Theory Seminar Zeuthen, DESY

Multi-Summation and Difference Field Theory for Particle Physics

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PART 1:
Motivation and the Basic Algorithms

Multi-summation and particle physics:

Feynman parameter integrals with one mass M and local operator insertions in $4+\varepsilon$-dimensional Minkowski space:

$$D_\varepsilon(n)=\int\frac{d^Dp_1}{(2\pi)^D}\dots\int\frac{d^Dp_k}{(2\pi)^D}\frac{N(p_1,\dots,p_k;p;M^2;\Delta,n)}{(-p_1^2+m_1^2)^{l_1}\dots(-p_k^2+m_k^2)^{l_k}}\prod_V\delta_V$$

$\downarrow$ (Blümlein/Stan/Schneider 2012)

Definite hypergeometric multi-sums:

$$D_\varepsilon(n)=\sum_{k_1=l_1}^{L_1(n)}\dots\sum_{k_v=l_v}^{L_v(n,k_1,\dots,k_{v-1})}\sum_{i=1}^{l}f_i(\varepsilon,n,k_1,\dots,k_v)$$

f_i: proper hypergeometric series in terms of Γ-functions
$L_v(n,k_1,\dots,k_{v-1})$: integer linear relation in the parameters (or ∞)

$\downarrow$ (symbolic summation)

$$D_\varepsilon(n)=\varepsilon^{-3}F_{-3}(n)+\varepsilon^{-2}F_{-2}(n)+\varepsilon^{-1}(n)F_{-1}(n)+\dots$$

A warm up example

$$\text{GIVEN } F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}
+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg)}_{f(n,k,j)}.$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

A warm up example

$$\text{GIVEN } F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(n,k,j)}.$$

FIND the first coefficients of the ϵ-expansion

$$F(N)=F_0(n)+\varepsilon F_1(n)+\dots$$

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A warm up example

GIVEN $F(n) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)} \times$

$$\times \Bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)} \Bigg).$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(n,k,j) = f_0(n,k,j) + \varepsilon f_1(n,k,j) + \dots$$

Arose in the context of
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A warm up example

GIVEN $$F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}\right.+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(n,k,j)}.$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

where

$$S_1(N)=\sum_{i=1}^{N}\frac{1}{i}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!}+\frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j) = g(j+1) - g(j)}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j) = g(j+1) - g(j)}$$

Sigma (based on a refined version of M. Karr's difference fields (1981)) computes

$$g(j) = \frac{(j+k+1)(j+N+1)j!k!(j+k+N)!\big(S_1(j)-S_1(j+k)-S_1(j+N)+S_1(j+k+N)\big)}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) =$$ ▸ Sigma

FIND $g(j)$:

$$\boxed{f(N,k,j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\boxed{\sum_{j=0}^{a} f(N,k,j) = g(a+1) - g(0)}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{a} f(N,k,j) = \underbrace{\frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!} + \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!} + \frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}}_{a\,\to\,\infty}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{j=0}^{\infty} f(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\Big(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\,(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}\Big)}^{f(N,k,j)}$$

$$\sum_{k=1}^{a}\sum_{j=0}^{\infty} f(N,k,j) = \sum_{k=1}^{a}\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!}$$

$$=$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} | c \in \mathbb{R}\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = \frac{1}{2} \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

Sigma

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)=\frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)=\frac{1}{96n(n+1)}\Big(S_1(n)^4+(12\zeta_2+54S_2(n))S_1(n)^2$$

$$+104S_3(n)S_1(n)-48S_{2,1}(n)S_1(n)+51S_2(n)^2+36\zeta_2S_2(n)$$

$$+126S_4(n)-48S_{3,1}(n)-96S_{1,1,2}(n)\Big)$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)+\varepsilon^3\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)+\dots$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)=\frac{1}{960n(n+1)}\Big(S_1(n)^5+(20\zeta_2+130S_2(n))S_1(n)^3+$$

$$(40\zeta_3+380S_3(n))S_1(n)^2+\left(135S_2(n)^2+60\zeta_2S_2(n)+510S_4(n)\right)S_1(n)$$

$$-240S_{3,1}(n)S_1(n)-240S_{1,1,2}(n)S_1(n)+160\zeta_2S_3(n)+S_2(n)(120\zeta_3$$

$$+380S_3(n))+624S_5(n)+\left(-120S_1(n)^2-120S_2(n)\right)S_{2,1}(n)$$

$$-240S_{4,1}(n)-240S_{1,1,3}(n)+240S_{2,2,1}(n)\Big)$$

More generally: Sigma's summation spiral

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

More generally: Sigma's summation spiral

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

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FIND all solutions expressible by indefinite nested products and sums
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3. Find a "closed form"

A(n)=combined solutions.

