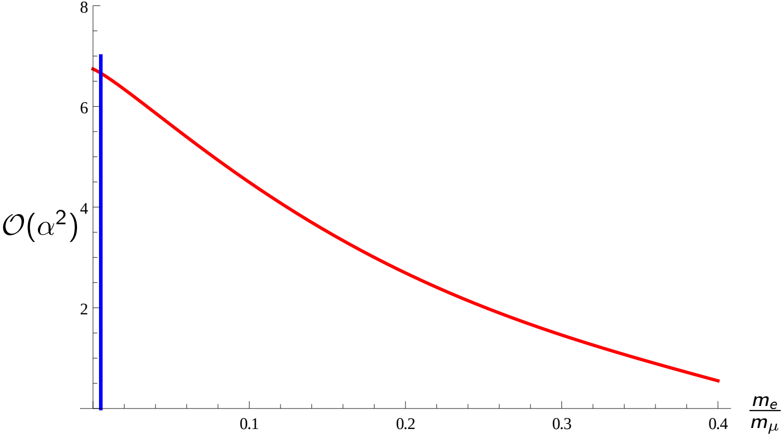
The decay rate of the muon is used to determine the most precise value of $G_F$.

The lifetime was recently measured at MuLan with a precision of 1ppm.

$$G_F = 1.1663787(6) \times 10^{-5}\ \mathrm{GeV}^{-2} \quad (0.5\mathrm{ppm})$$

At this precision, the $\mathcal{O}(\alpha^2)$ corrections were very important.

# Recent Precision in the Decay Rate




An unexpected correction at order $\frac{m_e}{m_\mu}$ was significant for the measurement.

# Do we need $\mathcal{O}(\alpha^3)$?

Compare the order of magnitude with the linear mass corrections at $\mathcal{O}(\alpha^2)$.

$$\alpha^2 \frac{m_e}{m_\mu} \approx \alpha^2 \frac{1}{200} \qquad \alpha^3 \approx \alpha^2 \frac{1}{137}$$

If the coefficient is large ($\approx 10$), the $\mathcal{O}(\alpha^3)$ corrections will be important.

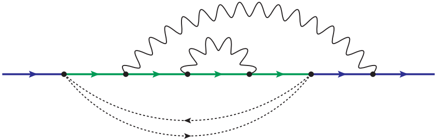
# The Calculations

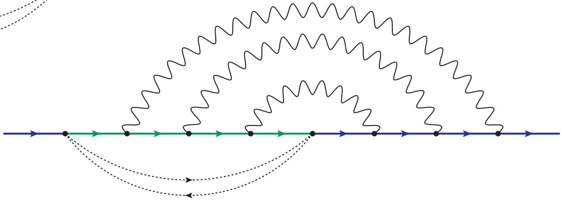
We want to compute the $\mathcal{O}(\alpha_S^2)$ corrections to the $b$-quark decay rate around the limit $m_c = m_b$.

We will find that the method provides some significant simplifications that make the $\mathcal{O}(\alpha^3)$ corrections to $\mu$-decay feasible.

# Method

The optical theorem is used, making these four and five loop calculations.

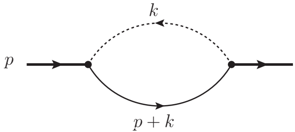




# Asymptotic Expansion

Asymptotic expansion is used to separate the two scales $M$ and $M\delta = M(1 - \frac{m}{M})$.

Regions are considered where the loop momenta are either hard (large) or soft (small).




$$\frac{1}{(k^2)^a[(k+p)^2 + m^2]^b}$$

# Asymptotic Expansion

Hard Region:

$$\begin{aligned}(k+p)^2+m^2 &= k^2+2p\cdot k-M^2+M^2(1-\delta)^2\\ &= \underbrace{k^2+2p\cdot k}_{1}-\underbrace{2M^2\delta}_{\delta}+\underbrace{M^2\delta^2}_{\delta^2}\\ &\approx k^2+2p\cdot k\end{aligned}$$

Integrals look like:

$$\frac{1}{(k^2)[(k+p)^2+m^2]}\approx\frac{1}{(k^2)[k^2+2p\cdot k]}\left(1+\frac{2M^2\delta-M^2\delta^2}{[k^2+2p\cdot k]}+\ldots\right)$$

All on-shell integrals. Real!

# Hard Integrals

A region that only includes hard integrals is real.

Since we only want the imaginary part, they do not contribute.

We only need two(three)-loop on-shell integrals at NNLO(NNNLO).

