Reduze 2, matroids, symbols
and their application to
Top quark pairs at two loops

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Theory Seminar
DESY Zeuthen
15 November 2012
**Top pair production at the LHC**

- **LHC precision** below NLO accuracy already
- **full WWbb** at NLO
- recent **threshold/resummations, approx. NNLO**: Beneke, Falgari, Klein, Piclum, Schwinn, Ubiali, Yan; Moch, Uwer, Vogt; Cacciari, Czakon, Mangano, Mitov, Nason; Ahrens, Ferroglia, Neubert, Pecjak, Yang; Kidonakis
- full NNLO for $q\bar{q}$: Bärnreuther, Czakon, Mitov ’12
Gauge invariant subsets in two-loop contributions

\textbf{gg channel:} 789 two-loop diagrams (+ ghost init.) contrib. to 16 coeff.:

\[ 2 \text{Re} \left\langle M^{(0)} | M^{(2)} \right\rangle = 2 C_F N_c \left( N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D \right) \]

\[ + N_c^2 n_l E_l + n_l F_l + \frac{n_l}{N_c^2} G_l + N_c n_l^2 H_l + \frac{n_l^2}{N_c} I_l \]

\[ + N_c^2 n_h E_h + n_h F_h + \frac{n_h}{N_c^2} G_h + N_c n_h^2 H_h + \frac{n_h^2}{N_c} I_h \]

\[ + N_c n_l n_h H_{lh} + \frac{n_l n_h}{N_c} I_{lh} \]

\textbf{q\bar{q} channel:} 218 two-loop diagrams, 10 coefficients

eample: for leading \( N_c \) coefficient \( A \) we need:

- 300 two-loop diagrams (+ ghost initiated), e.g.:

\[ \text{example diagrams with two independent ratios of scales} \]

- up to: 4-point, 7 propagators, 4 loop momenta in numerator
Steps toward complete NNLO calculation
for $2 \text{Re} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle$:

**small mass expansion**:
- Czakon, Mitov, Moch (2006) for $q\bar{q}, \, gg$

**IR poles**:
- Ferroglia, Neubert, Pecjak, Yang (2009) for $q\bar{q}, \, gg$

$q\bar{q}$ with **full dependence** on $s, \, t, \, m_t, \, \mu$:
- numerical result for all contributions: Czakon (2008)
- analytical result for fermionic: Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)
- analytical result for leading $N_c$: Bonciani, Ferroglia, Gehrmann, Studerus (2009)

$gg$ with **full dependence** on $s, \, t, \, m_t, \, \mu$:
- numerical result for all contributions: Czakon, Bärnreuther (in preparation)
Organisation of our calculation

Recipe

1. generate **Feynman diagrams** with QGRAF by Nogueira
2. build **interference** terms
3. **reduce** scalar integrals to masters with **parallel Laporta**
4. solve **masters** with differential equations
5. renormalize: \( \overline{MS} \), pole mass

⇒ **analytical result** in terms of generalized polylogarithms, allows fast **numerical evaluation, expansions, ...**

various tasks automatized in computer program **Reduze 2**
Reduze 2 - Distributed Feynman Integral Reduction

A.v.M., Studerus
arXiv:1201.4330
http://projects.hepforge.org/reduze
**Reduze 2 and external software**

- **key features**
  - fully parallelized reductions, resumable
  - topological analysis of diagrams and sectors

- **open source C++**

- **libraries / programs** used (no proprietary mandatory):
  - GiNaC by Bauer, Frink, Kreckel
  - yaml-cpp
  - optional: MPI
  - optional: Berkeley DB
  - optional: Fermat CAS by Lewis (closed source)

- **interfaces**
  - input: QGRAF, YAML
  - output: FORM, Mathematica, Maple
**Reductions using integration by parts (IBP) identities**

**example:** massive 1-loop tadpole

\[
\int d^d k \frac{1}{k^2 - m^2}
\]

calculating IBP identity

\[
0 = \int d^d k \frac{\partial}{\partial k_\mu} \left( k_\mu \frac{1}{k^2 - m^2} \right)
= \int d^d k \left( \frac{d}{k^2 - m^2} - \frac{2k^2}{(k^2 - m^2)^2} \right)
= \int d^d k \left( \frac{d}{k^2 - m^2} - \frac{2(k^2 - m^2 + m^2)}{(k^2 - m^2)^2} \right)
= (d - 2) \int d^d k \frac{1}{k^2 - m^2} - 2m^2 \int d^d k \frac{1}{(k^2 - m^2)^2}
\]
gives directly reduction of integral with additional numerator