The basic summation algorithm

(a simplified version of Karr's algorithm, 1981)

Recall some basic notions

- $(\mathbb{K}, +, *, 0, 1)$ is a field iff
  - $(\mathbb{K}, +, 0)$ and $(\mathbb{K}^* = \mathbb{K} \setminus \{0\}, *, 1)$ are commutative groups
  - for all $a, b, c \in \mathbb{K}:\ a * (b + c) = a * b + a * c$ (distributivity law)

From now on we suppress the operations and just say that $\mathbb{K}$ is a field.

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- Examples: the rational numbers $\mathbb{Q}$ or the rational function field $\mathbb{Q}(x)$, i.e., x is transcendental over $\mathbb{Q}$ ($\mathbb{Q}[x]$ is the polynomial ring with coefficients from $\mathbb{Q}$) and

$$\mathbb{Q}(x) = \{\frac{p}{q} : p, q \in \mathbb{Q}[x] \text{ with } q \neq 0\}$$

  Usually we assume that $\frac{p}{q}$ is reduced, i.e., $\gcd(p, q) = 1$.

  Note: If $\mathbb{K} = \mathbb{Q}(x)$ is already a rational function field, we can construct on top again a rational function field $\mathbb{K}(y) = \mathbb{Q}(x)(y)$, etc.

Recall some basic notions

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  - $(\mathbb{K}, +, 0)$ and $(\mathbb{K}^* = \mathbb{K} \setminus \{0\}, *, 1)$ are commutative groups
  - for all $a, b, c \in \mathbb{K}:\ a * (b + c) = a * b + a * c$ (distributivity law)

  From now on we suppress the operations and just say that $\mathbb{K}$ is a field.

- Let $\mathbb{F}$ be a field. A map $\sigma : \mathbb{F} \to \mathbb{F}$ is a field automorphism iff
  - σ is bijective
  - for all $a, b \in \mathbb{F}$: $\sigma(a + b) = \sigma(a) + \sigma(b)$ and $\sigma(a\,b) = \sigma(a)\sigma(b)$
  - for all $a \in \mathbb{F}^*$: $\sigma(-a) = -\sigma(a)$ and $\sigma(\frac{1}{a}) = \frac{1}{\sigma(a)}$
  - $\sigma(0) = 0$ and $\sigma(1) = 1$

  Note: various of these properties are redundant, i.e., can be proven by a subset of these axioms.

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (S_1(k) - 1)k.$$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$

$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

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$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

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This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1)
\end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

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\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
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\end{aligned}$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c\in\mathbb{K} \\
\sigma(t_1) &= a_1\,t_1 + f_1, \quad a_1\in\mathbb{K}^*, \qquad f_1\in\mathbb{K} \\
\sigma(t_2) &= a_2\,t_2 + f_2, \quad a_2\in\mathbb{K}(t_1)^*, \qquad f_2\in\mathbb{K}(t_1) \\
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\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c\in\mathbb{K}(t_1)(t_2)\dots(t_e)|\sigma(c)=c\} = \mathbb{K}.$$

GIVEN $f\in\mathbb{F}$;
FIND $g\in\mathbb{F}$ such that

$$\sigma(g)-g=f.$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $\Pi\Sigma^*$-field for the summand

$$\boxed{\mathrm{const}_\sigma \mathbb{F} = \mathbb{Q}}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

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FIND $g \in \mathbb{Q}(k)(h)$:

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g = \sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g = \sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

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FIND $g'\in\mathbb{Q}(k)[h]$ with

$$\frac{1}{\sigma(d)}\sigma(g')-\frac{1}{d}g=\sigma(\frac{g'}{d})-\frac{g'}{d}=h.$$