# Asymptotic Expansion

Soft Region:

$$\begin{aligned}(k+p)^2+m^2 &= k^2+2p\cdot k-M^2+M^2(1-\delta)^2\\ &= \underbrace{2p\cdot k-2M^2\delta}_{\delta}+\underbrace{k^2+M^2\delta^2}_{\delta^2}\\ &\approx 2p\cdot k-2M^2\delta\end{aligned}$$

Integrals look like:

$$\frac{1}{(k^2)[(k+p)^2+m^2]}\approx\frac{1}{(k^2)[2p\cdot k-2M^2\delta]}\left(1-\frac{k^2+M^2\delta^2}{[2p\cdot k-2M^2\delta]}\right)$$

All eikonal integrals. Have an imaginary part.

# Soft Integrals

The general form of an integral is

$$I(a,b,c,d) = \int \frac{d^D k}{(2\pi)^D} \frac{(k\cdot q)^d}{(k^2)^{a+\epsilon}[2p\cdot k]^b[2p\cdot k+\Delta]^c}$$
$$\propto \Delta^{i+j\epsilon}$$

$\Delta$ is either constant or of the form $2p\cdot k' + \Delta'$.

This leads to an iterative scheme for computing soft integrals.

# 3-Gluon Vertices

More difficult to calculate because of the extra massless propagator.

$$\frac{1}{(q^2)(q+k)^2(k^2)}$$

Iterative method can't be used anymore.

Laporta reduction is used to get rid of one of the massless propagators.

# Partial Fractions

In some cases, partial fraction decomposition is needed to simplify the integrals.

$$\frac{1}{[2p\cdot k_1 - 2M^2\delta][2p\cdot(k_1+k_2) - 2M^2\delta]}$$

$$= \frac{1}{[2p\cdot k_2]}\left(\frac{1}{[2p\cdot k_1 - 2M^2\delta]} - \frac{1}{[2p\cdot(k_1+k_2) - 2M^2\delta]}\right)$$

At NNLO, this will always reduce to integrals that can be evaluated.

# 4-Loop Soft Integrals

At NNNLO this iterative scheme does not work on some diagrams.

We end up with integrals like:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2)^a [2p \cdot k + \Delta_1]^b [2p \cdot k + \Delta_2]^c}$$

In this case, IBP is used to reduce everything to a set of master integrals.

# Programs

Some diagram generation was done using qgraf for the $b$-decay and M. Czakon's DiaGen for the $\mu$-decay.

FIRE was used for Laporta reduction of 3-gluon vertex contributions.

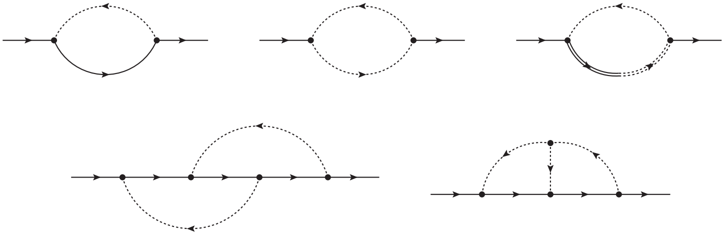
M. Czakon's IdSolver is being used for IBP reduction of soft integrals.

Other aspects of the calculation are computed using TFORM.

# Semileptonic $b$-decay

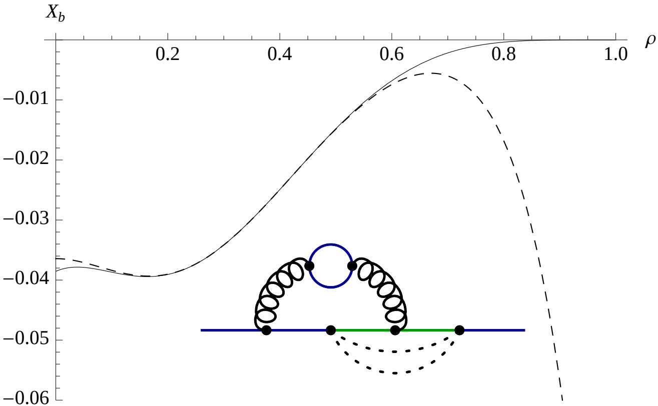
There are 39 four-loop diagrams.

All can be reduced to five master integrals of at most two loops.

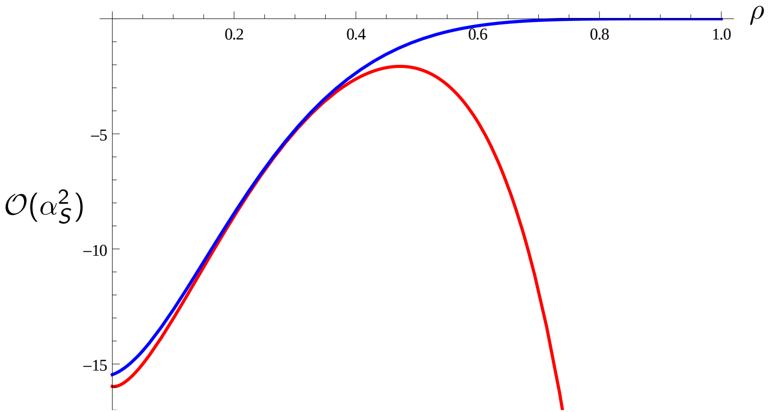


Computed analytically to $\mathcal{O}(\delta^{11})$.