\[
\int d^d k \frac{1}{(k^2 - m^2)^2} = \frac{(d - 2)}{2m^2} \int d^d k \frac{1}{k^2 - m^2}
\]
Example of reduction

in reality we are after something like this

⇒

need some computer program

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**Laporta’s algorithm**

**Index integrals:**

- **define** integral family: set of propagators \( \{1/D_1, \ldots, 1/D_N\} \) such that:
  - all scalar products with loop momenta are linear combinations of \( D_i \)
- **sector** defined by subset of propagators (denominators)
- counting propagator exponents indexes integrals:

  \[
  \int d^d k_1 \cdots d^d k_L \frac{1}{D_1^{n_1} \cdots D_N^{n_N}} \mapsto \{n_1, \ldots, n_N\} \quad \text{with } n_i \in \mathbb{Z}
  \]

**Laporta algorithm**

- define **ordering** for integrals \( I(n_1, \ldots, n_N) \)
- generate **integration by parts identities** (IBPs): sparse system of equations
- solve linear system of equations

**Based on:**

- Chetyrkin, Tkachov ’81

**Public implementations:**

- Anastasiou: AIR, Smirnov: FIRE, Studerus: Reduze 1
- parallel variant: AvM, Studerus: Reduze 2
**Reduze 2: parallelization of Laporta-Algorithm**

- generate the system of equations
- sort equations in blocks with the **same leading integral**

\[
\begin{align*}
I_5 + c_{14} I_4 + c_{13} I_3 &= 0 \\
I_5 + c_{24} I_4 + c_{22} I_2 &= 0 \\
I_5 + c_{33} I_3 + c_{32} I_2 &= 0 \\
I_3 + c_{42} I_2 &= 0 \\
I_3 + c_{51} I_1 &= 0 \\
I_2 + c_{61} I_1 &= 0
\end{align*}
\]

- send blocks to workers
Reduze 2: parallel execution of jobs

consider reduction of multiple sectors:

⇒ parallel reduction of several sectors ("jobs"), balance workers between them
Reduze 2: distributed Laporta algorithm

Reduze 2 / Tops at two loops

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Removing ambiguities for integrals

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Reduze 2: sectors, graphs and matroids

\[ \text{integral A (propagators)} \xleftarrow{\text{construct}} \exists \text{shift} \xrightarrow{\text{construct}} \text{integral B (propagators)} \]

\[ \text{graph A} \xleftarrow{\text{isomorphic}} \text{graph B} \]

Problem: graphs not unique!

Example:

\[ \text{same propagators} \xrightarrow{\text{not isomorphic}} \text{solution: select unique representative by allowing for twists} \]

Idea based on theorem by Bogner, Weinzierl (2010):

First Symanzik polynomials

\[ U \text{isomorphic} \iff \text{graphs isomorphic up to twists} \]

Proof based on Whitney's theorem for isomorphisms of graph matroids.
**Reduze 2: sectors, graphs and matroids**

- integral $A$ (propagators) \(\xrightarrow{\exists \text{shift ?}}\) integral $B$ (propagators)

- graph $A$ \(\xrightarrow{\text{isomorphic ?}}\) graph $B$

- **Problem**: graphs not unique!

- e.g.: [Graph A] \(\xrightarrow{\text{same propagators but not isomorphic}}\) [Graph B]
**Reduze 2: sectors, graphs and matroids**

- **integral A** (propagators) \[\exists \text{shift} \quad \rightarrow \quad \text{integral B} \quad \text{(propagators)}\]

- **construct**

- **graph A** \[\leftarrow \quad \exists \text{isomorphic} \quad \rightarrow \quad \text{graph B}\]

- **problem**: graphs not unique!

