Reduze 2, matroids, symbols

and their application to

Top quark pairs at two loops

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PRiSMA

Theory Seminar
DESY Zeuthen
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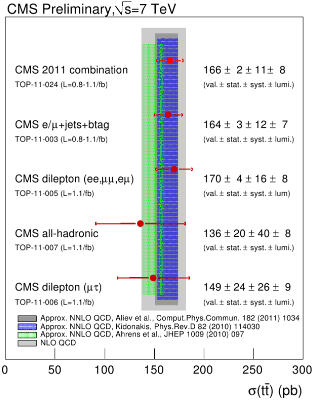
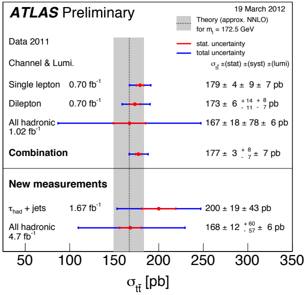
Top pair production at the LHC





- **LHC precision** below NLO accuracy already
- **full** *WWbb* at NLO
- recent **threshold/resummations**, **approx. NNLO**: Beneke, Falgari, Klein, Piclum, Schwinn, Ubiali, Yan; Moch, Uwer, Vogt; Cacciari, Czakon, Mangano, Mitov, Nason; Ahrens, Ferroglia, Neubert, Pecjak, Yang; Kidonakis
- full NNLO for $q\bar{q}$: Bärnreuther, Czakon, Mitov '12

Gauge invariant subsets in two-loop contributions

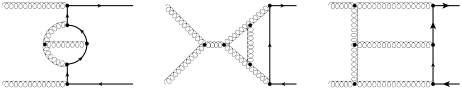
gg **channel**: 789 two-loop diagrams (+ ghost init.) contrib. to 16 coeff.:

$$2\,\mathrm{Re}\left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \right\rangle = 2C_F N_c \Big(N_c^3 \mathbf{A} + N_c \mathbf{B} + \frac{1}{N_c}\mathbf{C} + \frac{1}{N_c^3}\mathbf{D}$$
$$+ N_c^2 n_l \mathbf{E_l} + n_l \mathbf{F_l} + \frac{n_l}{N_c^2}\mathbf{G_l} + N_c n_l^2 \mathbf{H_l} + \frac{n_l^2}{N_c}\mathbf{I_l}$$
$$+ N_c^2 n_h \mathbf{E_h} + n_h \mathbf{F_h} + \frac{n_h}{N_c^2}\mathbf{G_h} + N_c n_h^2 \mathbf{H_h} + \frac{n_h^2}{N_c}\mathbf{I_h}$$
$$+ N_c n_l n_h \mathbf{H_{lh}} + \frac{n_l n_h}{N_c}\mathbf{I_{lh}} \Big)$$

$q\bar{q}$ **channel**: 218 two-loop diagrams, 10 coefficients

example: for **leading N_c coefficient** A we need:

- 300 two-loop diagrams (+ ghost initiated), e.g.:



- two independent ratios of scales
- up to: 4-point, 7 propagators, 4 loop momenta in numerator

Steps toward complete NNLO calculation

for $2\,Re\,\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle$:

small mass expansion:

- Czakon, Mitov, Moch (2006) for $q\bar{q}$, gg

IR poles:

- Ferroglia, Neubert, Pecjak, Yang (2009) for $q\bar{q}$, gg

$q\bar{q}$ with **full dependence** on s, t, m_t, μ:

- numerical result for all contributions:
  Czakon (2008)
- analytical result for fermionic:
  Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)
- analytical result for leading N_c:
  Bonciani, Ferroglia, Gehrmann, Studerus (2009)

gg with **full dependence** on s, t, m_t, μ:

- analytical result for leading N_c:
  Bonciani, Ferroglia, Gehrmann, A.v.M., Studerus (2010)
- analytical result for light fermionic:
  Bonciani, Ferroglia, Gehrmann, A.v.M., Studerus (in preparation)
- numerical result for all contributions:
  Czakon, Bärnreuther (in preparation)

Organisation of our calculation

Recipe

1. generate **Feynman diagrams** with `QGRAF` by Nogueira
2. build **interference** terms
3. **reduce** scalar integrals to masters with **parallel Laporta**
4. **solve masters** with differential equations
5. **renormalize**: $\overline{MS}$, pole mass

⇒ **analytical result** in terms of generalized polylogarithms, allows fast **numerical evaluation**, **expansions**, ...

various tasks automatized in computer program `Reduze 2`

```
________      ________
___  __ \_________  /___  _________
__  /_/ /  _ \  __  /_  / / /__  /  _ \
_  _, _//  __/ /_/ / / /_/ /__  /_  __/
/_/ |_| \___/\__,_/  \__,_/ _____\___/
```

Reduze 2 - Distributed Feynman Integral Reduction

A.v.M., Studerus

arXiv:1201.4330

http://projects.hepforge.org/reduze

Reduze 2 and external software

- **key features**
  - fully parallized reductions, resumable
  - topological analysis of diagrams and sectors
- **open source** C++
- **libraries / programs** used (no proprietary mandatory):
  - GiNaC by Bauer, Frink, Kreckel
  - yaml-cpp
  - optional: MPI
  - optional: Berkeley DB
  - optional: Fermat CAS by Lewis (closed source)
- **interfaces**
  - input: QGRAF, YAML
  - output: FORM, Mathematica, Maple