Degree bound: COMPUTE $b\geq 0$:

$$\forall g\in\mathbb{Q}(k)[h]\quad \sigma(g)-g=h \quad\Rightarrow\quad \deg(g)\leq b.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

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Degree bound: COMPUTE $b \geq 0$:

$b = 2$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

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ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

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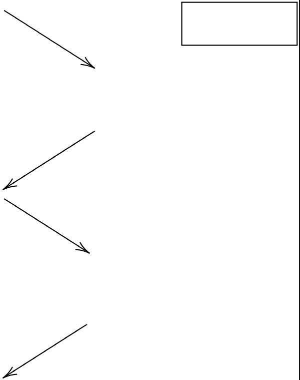

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

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$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

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$\leftarrow$ $g_0 = -k$, $d = 0$

Difference equations in difference fields

Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f\in\mathbb{F}$.
- Find $g\in\mathbb{F}$:

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$\downarrow$ $\uparrow$

Parameterized Telescoping

- Given $f_0, \dots, f_d \in \mathbb{F}$.
- Find all $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \dots + c_d\, f_d.}$$

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Parameterized first order linear difference equation

- Given $f_0,\dots,f_d\in\mathbb{F}$, $a_0,a_1\in\mathbb{F}$.
- Find all $c_0,\dots,c_d\in\mathbb{K}$, $g\in\mathbb{F}$:

$$\boxed{a_1\,\sigma(g)+a_0\,g=c_0\,f_0+\dots+c_d\,f_d.}$$

More generally: **Parameterized linear difference equations**

- Given $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1, \dots, a_m \in \mathbb{F}$ $(a_0 a_m \neq 0)$.
- Find all $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{a_m \sigma^m(g) + \dots + a_1\, \sigma(g) + a_0\, g = c_0\, f_0 + \dots + c_d\, f_d.}$$

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- Given $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1, \dots, a_m \in \mathbb{F}$ $(a_0 a_m \neq 0)$.
- Find all $(c_0, \dots, c_d, g) \in \mathbb{K}^{d+1} \times \mathbb{F}$

$$\boxed{a_m \sigma^m(g) + \dots + a_1\, \sigma(g) + a_0\, g = c_0\, f_0 + \dots + c_d\, f_d.}$$

More generally: **Parameterized linear difference equations**

- Given $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1, \dots, a_m \in \mathbb{F}$ $(a_0 a_m \neq 0)$.
- Compute $V := \Big\{(c_0, \dots, c_d, g) \in \mathbb{K}^{d+1} \times \mathbb{F}$:

$$\boxed{a_m \sigma^m(g) + \dots + a_1\, \sigma(g) + a_0\, g = c_0\, f_0 + \dots + c_d\, f_d.}\Big\}$$

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Remark:

- V is a subspace of $\mathbb{K}^{d+1} \times \mathbb{F}$ over $\mathbb{K}$
- $\dim_{\mathbb{K}} V \leq m + d + 1$
- $\boxed{\text{Compute a basis of } V}$

PART 2: The Summation Paradigms and Difference Field Theory

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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We compute

$$g(k) = (S_1(k) - 1)k.$$

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for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$
$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

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FIND a closed form for

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A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $\Pi\Sigma^*$-field for the summand

$\boxed{\text{const}_\sigma\mathbb{F} = \mathbb{Q}}$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

Creative telescoping in difference fields

Simplify

$$A(n) := \sum_{k=1}^{n} \binom{n}{k} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(n)(k)(h)(b)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1},$$

$$\sigma(b) = \frac{n-k}{k+1}\, b, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}.$$

Creative telescoping

REPRESENT $f(n,k)$ in $\mathbb{F}$:

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☹

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REPRESENT $f(n+i,k)$ in $\mathbb{F}$:

$$f(n,k) = S_1(k)\binom{n}{k} \quad\longleftrightarrow\quad h\,b =: f_0$$

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FIND $g \in \mathbb{F}$ and $c_0, c_1 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g) - g = c_0 f_0 + c_1 f_1}$$