- e.g.: ![graph example]

- **solution**: select unique representative by allowing for **twists**

- idea based on **theorem** by Bogner, Weinzierl (2010):

  - first Symanzik polynomials $\mathcal{U}$ isomorphic $\iff$ graphs isomorphic up to twists

- proof based on Whitney’s theorem for isomorphisms of graph matroids
**Definition of matroid**

A matroid is a pair \((E, \mathcal{I})\) where \(E\) finite ground set, \(\mathcal{I}\) collection of subsets of \(E\), the "independent sets", and

- \(\emptyset \in \mathcal{I}\)
- if \(I \in \mathcal{I}\) and \(I' \subset I\) then also \(I' \in \mathcal{I}\)
- if \(I_1, I_2 \in \mathcal{I}\), \(|I_1| < |I_2|\) then \(\exists e \in I_2 - I_1\) with \(I_1 \cup \{e\} \in \mathcal{I}\)

- generalizes notion of **linear dependency**
- graph matroid: dependencies of **edges**
- application to Feynman graph: propagators relevant, no reference to vertices

Propagators of two vacuum diagrams related by shift \(\Leftrightarrow\) graph matroids isomorphic
Algorithm: shift finder

1. generate graph for sector
2. colour edges according to masses
3. connect external legs with a new vertex
4. decompose into triconnected components (Hopcroft, Tarjan '73; Gutwenger, Mutzel '01)
5. minimize graph by twists
6. check for graph isomorphism (McKay '81)

Example:

Tree of triconnected components
(dashed "virtual edges" mark positions for Tutte twists)
**Usage**

**Input files:**

```yaml
# kinematics.yaml
kinematics:
  incoming_momenta: [p1, p2]
  outgoing_momenta: [p3, p4]
  momentum_conservation: [p4, p1 + p2 - p3]
kinematic_invariants:
  - [mt, 1]
  - [s, 2]
  - [t, 2]
scalar_product_rules:
  - [[p1,p1], 0]
  - [[p2,p2], 0]
  - [[p3,p3], mt^2]
  - [[p1+p2, p1+p2], s]
  - [[p1-p3, p1-p3], t]
  - [[p2-p3, p2-p3], -s-t+2*mt^2] # == u
symbol_to_replace_by_one: mt
```

```yaml
# integralfamilies.yaml
integralfamilies:
  - name: planarbox
    loop_momenta: [k1, k2]
    propagators:
      - [k1, 0]
      - [k2, 0]
      - [k1-k2, 0]
      - [k1-p1, 0]
      - [k2-p1, 0]
      - [k1-p1-p2, 0]
      - [k2-p1-p2, 0]
      - [k1-p3, "mt^2"]
      - [k2-p3, "mt^2"]
    permutation_symmetries:
      - [1, 6], [2, 7] # ki<->-ki+p1+p2, p1<->p2, p3<->p4
      - [1, 2], [4, 5], [6, 7], [8, 9] # k1<->k2
```

**automatic determination of sector properties:**
- zero sectors
- unphysical sectors
- generation of graphs for physical sectors
- symmetry shifts of sectors
- shifts which identifies different sectors (also between different integral families)
- handles crossings of external momenta
- upcoming: generation of maximally symmetric family for given graph
APPLICATION EXAMPLE

auto-generated shifts for non-planar double box family:

```plaintext
figures : less
```

```plaintext
... sector_mappings:
  name: box2n
  zero_sectors:
    t=0: [0]
    t=1: [1, 4, 16, 64, 256]
    t=2: [3, 5, 6, 12, 17, 18, 20, 24, 48, 65, 66, 68, 72, 80, 96, 192, 257, 260, 272, 920]
    ... truncated 
  sectors_without_graph:
    t=3: [22, 28, 52, 82, 84, 86, 104, 112, 276]
    t=4: [23, 29, 30, 53, 54, 60, 83, 85, 89, 90, 92, 105, 106, 108, 113, 114, 115, 120, 21
    ... truncated ]
  sector_relations:
    3: [[box2p, 3], [[k1, k1, [k2, k2]]]
    6: [[box2p, 3], [[k1, -p1+k2], [k2, k1]]]
    7: [[box2p, 11], [[k1, k1-p1], [k2, k2]]]
    12: [[box2p, 3], [[k1, k1-p1], [k2, k2-p2]]]
    13: [[box2p, 11], [[k1, k1-p1], [k2, k2-p2]]]
    15: [[box2p, 3], [[k1, -p1+k2], [k2, k2-p2-p3]]]
    18: [[box2p, 3], [[k1, k2+k3], [k2, k1]]]
    19: [[box2p1x3x24, 11], [[k1, k1+p3], [k2, k2]]]
    24: [[box2p, 3], [[k1, k2+p3], [k2, k1-p2]]]
    25: [[box2p1x3x24, 11], [[k1, k1+p3], [k2, k2-p2]]]
    26: [[box2p1x3x34, 11], [[k1, k2+p3], [k2, k1-p2]]]
    27: [[box2p1x3, 18], [[k1, k2+p3], [k2, k1-p2+p3]]]
  ... truncated 
  crossed_sector_relations:
    x12:
    207: [[box2n, 207], [[k1, -k2-p2], [k2, -k1-p1]]]
    335: [[box2n, 335], [[k1, -k2-p2], [k2, -k1-p1]]]
    463: [[box2n, 463], [[k1, -k2-p2], [k2, -k1-p1]]]
    x1234:
    335: [[box2n, 335], [[k1, -k2-p2], [k2, k1+p1-k2]]]
    x1234:
    335: [[box2n1x4x23, 335], [[k1, -k1+p1+k2-p3]]]
    x1243:
    335: [[box2n1x4x23, 335], [[k1, -k1+p1+k2-p3], [k2, -k1+p1+k2-p3]]]
    x1243:
    335: [[box2n1x2x34, 335], [[k2, k1-k2+p2]]]
  sector_symmetries:
    207:
    - [[k1, -k1-p1], [k2, -k2-p2]]
    - [[k1, -k2-p2], [p1, p2], [k2, -k1-p1], [p2, p1], [p3, p1+p2-p3]]
    - [[k1, k2], [p1, p2], [k2, k1], [p2, p1], [p3, p1+p2-p3]]
    463:
    - [[k1, k2], [p1, p2], [k2, k1], [p2, p1], [p3, p1+p2-p3]]
```