Reductions using integration by parts (IBP) identities

example: massive 1-loop tadpole

$$\int d^d k \frac{1}{k^2 - m^2}$$

calculating **IBP** identity

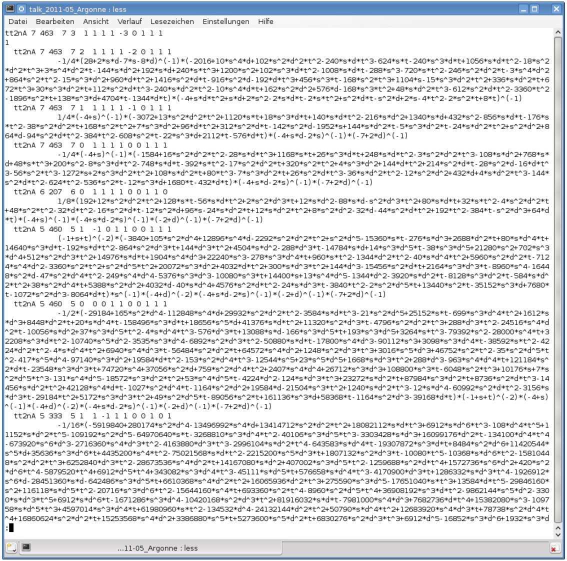
$$\begin{aligned}
0 = & \int d^d k \frac{\partial}{\partial k_\mu} \left(k_\mu \frac{1}{k^2 - m^2} \right) \\
= & \int d^d k \left(\frac{d}{k^2 - m^2} - \frac{2k^2}{(k^2 - m^2)^2} \right) \\
= & \int d^d k \left(\frac{d}{k^2 - m^2} - \frac{2(k^2 - m^2 + m^2)}{(k^2 - m^2)^2} \right) \\
= & (d-2) \int d^d k \frac{1}{k^2 - m^2} - 2m^2 \int d^d k \frac{1}{(k^2 - m^2)^2}
\end{aligned}$$

gives directly **reduction** of integral with additional numerator

$$\int d^d k \frac{1}{(k^2 - m^2)^2} = \frac{(d-2)}{2m^2} \int d^d k \frac{1}{k^2 - m^2}$$

Example of reduction

in reality we are after something like this



⇒ need some computer program

Laporta's algorithm

index integrals:

- define **integral family**: set of propagators $\{1/D_1, \ldots, 1/D_N\}$ such that: all scalar products with loop momenta are linear combinations of D_i
- **sector** defined by subset of propagators (denominators)
- counting propagator exponents indexes integrals:

$$\text{Feynman integrals of some topologies} \quad \rightarrow \quad \mathbb{Z}^N$$

$$\int d^d k_1 \cdots d^d k_L \frac{1}{D_1^{n_1} \cdots D_N^{n_N}} \mapsto \{n_1, \ldots, n_N\} \quad \text{with } n_i \in \mathbb{Z}$$

Laporta algorithm

1. define **ordering** for integrals $I(n_1, \ldots, n_N)$
2. generate **integration by parts identities** (IBPs): **sparse system** of equations
3. **solve linear system** of equations

based on:

- Chetyrkin, Tkachov '81

public implementations:

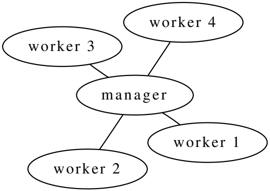
- Anastasiou: `AIR`, Smirnov: `FIRE`, Studerus: `Reduze 1`
- parallel variant: AvM, Studerus: `Reduze 2`

Reduze 2: parallelization of Laporta-Algorithm

- generate the system of equations
- sort equations in blocks with the **same leading integral**

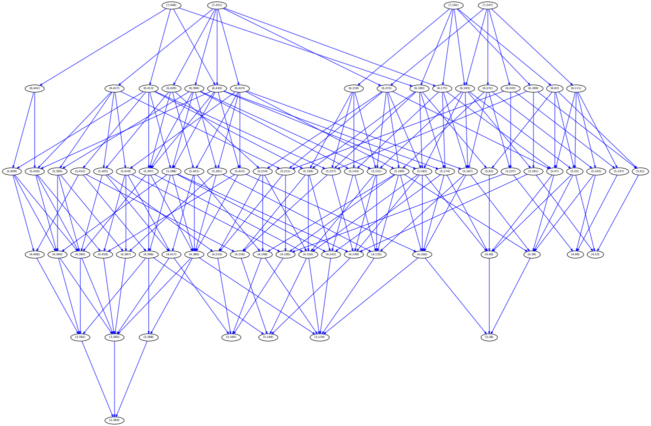
$$
\begin{array}{|l|}
\hline
\mathbf{I}_5 + c_{14}\mathbf{I}_4 + c_{13}\mathbf{I}_3 = 0 \\
\mathbf{I}_5 + c_{24}\mathbf{I}_4 + c_{22}\mathbf{I}_2 = 0 \\
\mathbf{I}_5 + c_{33}\mathbf{I}_3 + c_{32}\mathbf{I}_2 = 0 \\
\hline
\mathbf{I}_3 + c_{42}\mathbf{I}_2 = 0 \\
\mathbf{I}_3 + c_{51}\mathbf{I}_1 = 0 \\
\hline
\mathbf{I}_2 + c_{61}\mathbf{I}_1 = 0 \\
\hline
\end{array}
$$