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FIND $g\in\mathbb{F}$ and $c_0,c_1,c_2\in\mathbb{Q}(n)$:

$$\boxed{\sigma(g)-g=c_0f_0+c_1f_1+c_2f_2}$$

We compute

$$c_0:=4\,(1+n),\quad c_1:=-2\,(3+2\,n),\quad c_2:=2+n,$$

$$g:=\frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b)}{(1-k+n)\,(2-k+n)}.$$

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This gives

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k)}$$

with

$$c_0(n) := 4\,(1+n), \quad c_1(n) := -2\,(3+2\,n), \quad c_2(n) := 2+n,$$

$$g(n,k) := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)S_1(k))\binom{n}{k})}{(1-k+n)\,(2-k+n)}.$$

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Summing over k from 0 to n gives

$$\boxed{g(n,n+1)-g(n,0)} = \begin{array}{l} c_0(n)\,\mathsf{A}(n) + \\ c_1(n)\,\left[\mathsf{A}(n+1) - f(n+1,n+1)\right] \\ c_2(n)\,\left[\mathsf{A}(n+2) - f(n+2,n+1) - f(n+2,n+2)\right]. \end{array}$$

for $A(n) = \sum_{k=0}^{n}\binom{n}{k}S_1(k)$

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Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)S(n) - 2(3+2n)S(n+1) + (2+n)S(n+2)}$$

for $A(n) = \sum_{k=0}^{n}\binom{n}{k}S_1(k)$ Example

Summation paradigms

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

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$$a_d(n)A(n+d) + \dots + a_0(n)A(n) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
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Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

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$$||$$

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e.g., Petkovšek's Hyper

$$\prod_{j=\lambda}^{n} b_1(j-1)$$

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$$c(n)\Big(S - b_d(n)\Big)\ldots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

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d linearly independent solutions

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$$\vdots$$

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Example

$$\vdots$$

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation for simplification

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4. Find a "closed form"

A(n)=combined solutions.

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K \in \mathbb{N}$, $r_i, s_i \in \mathbb{Q}$, and p_i, q_i are polynomials in $x_1, \dots, x_7$.

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$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+F_{-1}(n)\varepsilon^{-1}+\boxed{F_0(n)}\varepsilon^{0}+\dots$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

Vermaseren, Moch: 3-5 CPU years (2004)

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$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\,\varepsilon^0 + \dots$$

$$\downarrow$$

Initial values $F_0(i)$, $i = 1,\dots,5114$

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Initial values $F_0(i)$, $i = 1, \dots, 5114$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

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$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\,F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \cdots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\, F_0(n+35) = 0$$

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$A_0 =$ 46409443092113136725039802237162641242004070859938540024124603151 94
9576502126934497104844629972221629340528573833320076715019401639150 1666
2795021380735610971095204560396627338875778269758860220127798356053 2017
3748759267144591132576514527194521425546215314730842059721076159532 9365
5156345299861313538471891130525329905319889360640146402160891162097 4192
0900166802995162078018294725826293945080115451177452783250387434166 1898
8916752210737846879797981026538551064393704386755756346752374040609 4658
9910046793335373195964562497752442467299065442773230988168534648377 1128
6902083714745202440152816907940693366534447618126024334417209769163 6706
6280305967553580902716969306447414771961021984962848689607964231297 5136
2077687686774188348836384694485449648262937243682969905539136917885 0397
0038163801161230267958089748807664772131119306347353167877796207576 5995 1
5202809978299053753901432067359626151

(885 decimal digits)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

$\downarrow$

Initial values $F_0(i)$, $i = 1,\dots,5114$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$ Sigma

CLOSED FORM

(J. Blümlein, M. Kauers, S. Klein, CS; Comput. Phys. Comm. 180, pp. 2143-2165. 2009; arXiv 0902.4091)

Difference field theory for symbolic summation

Problem description

A sequence domain:

$$(X, \mathrm{ev}, \mathfrak{d})$$

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set of terms

Example

- $X = \mathbb{K}(x)$.