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Available jobs in Reduze 2

List of available job types:

apply_crossings: Generates reduction results for crossed sectors.
cat_files: Concatenates files.
collect_integrals: Collects all integrals appearing in the input file.
compute_diagram_interferences: Computes interferences of diagrams.
compute_differential_equations: Computes derivatives of integrals wrt invariants.
export: Exports to FORM, Mathematica or Maple format.
find_diagram_shifts: Matches diagrams to sectors via graphs.
find_diagram_shifts_alt: Matches diagrams to sectors via combinatorics.
generate_identities: Generates identities like IBPs for given seeds.
generate_seeds: Generates integrals from a sector.
insert_reductions: Inserts reductions in expressions.
normalize: Simplifies linear combinations and equations.
print_reduction_info_file: Analyzes reductions in a file.
print_reduction_info_sectors: Analyzes reductions available for sectors.
print_sector_info: Prints diagrams and other information for sectors.
reduce_files: Reduces identities in given files.
reduce_sectors: Reduces integrals from a selection of sectors.
run_reduction: Low-level job to run a reduction.
select_reductions: Selects reductions for integrals.
setup_sector_mappings: Finds shifts between sectors via graphs.
setup_sector_mappings_alt: Finds shifts between sectors via combinatorics.
sum_terms: Sums terms.
test: Empty job template.
verify_same_terms: Verifies two files contain the same terms.

Available jobs in Reduze 2

detailed description: see builtin on-line help
Top quark pair production at two loops: gluon channel results
Results for \( gg \) leading \( N_c \): expansions

**Threshold expansion:**

\[ \beta = \sqrt{1 - 4m^2/s}, \]
\[ \text{c.m. scatt. angle} = \pi/2 \]

**Small mass expansion:**

\[ \eta = \frac{s}{4m_t^2} - 1, \]
\[ \text{c.m. scatt. angle} = \pi/2 \]

(careful with phase space for small \( m \) expansion:
don’t introduce forw.-backw. asymmetry, \( \phi = -(t - m_t^2)/s = \text{const} \))

Light fermion loops in $gg \to t\bar{t}$

- **analytical** result for light fermion loops in $gg \to t\bar{t}$:
  Bonciani, Ferroglia, Gehrmann, A.v.M., Studerus (in preparation)

- required calculation of **11 new master integrals**
  - differential equations + Mellin-Barnes for const.
  - used: MB.m by Czakon '05, planar: AMBRE by Gluza, Kajda, Riemann '07 + Yundin '10

- most difficult integrals: **massive non-planar double box**

  ▶ 3 master integrals (+ subtopo MIs)
  ▶ analytical solution for massless case: Tausk 1999
  ▶ $3 + 1$ scale problem: $s, t, u$ and $m$ appear naturally
Analytical solution for massive non-planar double box

solved with differential equations:
A.v.M, Studerus (in preparation)

- all masters up to and including finite parts
- determined integration constants from
  - regularity conditions
  - symmetry conditions
  - Mellin-Barnes (cmp. with SecDec by Carter, Heinrich ‘10)
    in kinematical limits (finite const. up to 3-fold scaleless M.B.)

found result in terms of \((x, y)\):