- send blocks to workers



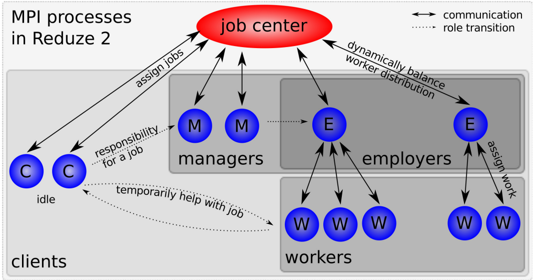
Reduze 2: parallel execution of jobs

consider reduction of multiple sectors:

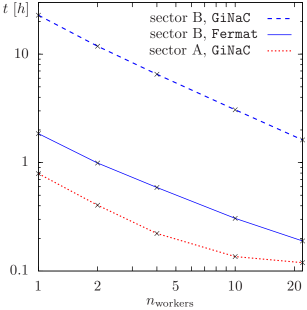


⇒ parallel reduction of several sectors ("jobs"), balance workers between them

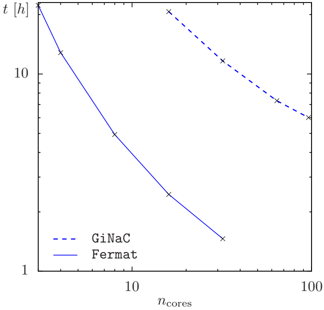
Reduze 2: distributed Laporta algorithm



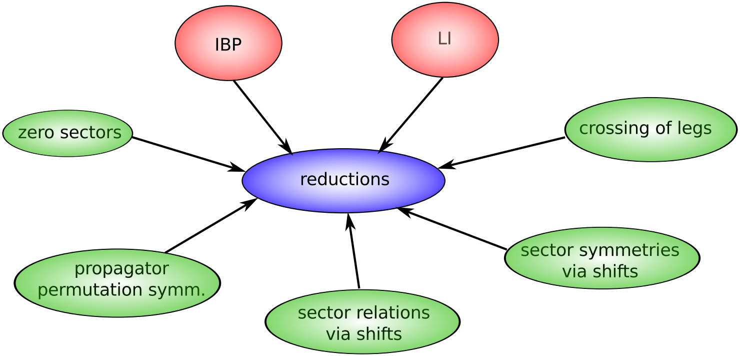
reduction of single sector:



reduction of sector selection:



Removing ambiguities for integrals



Reduze 2: sectors, graphs and matroids

integral A (propagators) ⟵ ∃shift ? ⟶ **integral B** (propagators)

↓ *construct* ↓ *construct*

graph A ⟵ isomorphic ? ⟶ **graph B**

integral A (propagators) ⟵ ∃shift ? ⟶ **integral B** (propagators)

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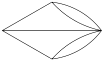
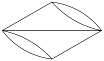
graph A ⟵ isomorphic ? ⟶ **graph B**

- **problem**: graphs not unique !

e.g.:  ⟵ same propagators but not isomorphic ⟶ 

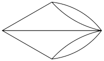
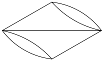
Reduze 2: sectors, graphs and matroids

integral A (propagators) ⟵ ∃shift ? ⟶ **integral B** (propagators)

↓ *construct* ↓ *construct*

graph A ⟵ isomorphic ? ⟶ **graph B**

- **problem**: graphs not unique !

e.g.:  ⟵ same propagators but not isomorphic ⟶ 

- **solution**: select unique representative by allowing for **twists**

idea based on **theorem** by Bogner, Weinzierl (2010):

first Symanzik polynomials $\mathcal{U}$ isomorphic $\Leftrightarrow$ graphs isomorphic **up to twists**

proof based on Whitney's theorem for isomorphisms of graph matroids

Definition of matroid

A matroid is a pair $(E, \mathcal{I})$ where E finite ground set, $\mathcal{I}$ collection of subsets of E, the **"independent sets"**, and

- $\emptyset \in \mathcal{I}$
- if $I \in \mathcal{I}$ and $I' \subset I$ then also $I' \in \mathcal{I}$
- if $I_1, I_2 \in \mathcal{I}$, $|I_1| < |I_2|$ then $\exists e \in I_2 - I_1$ with $I_1 \cup \{e\} \in \mathcal{I}$

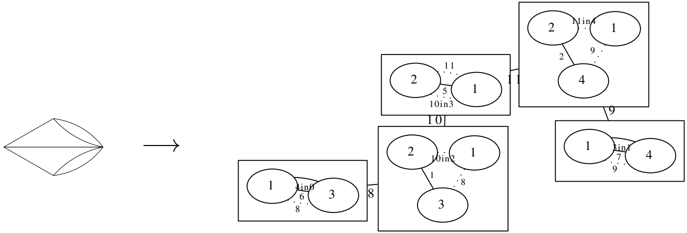
- generalizes notion of **linear dependency**
- graph matroid: dependencies of **edges**
- application to Feynman graph: propagators relevant, no reference to vertices

propagators of two vacuum diagrams related by shift $\Leftrightarrow$ graph matroids isomorphic

Algorithm: shift finder

1. generate graph for sector
2. colour edges according to masses
3. connect external legs with a new vertex
4. decompose into triconnected components (Hopcroft, Tarjan '73; Gutwenger, Mutzel '01)
5. minimize graph by twists
6. check for graph isomorphism (McKay '81)

example:



tree of triconnected components
(dashed "virtual edges" mark positions for Tutte twists)