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$$(X, \mathrm{ev}, \mathfrak{d})$$

set of terms   evaluation $X \times \mathbb{N} \to \mathbb{K}$

Example

- $X = \mathbb{K}(x)$.
- For $f = \frac{p}{q} \in \mathbb{K}(x)$ with $\gcd(p, q) = 1$,

$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0; \end{cases}$$

Problem description

A sequence domain:

$$(X, \mathrm{ev}, \mathfrak{d})$$

set of terms — evaluation $X \times \mathbb{N} \to \mathbb{K}$ — measure of simplicity $X \to \mathbb{N}$

Example

- $X = \mathbb{K}(x)$.
- For $f = \frac{p}{q} \in \mathbb{K}(x)$ with $\gcd(p,q) = 1$,

$$\mathrm{ev}(f,k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0; \end{cases}$$

- $$\mathfrak{d}(f) = \begin{cases} 0 & \text{if } f \in \mathbb{K} \\ 1 & \text{if } f \in \mathbb{K}(x) \setminus \mathbb{K} \end{cases}$$

Sum sequence domain

Given $(X, \mathrm{ev}, \mathfrak{d})$, we get the sum sequence domain $(\mathsf{Sum}(X), \mathrm{ev}, \mathfrak{d})$.

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$\mathsf{Sum}(X)$ is the term algebra over X with the signature

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Example. Elements from $\mathsf{Sum}(\mathbb{Q}(x))$ are:

$$\frac{1}{x}$$

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$$\frac{1}{x} \qquad \mathsf{Sum}(7, \frac{1}{x})$$

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$$
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$$\mathsf{Sum}\Big(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}(1, \tfrac{1}{x})^2 \oplus \mathsf{Sum}(1, \tfrac{1}{x^2})\big)\big) \oplus \mathsf{Sum}(1, \mathsf{Sum}(1, \tfrac{1}{x}))\big)\Big)$$

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$$\mathrm{ev}(\overbrace{\mathsf{Sum}(7, \frac{1}{x})}^{\mathsf{H}}, k) =$$

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$$\text{ev}(\overbrace{\mathsf{Sum}(7, \frac{1}{x})}^{\mathsf{H}}, k) = \sum_{i=7}^{k} \text{ev}(\frac{1}{x}, i) = \sum_{i=7}^{k} \frac{1}{i}$$

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Example.

$$\mathrm{ev}(\overbrace{\mathsf{Sum}(7, \frac{1}{x})}^{\mathsf{H}}, k) = \sum_{i=7}^{k} \mathrm{ev}(\frac{1}{x}, i) = \sum_{i=7}^{k} \frac{1}{i} \qquad \mathfrak{d}(H) = 1 + \mathfrak{d}(\frac{1}{x}) = 2$$

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Example.

$$\mathrm{ev}(\mathsf{Sum}\overbrace{\Big(1, \tfrac{1}{x} \otimes (\mathsf{Sum}\ (1, \tfrac{1}{x} \otimes (\mathsf{Sum}\ (1, \tfrac{1}{x})^2 \oplus \mathsf{Sum}(1, \tfrac{1}{x^2}))) \oplus \mathsf{Sum}(1, \mathsf{Sum}(1, \tfrac{1}{x})))\Big)}^{\mathsf{A}}, k)$$

$$= \sum_{r=1}^{k} \frac{\displaystyle\sum_{l=1}^{r} \frac{\Big(\displaystyle\sum_{i=1}^{l} \frac{1}{i}\Big)^2 + \displaystyle\sum_{i=1}^{l} \frac{1}{i^2}}{l} + \sum_{l=1}^{r} \frac{\displaystyle\sum_{i=1}^{l} \frac{1}{i}}{l}}{r}$$

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Example.

$$\mathrm{ev}(\overbrace{\mathsf{Sum}_1\Big(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}_2(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}_3(1, \tfrac{1}{x})^2 \oplus \mathsf{Sum}(1, \tfrac{1}{x^2})\big)\big) \oplus \mathsf{Sum}(1, \mathsf{Sum}(1, \tfrac{1}{x}))\big)\Big)}^{A}, k)$$

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The **optimal depth** of $A \in \mathsf{Sum}(X)$ is

$$\min\Big\{\mathfrak{d}(B) | B \in \mathsf{Sum}(X) \text{ such that } \mathrm{ev}(A,k) = \mathrm{ev}(B,k) \text{ for some } k \text{ on}\Big\}.$$

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Problem: Depth Optimal Simplification.