- linear combinations of GPLs with rational prefactors
- transcendality up to 4
- 805 GPLs total, 166 two-dim.
- GPLs with argument \(y\): weights \(\{0, -1, -x, -1/x, -1/x - x, -1/x - x + 1\}\)
- GPLs with argument \(x\): weights \(\{0, \pm 1, \pm i, (1 \pm i\sqrt{3})/2\}\)
  - for univariate (cyclotomic) cmp. Ablinger, Blümlein, Schneider ’11
Generalised polylogarithms

Remiddi, Gehrmann; Goncharov:

**Definition of generalised polylogarithms (GPLs)**

\[
G(\vec{0}_n; x) = \frac{1}{n!} \log^n(x),
\]

\[
G(a; x) = \int_0^x dt f_a(t), \quad \text{for } a \neq 0
\]

\[
G(a, \vec{b}; x) = \int_0^x dt f_a(t) G(\vec{b}; t), \quad \text{for } a \neq 0
\]

with weight functions for (complex) weight \(a\):

\[
f_a(x) = \frac{1}{x - a}
\]

- \(G(a \neq 0; x) = \log \left( \frac{a-x}{a} \right)\), \(G(\vec{0}_{n-1}, 1; x) = \text{Li}_n(x)\), \(G(\vec{0}_n, \vec{1}_p; x) = S_{n,p}(x)\)
- for weights 0, 1, −1, GPLs specialize to harmonic polylogarithms (HPLs), Remiddi, Vermaseren (1999)
- shuffle algebra, **symbols** (see \(N = 4\) remainder func), ...
**Simplification via symbols**

**result** for $1/\epsilon$ of scalar master:

- using GPLs with variables $(x, y)$: **62 different GPLs** (23 two-dim.)
- choosing different arguments via **symbols**, this **simplifies to**:

\[
M_1|_{1/\epsilon} = \frac{1}{16m^2y_1z_1(y_1+z_1)} \times \\
\times \left( \text{Li}_3 \left( \frac{y_1z_1}{y_1+z_1} \right) \right) \\
+ \text{Li}_2 \left( \frac{y_1z_1}{y_1+z_1} \right) \left( \log(-y_1 - z_1) - \log y_1 - \log z_1 \right) \\
+ \text{polynomial in } \log(y_1), \log(z_1), \log(-y_1 - z_1), \log \left( 1 - \frac{y_1z_1}{y_1+z_1} \right) \\
\]

with $y_1 := -t/m^2 + 1$, 
$z_1 := -u/m^2 + 1$

- recent progress with symbols: Duhr, Gangl, Rhodes '11, Duhr '12
Symbol calculus for multiple polylogarithms
Motivation for symbol calculus

**symbol calculus:**
- tool to exploit functional relations between polylogarithms in a systematic way
  - works: simplify and transform functions; Zagier ('91), Goncharov ('95), Gangl ('02)
  - wish: perform integrations, see Chavez, Duhr '12 and talks by C. Bogner

**first application** in theoretical physics:
- $N = 4$ supersymmetric Yang-Mills two-loop six-point remainder function
  Del Duca, Duhr, Smirnov (09): $\mathcal{O}(10^3)$ multi-dimensional polylogs
- simplifies to few-line expression with few $\text{Li}_4$, $\text{Li}_2$ and $\ln !$
  found with symbols by Goncharov, Spradlin, Vergu, Volovich ('10)
symbols also very welcome for QCD, e.g.:

- e.g. 

- corner integral alone gives $\mathcal{O}(10^3)$ GPLs, many two-dimensional
- numerical evaluation slow, stability problematic
- switching basis functions, expansions, analytical continuation difficult
- simplifications guided by symbols possible; Ferroglia, AvM, Studerus

crucial progress very recently:

- construction of functions for given symbol; Duhr, Gangl, Rhodes (’11)
  - applied to HPLs; Bühler, Duhr (’11)

- extended symbol calculus from coproduct: constants;
  Goncharov (’02), Brown (’11), Duhr (’12)
  demonstrated for QCD amplitude; Duhr (’12)
**Definition of symbol map**

Let $G$ be a generalized polylogarithm with

$$dG = \sum_i \hat{G}_i \, d\ln(R_i)$$

where $R_i$ is a rational function of the polylog arguments. The symbol map $S$

$$S(G) = \sum_i S(\hat{G}_i) \otimes R_i,$$

associates a tensor with the polylogarithm.