Usage

Input files:

```
# kinematics.yaml

kinematics:
  incoming_momenta: [p1, p2]
  outgoing_momenta: [p3, p4]
  momentum_conservation: [p4, p1 + p2 - p3]
  kinematic_invariants:
    - [mt, 1]
    - [s,  2]
    - [t,  2]
  scalarproduct_rules:
    - [[p1,p1],  0]
    - [[p2,p2],  0]
    - [[p3,p3],  mt^2]
    - [[p1+p2, p1+p2], s]
    - [[p1-p3, p1-p3], t]
    - [[p2-p3, p2-p3], -s-t+2*mt^2] # == u
  symbol_to_replace_by_one: mt
```

```
# integralfamilies.yaml

integralfamilies:
  - name: planarbox
    loop_momenta: [k1, k2]
    propagators:
      - [ k1, 0 ]
      - [ k2, 0 ]
      - [ k1-k2, 0 ]
      - [ k1-p1, 0 ]
      - [ k2-p1, 0 ]
      - [ k1-p1-p2, 0 ]
      - [ k2-p1-p2, 0 ]
      - [ k1-p3, "mt^2" ]
      - [ k2-p3, "mt^2" ]
    permutation_symmetries:
      - [ [ 1, 6 ], [ 2, 7 ] ]  # ki<->-ki+p1+p2, p1<->p2, p3<->p4
      - [ [ 1, 2 ], [ 4, 5 ], [ 6, 7 ], [ 8, 9] ]  # k1<->k2
```

automatic determination of sector properties:

- zero sectors
- unphysical sectors
- generation of graphs for physical sectors
- symmetry shifts of sectors
- shifts which identifies different sectors (also between different integral families)
- handles crossings of external momenta
- upcoming: generation of maximally symmetric family for given graph

Application example

auto-generated shifts for non-planar double box family:

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```
---
sectormappings:
  name: box2n
  zero_sectors:
    t=0: [0]
    t=1: [1, 4, 16, 64, 256]
    t=2: [3, 5, 6, 12, 17, 18, 20, 24, 48, 65, 66, 68, 72, 80, 96, 192, 257, 260, 272, 320]
    [ ... truncated ]
  sectors_without_graph:
    t=3: [22, 28, 52, 82, 84, 88, 104, 112, 276]
    t=4: [23, 29, 30, 53, 54, 60, 83, 85, 86, 89, 90, 92, 105, 106, 108, 113, 114, 116, 120, 21
    t=5: [31, 55, 61, 87, 91, 93, 94, 107, 109, 110, 115, 117, 118, 121, 122, 124, 211, 213, 21
    [ ... truncated ]
  sector_relations:
    3: [[box2p, 3], [[k1, k1], [k2, k2]]]
    6: [[box2p, 3], [[k1, -p1+k2], [k2, k1]]]
    7: [[box2p, 11], [[k1, k1-p1], [k2, k2]]]
    12: [[box2p, 3], [[k1, k1-p1], [k2, k2-p2]]]
    13: [[box2p, 11], [[k1, k1-p1], [k2, k2-p2]]]
    15: [[box2p, 58], [[k1, -p1+k2], [k2, k1-p1-p2]]]
    18: [[box2p, 3], [[k1, k2+p3], [k2, k1]]]
    19: [[box2px13x24, 11], [[k1, k1+p3], [k2, k2]]]
    24: [[box2p, 3], [[k1, k2+p3], [k2, k1-p2]]]
    25: [[box2px13x24, 11], [[k1, k1+p3], [k2, k2-p2]]]
    26: [[box2px12x34, 11], [[k1, k2+p3], [k2, k1-p2]]]
    27: [[box2px13, 58], [[k1, k2+p3], [k2, k1-p2+p3]]]
    [ ... truncated ]
  crossed_sector_relations:
    x12:
      207: [[box2n, 207], [[k1, -k2-p2], [k2, -k1-p1]]]
      335: [[box2n, 335], [[k1, -k2-p2], [k2, -k1-p1]]]
      463: [[box2n, 463], [[k1, -k2-p2], [k2, -k1-p1]]]
    x123:
      335: [[box2n, 335], [[k1, -k2-p2], [k2, k1+p1-k2]]]
    x1234:
      335: [[box2nx14x23, 335], [[k1, -k1+p1+k2-p3]]]
    x124:
      335: [[box2nx14x23, 335], [[k1, -k1+p1+k2-p3], [k2, -k1+p1+p2-p3]]]
    x1243:
      335: [[box2nx12x34, 335], [[k2, k1-k2+p2]]]
  sector_symmetries:
    207:
      - [[k1, -k1-p1], [k2, -k2-p2]]
      - [[k1, -k2-p2], [p1, p2], [k2, -k1-p1], [p2, p1], [p3, p1+p2-p3]]
      - [[k1, k2], [p1, p2], [k2, k1], [p2, p1], [p3, p1+p2-p3]]
    463:
      - [[k1, k2], [p1, p2], [k2, k1], [p2, p1], [p3, p1+p2-p3]]
(END)
```


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Available jobs in Reduze 2

andreas : bash

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```
andreas@chili:~$ reduze -h jobs

List of available job types:

apply_crossings:                  Generates reduction results for crossed sectors.
cat_files:                        Concatenates files.
collect_integrals:                Collects all integrals appearing in the input file.
compute_diagram_interferences:    Computes interferences of diagrams.
compute_differential_equations:   Computes derivatives of integrals wrt invariants.
export:                           Exports to FORM, Mathematica or Maple format.
find_diagram_shifts:              Matches diagrams to sectors via graphs.
find_diagram_shifts_alt:          Matches diagrams to sectors via combinatorics.
generate_identities:              Generates identities like IBPs for given seeds.
generate_seeds:                   Generates integrals from a sector.
insert_reductions:                Inserts reductions in expressions.
normalize:                        Simplifies linear combinations and equations.
print_reduction_info_file:        Analyzes reductions in a file.
print_reduction_info_sectors:     Analyzes reductions available for sectors.
print_sector_info:                Prints diagrams and other information for sectors.
reduce_files:                     Reduces identities in given files.
reduce_sectors:                   Reduces integrals from a selection of sectors.
run_reduction:                    Low-level job to run a reduction.
select_reductions:                Selects reductions for integrals.
setup_sector_mappings:            Finds shifts between sectors via graphs.
setup_sector_mappings_alt:        Finds shifts between sectors via combinatorics.
sum_terms:                        Sums terms.
test:                             Empty job template.
verify_same_terms:                Verifies two files contain the same terms.

andreas@chili:~$
```