# Corrections from Heavy Loops



# $\mathcal{O}(\alpha_S^2)$ Results



# Muon Decay

There are 249 five-loop diagrams.

The iterative method used at NNLO breaks here so we must turn to IBP.

This requires the generation of 3 897 849 400 IBP identities.

Run on WestGrid computers using $\sim$750 processors and 3.12 years of CPU time.

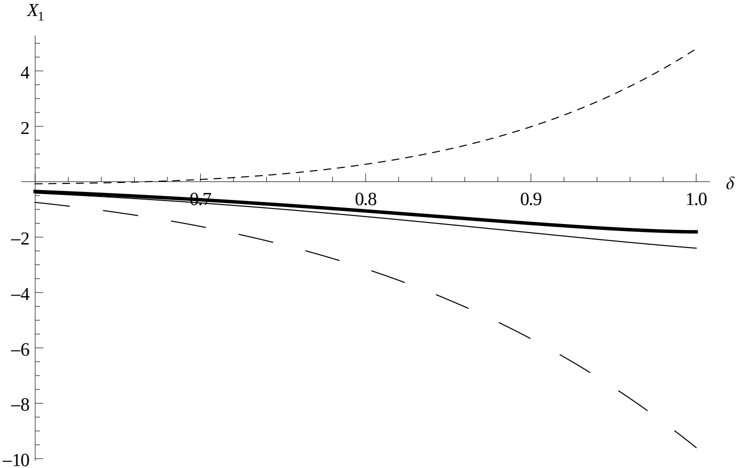
# Results

A single diagram can use on the order of 1TB of temporary storage.

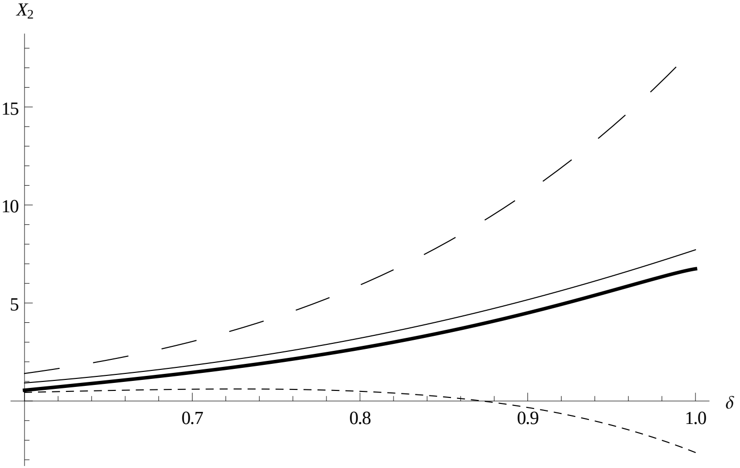
The calculation is still running but results have been obtained for the first two orders.

It is possible to use these results to obtain an estimate of the full correction.

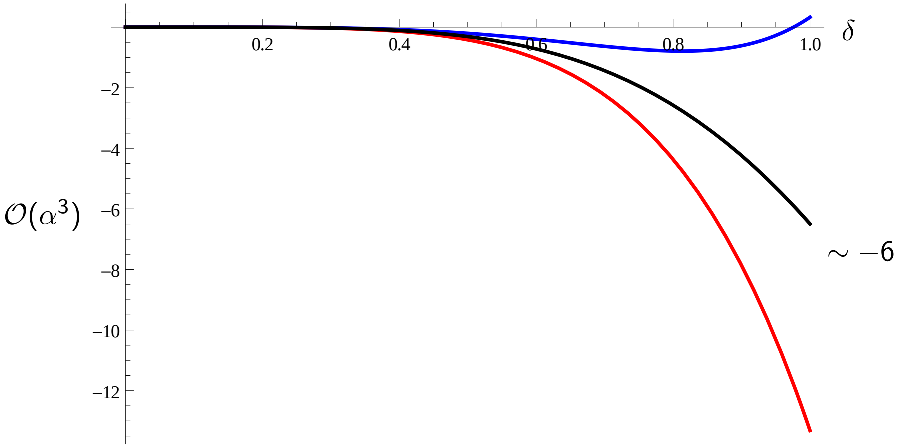
# Estimates (NLO)



# Estimates (NNLO)



# Estimates (NNNLO)



# Summary

The expansion from equal mass has some benefits even though it can be un-intuitive.

It provides the only method of computing $\mathcal{O}(\alpha^3)$ corrections to the muon decay rate.

This can be extended to the $\mathcal{O}(\alpha_S^3)$ corrections to the $b$-decay when it is needed.

It can also be applied to the calculation of the angular distribution between the heavy and light particle.