**Examples:**
- $S(\ln x) = x$
- $S(\text{Li}_3 x) = -(1 - x) \otimes x \otimes x$
- $S(G(1, 0, -1, -1, x)) = (1 + x) \otimes (1 + x) \otimes x \otimes (1 - x)$

**Rules for symbols**
- $R_1 \cdots \otimes (R_a R_b) \otimes \cdots R_k = R_1 \cdots \otimes R_a \otimes \cdots R_k + R_1 \cdots \otimes R_b \otimes \cdots R_k$ (log law)
- $R_1 \cdots \otimes (cR_a) \otimes \cdots R_k = R_1 \cdots \otimes R_a \otimes \cdots R_k$ for constant $c$
- preserves shuffle product
**Example for symbol calculus**

**goal:** derive "simplification formula" for $\text{Li}_2(1/x)$ with $0 < x < 1$, $\text{Im} \, x = \varepsilon$

$$
S(\text{Li}_2(1/x)) = -(-1 + 1/x) \otimes (1/x) \\
= (1 - x) \otimes x - x \otimes x \\
= S(-\text{Li}_2(x) - (1/2) \ln^2 x)
$$

reproduces the highest degree part of the full answer

$$
\text{Li}_2(1/x) = -\text{Li}_2(x) - (1/2) \ln^2 x + i\pi \ln x - (2/3)\pi^2
$$

**note:** works at **highest degree** only

- $S(\ln(-x)) = S(\ln(x))$: no info on discontinuity
- $S(\pi) = S(\zeta_3) = 0$: no constants

"integrating the symbol" ⇒ **algorithmic reduction** (structured with shuffle eliminators): Duhr, Gangl, Rhodes ('11)
**SYSTEmatic REDUCTION TO BASIS FUNCTION WITH SYMBOLS**

$gg \rightarrow t\bar{t}$ **at two-loops**, ferm. contrib. (incl. non-planar part)

- pick specific coefficient in finite part: linear combination of polylogs
- pick uniform weight 4 part
- representation not unique (due to crossings etc.), imaginary polylogs
- here: 182 GPLs in total, 84 weight 4 non-$\text{Li}_k$ GPLs
- symbol contains slots (weight have unit roots):

  \[
  \{-1 + x, x, 1 + x, y, 1 + y, x + y, 1 + xy, 1 + x^2 + xy, 1 - x + x^2 + xy\}
  \]

**algorithmic conversion** at highest degree to new set of real basis function:

1. partition (4) (no shuffles), $\text{Li}_{22}(R_1, R_2), \text{Li}_4(R_1)$
2. partition (3,1): $\text{Li}_3(R_1) \ln(R_2)$
3. partition (2,2): $\text{Li}_2(R_1) \text{Li}_2(R_2)$
4. partition (2,1,1): $\text{Li}_2(R_1) \ln(R_2) \ln(R_3)$
5. partition (1,1,1,1) (rest): $\ln(R_1) \ln(R_2) \ln(R_3) \ln(R_4)$

**result:**

- 28 $\text{Li}_{22}$, 48 $\text{Li}_4$, 58 $\text{Li}_3$, 18 $\text{Li}_2$, 11 $\ln$
- all real
What about subleading degree terms (\textit{const} × polylog) ?

- accessible by \textbf{coproduct}: Goncharov ('02), Brown ('11)
- extended symbol calculus based on coproduct by Duhr ('12) with:

\[
\Delta(\pi) = \pi \otimes 1 \\
\Delta(\zeta_k) = \zeta_k \otimes 1 + 1 \otimes \zeta_k \quad \text{for } k \text{ (odd)}
\]

example: \(\Delta_{1,1} \left( \text{Li}_2(1/x) \right) = -\ln(1 - 1/x) \otimes \ln(1/x)\)

\[
= \ln(1 - x) \otimes \ln(x) - \ln(x) \otimes \ln(x) + i\pi \otimes \ln(x)
= \Delta_{1,1} \left( - \text{Li}_2(x) - \frac{1}{2} \ln^2(x) + i\pi \ln(x) \right)
\]

reproduces identity up to pure constant, fix by limits or numerical evaluation:

\[
\text{Li}_2(1/x) - \left( - \text{Li}_2(x) - \frac{1}{2} \ln^2(x) + i\pi \ln(x) \right) = -6.5797362673929 \cdots = -(2/3)\pi^2
\]

\textbf{results} for \(gg \rightarrow t\bar{t}\): \textbf{systematic reductions} give:

- considerable simplifications, heavy improvement in access wrt traditional techniques
- beyond classical polylogs, need \text{Li}_{22}
Conclusions

- $gg \rightarrow t\bar{t}$: analytical two-loop corrections
  - leading $N_c +$ light fermionic
  - massive non-planar double box: analytical solution

- **Reduze 2**: open source tool
  - parallelized reductions of Feynman integrals
  - graph matroid algorithm: determine shifts between sectors or diagrams

- **symbol + coproduct** refinement:
  - powerful algorithmic treatment of multiple polylogs
  - simplifications of QCD integrals with mass
  - outlook: direct solving of differential equations?
Supplementary Slides

5 Ingredients for full NNLO calculation

6 Mellin-Barnes for NP box

7 Leading poles and GPL conversions for NP box

8 Coproduct
**Ingredients for full NNLO calculation**

- **VV**: two-loop ME for $q\bar{q} \rightarrow t\bar{t}$ and $gg \rightarrow t\bar{t}$
  
- **RV**: one-loop ME for $t\bar{t} + 1$ parton  
  Dittmaier, Uwer, Weinzierl '07

- **RR**: tree level ME for $t\bar{t} + 2$ partons

- **Subtraction terms**: up to 2 unresolved partons needed  
  Gehrmann-De Ridder, Ritzmann '09; Daleo et al. '09; Boughezal et al. '10; Glover, Pires '10; Czakon '10, '11; Anastasiou, Herzog, Lazopoulos '10; Abelof, Gehrmann-De Ridder '11, '12; Gehrmann, Monni '11; Bierenbaum, Czakon, Mitov '11

- **Combined $q\bar{q}$**: Bärnreuther, Czakon, Mitov '12

**Consider VV (2 → 2 ingredients):**

$$
\sum_{\text{spin,colour}} |M|^2 = 16\pi^2\alpha_s^2 \left[ A_0 + \left( \frac{\alpha_s}{\pi} \right) A_1 + \left( \frac{\alpha_s}{\pi} \right)^2 A_2 + O(\alpha_s^3) \right]
$$

with

$$
A_0 = \langle M^{(0)} | M^{(0)} \rangle
$$

$$
A_1 = 2 \text{Re} \langle M^{(0)} | M^{(1)} \rangle
$$

$$
A_2 = \langle M^{(1)} | M^{(1)} \rangle + 2 \text{Re} \langle M^{(0)} | M^{(2)} \rangle
$$

$$
\langle M^{(1)} | M^{(1)} \rangle: \text{Kniehl, Körner, Merebashvili, Rogal '05-'08, Anastasiou, Aybat '08}
$$
Mellin-Barnes representation:

\[ I(a_9) = (-1)^{-1+a_9} \pi^d \frac{\Gamma(-2 + d/2)^2}{\Gamma(a_9)\Gamma(-4 + d)\Gamma(-6 - a_9 + 3d/2)} \]

\[ \int_{-i\infty}^{i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dz_2}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dz_3}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dz_4}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dz_5}{2\pi i} \]

\[ (-s)^{-6-a_9+d-z_1-z_2-z_5} (-t_1)^z_1 (-u_1)^z_2 (m^2)^z_5 \]

\[ \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5) \]

\[ \Gamma(1 + z_1 + z_3)\Gamma(1 + z_2 + z_3)\Gamma(1 + z_1 + z_4)\Gamma(1 + z_2 + z_4) \]

\[ \Gamma(4 - d/2 + z_1 + z_2 + z_3 + z_4)\Gamma(-5 - a_9 + d - z_1 - z_2 - z_3 - z_5) \]

\[ \Gamma(-5 - a_9 + d - z_1 - z_2 - z_4 - z_5)\Gamma(6 + a_9 - d + z_1 + z_2 + z_3 + z_4 + z_5) \]

\[ \Gamma(a_9 + z_1 + z_2 + 2z_5))/\Gamma(2 + z_1 + z_2 + z_3 + z_4)^2 \]

with

\[ a_9 = \text{power of massive propagator} \]
\[ t_1 = t + m^2 \]
\[ u_1 = u + m^2 \]
A non-planar massive double box