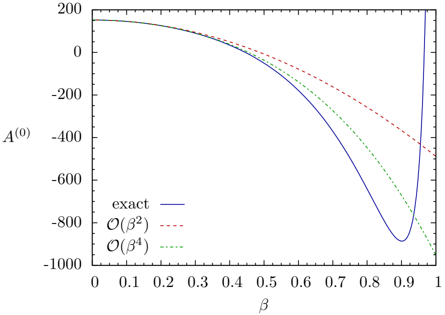
andreas : bash

detailed description: see builtin on-line help

Top quark pair production at two loops: gluon channel results

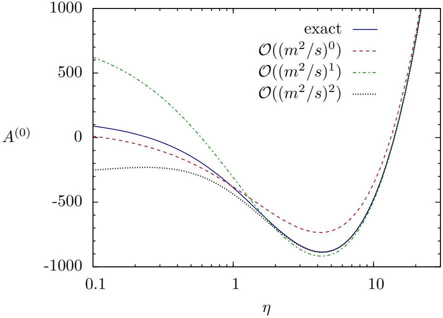
Results for gg leading N_c: expansions

threshold expansion:



$\beta = \sqrt{1 - 4m_t^2/s}$,
c.m. scatt. angle $= \pi/2$

small mass expansion:



$\eta = \frac{s}{4m_t^2} - 1$,
c.m. scatt. angle $= \pi/2$

(careful with phase space for small m expansion:
don't introduce forw.-backw. asymmetry, $\phi = -(t - m_t^2)/s = const$)

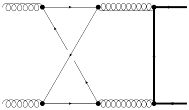
Bonciani, Ferroglia, Gehrmann, A.v.M., Studerus (2010)

LIGHT FERMION LOOPS IN $gg \to t\bar{t}$

- **analytical** result for **light fermion** loops in $gg \to t\bar{t}$:
  Bonciani, Ferroglia, Gehrmann, A.v.M., Studerus (in preparation)

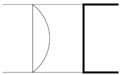
- required calculation of **11 new master integrals**
  - differential equations + Mellin-Barnes for const.
  - used: `MB.m` by Czakon '05, planar: `AMBRE` by Gluza, Kajda, Riemann '07 + Yundin '10

- most difficult integrals: **massive non-planar double box**

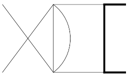


  - 3 master integrals (+ subtopo MIs)
  - analytical solution for massless case: Tausk 1999
  - 3 + 1 scale problem: s, t, u and m appear naturally

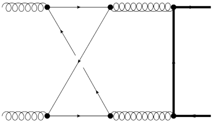
subtopo with s,t:



subtopo with s,u:



Analytical solution for massive non-planar double box



solved with differential equations:
A.v.M, Studerus (in preparation)

- all masters up to and including finite parts
- determined integration constants from
  - regularity conditions
  - symmetry conditions
  - Mellin-Barnes (cmp. with SecDec by Carter, Heinrich '10) in kinematical limits (finite const. up to 3-fold scaleless M.B.)

found **result** in terms of (x, y):

- linear combinations of GPLs with rational prefactors
- transcendality up to 4
- 805 GPLs total, 166 two-dim.
- GPLs with argument y: weights $\{0, -1, -x, -1/x, -1/x - x, -1/x - x + 1\}$
- GPLs with argument x: weights $\{0, \pm 1, \pm i, (1 \pm i\sqrt{3})/2\}$
  - for univariate (cyclotomic) cmp. Ablinger, Blümlein, Schneider '11

Generalised polylogarithms

Remiddi, Gehrmann; Goncharov:

Definition of generalised polylogarithms (GPLs)

$$G(\vec{0}_n; x) = \frac{1}{n!}\log^n(x),$$

$$G(a; x) = \int_0^x dt\ f_a(t), \qquad \text{for } a \neq 0$$

$$G(a, \vec{b}; x) = \int_0^x dt\ f_a(t) G(\vec{b}; t), \qquad \text{for } a \neq 0$$

with weight functions for (complex) weight a:

$$f_a(x) = \frac{1}{x - a}$$

- $G(a \neq 0; x) = \log\left(\frac{a-x}{a}\right)$, $G(\vec{0}_{n-1}, 1; x) = Li_n(x)$, $G(\vec{0}_n, \vec{1}_p; x) = S_{n,p}(x)$
- for weights $0, 1, -1$, GPLs specialize to harmonic polylogarithms (HPLs), Remiddi, Vermaseren (1999)
- shuffle algebra, **symbols** (see $N = 4$ remainder func), ...

Simplification via symbols

result for $1/\epsilon$ of scalar master:

- using GPLs with variables (x, y): **62 different GPLs** (23 two-dim.)
- choosing different arguments via **symbols**, this **simplifies to**:

$$
\begin{aligned}
M_1\big|_{1/\epsilon} = & \frac{1}{16 m^2 y_1 z_1 (y_1 + z_1)} \times \\
& \times \left(\mathrm{Li}_3 \left(\frac{y_1 z_1}{y_1 + z_1} \right) \right. \\
& \quad + \mathrm{Li}_2 \left(\frac{y_1 z_1}{y_1 + z_1} \right) \left(\log(-y_1 - z_1) - \log y_1 - \log z_1 \right) \\
& \quad \left. + \text{polynomial in } \log(y_1),\ \log(z_1),\ \log(-y_1 - z_1),\ \log\left(1 - \frac{y_1 z_1}{y_1 + z_1}\right) \right)
\end{aligned}
$$

with $y_1 := -t/m^2 + 1$,
$z_1 := -u/m^2 + 1$

- recent progress with symbols: Duhr, Gangl, Rhodes '11, Duhr '12

Symbol calculus for multiple polylogarithms

Motivation for symbol calculus

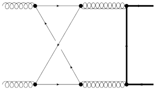
symbol calculus:

- tool to exploit functional relations between polylogarithms in a systematic way
  - works: simplify and transform functions; Zagier ('91), Goncharov ('95), Gangl ('02)
  - wish: perform integrations, see Chavez, Duhr '12 and talks by C. Bogner

first application in theoretical physics:

- $N = 4$ supersymmetric Yang-Mills two-loop six-point remainder function
  Del Duca, Duhr, Smirnov (09): $\mathcal{O}(10^3)$ multi-dimensional polylogs
- simplifies to few-line expression with few Li_4, Li_2 and ln !
  found with symbols by Goncharov, Spradlin, Vergu, Volovich ('10)

symbols also very welcome for **QCD**, e.g.:

- e.g. 
- corner integral alone gives $\mathcal{O}(10^3)$ GPLs, many two-dimensional
- numerical evaluation slow, stability problematic
- switching basis functions, expansions, analytical continuation difficult
- simplifications guided by symbols possible; Ferroglia, AvM, Studerus

crucial progress very recently:

- construction of functions for given symbol; Duhr, Gangl, Rhodes ('11)
  - applied to HPLs; Bühler, Duhr ('11)
- extended symbol calculus from coproduct: **constants**;
  Goncharov ('02), Brown ('11), Duhr ('12)
  demonstrated for QCD amplitude; Duhr ('12)

Definition of symbol map

Let G be a generalized polylogarithm with

$$\mathrm{d}G = \sum_i \hat{G}_i \, \mathrm{d} \ln(R_i)$$

where R_i is a rational function of the polylog arguments. The **symbol map $\mathcal{S}$**

$$\mathcal{S}(G) = \sum_i \mathcal{S}(\hat{G}_i) \otimes R_i \,,$$

associates a tensor with the polylogarithm.

examples:

- $\mathcal{S}(\ln x) = x$
- $\mathcal{S}(\mathrm{Li}_3\, x) = -((1-x) \otimes x \otimes x)$
- $\mathcal{S}(G(1,0,-1,-1,x)) = (1+x) \otimes (1+x) \otimes x \otimes (1-x)$

Rules for symbols

- $R_1 \cdots \otimes (R_a R_b) \otimes \cdots R_k = R_1 \cdots \otimes R_a \otimes \cdots R_k + R_1 \cdots \otimes R_b \otimes \cdots R_k$ (log law)
- $R_1 \cdots \otimes (c R_a) \otimes \cdots R_k = R_1 \cdots \otimes R_a \otimes \cdots R_k$ for constant c
- preserves shuffle product

Example for symbol calculus

goal: derive "simplification formula" for $\mathrm{Li}_2(1/x)$ with $0 < x < 1$, $\mathrm{Im}\, x = \varepsilon$

$$\begin{aligned}\mathcal{S}(\mathrm{Li}_2(1/x)) &= -(-1 + 1/x) \otimes (1/x) \\ &= (1 - x) \otimes x - x \otimes x \\ &= \mathcal{S}(-\mathrm{Li}_2(x) - (1/2)\ln^2 x)\end{aligned}$$

reproduces the highest degree part of the full answer

$$\mathrm{Li}_2(1/x) = -\mathrm{Li}_2(x) - (1/2)\ln^2 x + i\pi \ln x - (2/3)\pi^2$$

note: works at **highest degree** only

- $\mathcal{S}(\ln(-x)) = \mathcal{S}(\ln(x))$: no info on discontinuity
- $\mathcal{S}(\pi) = \mathcal{S}(\zeta_3) = 0$: no constants

"integrating the symbol" $\Rightarrow$ **algorithmic reduction** (structured with shuffle eliminators): Duhr, Gangl, Rhodes ('11)