Analytical solution

\[ \frac{1}{N(\epsilon)} I(1) = \frac{1}{\epsilon^4} \cdot \frac{y_1 + z_1}{32x_s^2 y_1 z_1} \]
\[ + \frac{1}{\epsilon^3} \cdot \frac{7(y_1 + z_1) - 6(y_1 + z_1) \log x_s + 3(y_1 - z_1)(\log y_1 - \log z_1)}{96x_s^2 y_1 z_1} \]
\[ + \frac{1}{\epsilon^2} \cdot \left( c_1 \log x_s + c_2 \log^2 x_s + c_3 \log y_1 + c_4 \log^2 y_1 + c_5 \log z_1 + c_6 \log^2 z_1 \right. \]
\[ \left. + c_7 \log x_s \log y_1 + c_8 \log x_s \log z_1 + c_9 \log y_1 \log z_1 \right) \]

- e.g. leading poles of scalar integral very simple:

- however: differential equations work with 2 + 1 on-shell variables \((s + t + u = 2m^2)\)

- GPL conversions involved, e.g.:

\[ G(-1, 0, -1; z) = G \left( -1, 0, -1; -\frac{1}{x} - x - y \right) \]
\[ = G \left( 1 - \frac{1}{x} - x, -\frac{1}{x} - x, 1 - \frac{1}{x} - x; y \right) + 42 \text{ more GPLs} \]
\[ = G \left( \frac{1 - y + \sqrt{-3 - 2y + y^2}}{2}, \frac{-y + \sqrt{-4 + y^2}}{2}, \frac{1 - y + \sqrt{-3 - 2y + y^2}}{2}; x \right) + 39 \text{ more GPLs} \]

- solved up to few integration constants
**Definition of the coproduct**

For a multiple polylogarithm

\[ I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t) \]

the coproduct \( \Delta \) is defined according to Goncharov (’02):

\[ \Delta (I(a_0; a_1, \ldots, a_n; a_{n+1})) = \sum_{0=i_1<\ldots<i_{k+1}=n} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_p+1-1}) \]

examples:

- \( \Delta(\ln(x)) = 1 \otimes \ln(x) + \ln(x) \otimes 1 \)
- \( \Delta(\text{Li}_2(x)) = 1 \otimes \text{Li}_2(x) - \ln(1-x) \otimes \ln(x) + \text{Li}_2(x) \otimes 1 \)
- \( \Delta(\ln(x) \ln(y)) = 1 \otimes (\ln(x) \ln(y)) + \ln(x) \otimes \ln(y) + \ln(y) \otimes \ln(x) + (\ln(x) \ln(y)) \otimes 1 \)

**Rules for the coproduct**

- coassociativity \((\text{id} \otimes \Delta) \Delta = (\Delta \otimes \text{id}) \Delta\)
- compatible with product: \( \Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \) where \((a_1 \otimes a_2) \cdot (b_1 \otimes b_2) \equiv (a_1 \cdot b_1) \otimes (a_2 \cdot b_2)\)

note: coproduct means ”decomposition”
Graded decomposition with the coproduct

- Hopf algebra of multiple polylogs **graded by weight**:

\[ \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \]

since coproduct preserves weight we may decompose

\[ \mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q \]

and define \( \Delta_{p,q} \) to be the part with values in \( \mathcal{H}_p \otimes \mathcal{H}_q \)

- iterated coproduct:

\[ \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \text{id}} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \text{id} \otimes \text{id}} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \]

and corresponding parts \( \Delta_{p,q,...,r} \)

- symbol \( S = \) maximally iterated coproduct \( \Delta_{1,...,1} \mod \pi \)
systematic procedure: back to $gg \rightarrow t\bar{t}$ at two-loops application:

1. fix maximum degree 4 part with symbol, subtract from original expression
2. take $\Delta_{1,1,1,1}$, use unique set of real ln, gives

$$i\pi \otimes \Delta_{1,1,1}(\text{something})$$

match ”something” at degree 3 to basis functions using symbols

3. take $\Delta_{2,1,1}$, use unique set of real ln, $\text{Li}_2$, gives

$$\pi^2 \otimes \Delta_{1,1}(\text{something})$$

match ”something” at degree 2 to basis functions using symbols

4. take $\Delta_{3,1}$, use unique set of real ln, $\text{Li}_2, \text{Li}_3$, gives

$$\zeta(3) \otimes \text{something1} + i\pi^3 \otimes \text{something2}$$

read off ”somethingi” (just logs) at degree 1

5. numerical evaluation, match against

$$i\pi \zeta_3, \quad \pi^4$$

(results requires matching of degree 3 and lower, obtained by recursion)

results for $gg \rightarrow t\bar{t}$:

- considerable simplifications for poles
- finite parts: heavy improvement in access wrt traditional techniques