Systematic reduction to basis function with symbols

$gg \to t\bar{t}$ **at two-loops**, ferm. contrib. (incl. non-planar part)

- pick specific coefficient in finite part: linear combination of polylogs
- pick uniform weight 4 part
- representation not unique (due to crossings etc.), imaginary polylogs
- here: 182 GPLs in total, 84 weight 4 non-Li_k GPLs
- symbol contains slots (weight have unit roots):

$$\{-1+x, x, 1+x, y, 1+y, x+y, 1+xy, 1+x^2+xy, 1-x+x^2+xy\}$$

algorithmic conversion at highest degree to new set of real basis function:

1. partition (4) (no shuffles), $\mathrm{Li}_{22}(R_1, R_2)$, $\mathrm{Li}_4(R_1)$
2. partition (3,1): $\mathrm{Li}_3(R_1)\ln(R_2)$
3. partition (2,2): $\mathrm{Li}_2(R_1)\,\mathrm{Li}_2(R_2)$
4. partition (2,1,1): $\mathrm{Li}_2(R_1)\ln(R_2)\ln(R_3)$
5. partition (1,1,1,1) (rest): $\ln(R_1)\ln(R_2)\ln(R_3)\ln(R_4)$

result:

- 28 Li_{22}, 48 Li_4, 58 Li_3, 18 Li_2, 11 ln
- all real

What about **subleading degree** *terms (const $\times$ polylog) ?*

- accessible by **coproduct**: Goncharov ('02), Brown ('11)
- extended symbol calculus based on coproduct by Duhr ('12) with:

$$\Delta(\pi) = \pi \otimes 1$$
$$\Delta(\zeta_k) = \zeta_k \otimes 1 + 1 \otimes \zeta_k \qquad \text{for } k \text{ (odd)}$$

example:
$$\begin{aligned}\Delta_{1,1}\left(\mathrm{Li}_2(1/x)\right) &= -\ln(1-1/x) \otimes \ln(1/x) \\ &= \ln(1-x) \otimes \ln(x) - \ln(x) \otimes \ln(x) + i\pi \otimes \ln(x) \\ &= \Delta_{1,1}\left(-\mathrm{Li}_2(x) - (1/2)\ln^2(x) + i\pi\ln(x)\right)\end{aligned}$$

reproduces identity up to pure constant, fix by limits or numerical evaluation:

$$\mathrm{Li}_2(1/x) - \left(-\mathrm{Li}_2(x) - (1/2)\ln^2(x) + i\pi\ln(x)\right) = -6.5797362673929\cdots = -(2/3)\pi^2$$

results for $gg \to t\bar{t}$: **systematic reductions** give:

- considerable simplifications, heavy improvement in access wrt traditional techniques
- beyond classical polylogs, need Li_{22}

Conclusions

- $gg \to t\bar{t}$: analytical two-loop corrections
  - leading N_c + light fermionic
  - massive non-planar double box: analytical solution
- Reduze 2: open source tool
  - parallelized reductions of Feynman integrals
  - graph matroid algorithm: determine shifts between sectors or diagrams
- symbol + coproduct refinement:
  - powerful algorithmic treatment of multiple polylogs
  - simplifications of QCD integrals with mass
  - outlook: direct solving of differential equations ?

Supplementary Slides

Ingredients for full NNLO calculation

- **VV**: two-loop ME for $q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$
- **RV**: one-loop ME for $t\bar{t}$ + 1 parton
  Dittmaier, Uwer, Weinzierl '07
- **RR**: tree level ME for $t\bar{t}$ + 2 partons
- **subtraction terms**: up to 2 unresolved partons needed
  Gehrmann-De Ridder, Ritzmann '09; Daleo et al. '09; Boughezal et al. '10; Glover, Pires '10; Czakon '10, '11; Anastasiou, Herzog, Lazopoulos '10; Abelof, Gehrmann-De Ridder '11, '12; Gehrmann, Monni '11; Bierenbaum, Czakon, Mitov '11
- **combined $q\bar{q}$**: Bärnreuther, Czakon, Mitov '12

consider **VV** ($2 \to 2$ ingredients):

$$\sum_{\text{spin,colour}} |\mathcal{M}|^2 = 16\pi^2\alpha_s^2 \left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{A}_2 + \mathcal{O}(\alpha_s^3)\right]$$

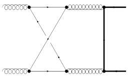
with

$$\mathcal{A}_0 = \left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \right\rangle$$

$$\mathcal{A}_1 = 2\,Re \left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \right\rangle$$

$$\mathcal{A}_2 = \left\langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \right\rangle + 2\,Re \left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \right\rangle$$

$\langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle$: Kniehl, Körner, Merebashvili, Rogal '05-'08, Anastasiou, Aybat '08



- Mellin-Barnes representation:

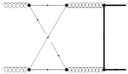
$$I(a_9) = (-1)^{-1+a_9}\pi^d \frac{\Gamma(-2+d/2)^2}{\Gamma(a_9)\Gamma(-4+d)\Gamma(-6-a_9+3d/2)}$$
$$\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_2}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_3}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_4}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_5}{2\pi i}$$
$$(-s)^{-6-a_9+d-z_1-z_2-z_5}(-t_1)^{z_1}(-u_1)^{z_2}(m^2)^{z_5}$$
$$\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5)$$
$$\Gamma(1+z_1+z_3)\Gamma(1+z_2+z_3)\Gamma(1+z_1+z_4)\Gamma(1+z_2+z_4)$$
$$\Gamma(4-d/2+z_1+z_2+z_3+z_4)\Gamma(-5-a_9+d-z_1-z_2-z_3-z_5)$$
$$\Gamma(-5-a_9+d-z_1-z_2-z_4-z_5)\Gamma(6+a_9-d+z_1+z_2+z_3+z_4+z_5)$$
$$\Gamma(a_9+z_1+z_2+2z_5))/\Gamma(2+z_1+z_2+z_3+z_4)^2$$

with

$$a_9 = \text{power of massive propagator}$$
$$t_1 = t+m^2$$
$$u_1 = u+m^2$$

A non-planar massive double box

Analytical solution



- e.g. leading poles of scalar integral very simple:

$$\frac{1}{N(\epsilon)} I(1) = \frac{1}{\epsilon^4} \cdot \frac{y_1 + z_1}{32 x_s^2 y_1 z_1}$$
$$+ \frac{1}{\epsilon^3} \cdot \frac{7(y_1 + z_1) - 6 * (y_1 + z_1) * \log x_s + 3(y_1 - z_1)(\log y_1 - \log z_1)}{96 x_s^2 y_1 z_1}$$
$$+ \frac{1}{\epsilon^2} \cdot \left(c_1 \log x_s + c_2 \log^2 x_s + c_3 \log y_1 + c_4 \log^2 y_1 + c_5 \log z_1 + c_6 \log^2 z_1\right.$$
$$\left. + c_7 \log x_s \log y_1 + c_8 \log x_s \log z_1 + c_9 \log y_1 \log z_1\right)$$

- however: differential equations work with 2 + 1 on-shell variables ($s + t + u = 2m^2$)
- GPL conversions involved, e.g.:

$$G(-1, 0, -1; z) = G\left(-1, 0, -1; -\frac{1}{x} - x - y\right)$$
$$= G\left(1 - \frac{1}{x} - x, -\frac{1}{x} - x, 1 - \frac{1}{x} - x; y\right) + 42 \text{ more GPLs}$$
$$= G\left(\frac{1 - y + \sqrt{-3 - 2y + y^2}}{2}, \frac{-y + \sqrt{-4 + y^2}}{2}, \frac{1 - y + \sqrt{-3 - 2y + y^2}}{2}; x\right) + 39 \text{ more GPLs}$$

- solved up to few integration constants

Definition of the coproduct

For a multiple polylogarithm

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{\mathrm{d}t}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t)$$

the coproduct Δ is defined according to Goncharov ('02):

$$\Delta\left(I(a_0; a_1, \ldots, a_n; a_{n+1})\right) = \sum_{0=i_1<\ldots<i_{k+1}=n} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}$$

examples:

- $\Delta(\ln(x)) = 1 \otimes \ln(x) + \ln(x) \otimes 1$
- $\Delta(\mathrm{Li}_2(x)) = 1 \otimes \mathrm{Li}_2(x) - \ln(1-x) \otimes \ln(x) + \mathrm{Li}_2(x) \otimes 1$
- $\Delta(\ln(x)\ln(y)) = 1 \otimes (\ln(x)\ln(y)) + \ln(x) \otimes \ln(y) + \ln(y) \otimes \ln(x) + (\ln(x)\ln(y)) \otimes 1$

Rules for the coproduct

- coassociativity $(\mathrm{id} \otimes \Delta)\,\Delta = (\Delta \otimes \mathrm{id})\,\Delta$
- compatible with product: $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$ where $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) \equiv (a_1 \cdot b_1) \otimes (a_2 \cdot b_2)$

note: coproduct means "decomposition"

Graded decomposition with the coproduct

- Hopf algebra of multiple polylogs **graded by weight**:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

since coproduct preserves weight we may decompose

$$\mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q$$

and define $\Delta_{p,q}$ to be the part with values in $\mathcal{H}_p \otimes \mathcal{H}_q$

- iterated coproduct:

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \mathrm{id}} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \mathrm{id} \otimes \mathrm{id}} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$$

and corresponding parts $\Delta_{p,q,\ldots,r}$

- symbol $\mathcal{S}$ = maximally iterated coproduct $\Delta_{1,\ldots,1}$ mod π

systematic procedure: back to $gg \to t\bar{t}$ **at two-loops** application:

1. fix maximum degree 4 part with symbol, substract from original expression
2. take $\Delta_{1,1,1,1}$, use unique set of real ln, gives

$$i\pi \otimes \Delta_{1,1,1}(\text{something})$$

   match "something" at degree 3 to basis functions using symbols
3. take $\Delta_{2,1,1}$, use unique set of real ln, Li_2, gives

$$\pi^2 \otimes \Delta_{1,1}(\text{something})$$

   match "something" at degree 2 to basis functions using symbols
4. take $\Delta_{3,1}$, use unique set of real ln, Li_2, Li_3, gives

$$\zeta(3) \otimes \text{something1} + i\pi^3 \otimes \text{something2}$$

   read off "somethingi" (just logs) at degree 1
5. numerical evaluation, match against

$$i\pi\zeta_3, \quad \pi^4$$

(requires matching of degree 3 and lower, obtained by recursion)

results for $gg \to t\bar{t}$:

- considerable simplifications for poles
- finite parts: heavy improvement in access wrt traditional techniques