Given $A \in \mathsf{Sum}(X)$;
find $B \in \mathsf{Sum}(X)$ and $\lambda \in \mathbb{N}$ such that

$$\mathrm{ev}(A,k) = \mathrm{ev}(B,k) \text{ for all } k \geq \lambda$$

and such that $\mathfrak{d}(B)$ is the **optimal depth** of A.

The **optimal depth** of $A \in \mathsf{Sum}(X)$ is

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Running example. *Given*

$$A = \mathsf{Sum}\Big(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}(1, \tfrac{1}{x} \otimes \big(\mathsf{Sum}(1, \tfrac{1}{x})^2 \oplus \mathsf{Sum}(1, \tfrac{1}{x^2})\big)\big) \oplus \mathsf{Sum}(1, \mathsf{Sum}(1, \tfrac{1}{x}))\big)\Big)$$

with

$$\operatorname{ev}(A,k) = \sum_{r=1}^{k} \frac{\displaystyle\sum_{l=1}^{r} \frac{\Big(\sum_{i=1}^{l} \frac{1}{i}\Big)^2 + \sum_{i=1}^{l} \frac{1}{i^2}}{l} + \sum_{l=1}^{r} \frac{\sum_{i=1}^{l} \frac{1}{i}}{l}}{r}$$

Find $B \in \mathsf{Sum}(\mathbb{Q}(x))$ and $\lambda \in \mathbb{N}$ such that

$$\operatorname{ev}(A,k) = \operatorname{ev}(B,k) \quad \forall k \geq \lambda$$

and such that the depth of B is optimal.

Karr's $\Pi\Sigma^*$-fields

- A difference field $(\mathbb{F}, \sigma)$ is a field $\mathbb{F}$ plus a field automorphism σ.

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- A difference field $(\mathbb{F}, \sigma)$ is a field $\mathbb{F}$ plus a field automorphism σ.

Example. $(\mathbb{Q}(x), \sigma)$ with the shift operator $\sigma(f) = f(x+1)$.

- The constant field is

$$\mathrm{const}_\sigma \mathbb{F} = \{k \in \mathbb{F} | \sigma(k) = k\}.$$

Example. $\mathrm{const}_\sigma \mathbb{Q}(x) = \mathbb{Q}$

Karr's $\Pi\Sigma^*$-fields

- A difference field $(\mathbb{E}, \sigma')$ is a difference field extension of $(\mathbb{F}, \sigma)$ if

$$\mathbb{F} \leq \mathbb{E} \quad \text{and} \quad \sigma'|_{\mathbb{F}} = \sigma.$$

  Note: From now on, we do not distinguish σ and σ' any longer

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- **Lemma.** Let $(\mathbb{F},\sigma)$ be a DF, and $a \in \mathbb{F}^*$ and $b \in \mathbb{F}$.
  Let t be transcendental over $\mathbb{F}$ ($\mathbb{F}(t)$ is a rational function field).
  Then there is a unique DF-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ such that

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$$\sigma(t) = a\,t + b.$$

Example. There is exactly one DF-extension $(\mathbb{Q}(x), \sigma)$ of $(\mathbb{Q}, \sigma)$ with $\sigma(x) = x + 1$.

Karr's $\Pi\Sigma^*$-fields

- A difference field $(\mathbb{E},\sigma')$ is a difference field extension of $(\mathbb{F},\sigma)$ if
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- **Lemma.** Let $(\mathbb{F},\sigma)$ be a DF, and $a\in\mathbb{F}^*$ and $b\in\mathbb{F}$.
Let t be transcendental over $\mathbb{F}$ ($\mathbb{F}(t)$ is a rational function field).
Then there is a unique DF-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ such that
$$\sigma(t)=a\,t+b.$$
- **Definition** Such an extension in which the constants remain unchanged ($\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$) is called
  - Π-extension if $\sigma(t)=a\,t$
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$$\left.\begin{array}{l}\text{– }\Pi\text{-extension if }\sigma(t)=a\,t\\ \text{– }\Sigma^*\text{-extension if }\sigma(t)=t+b\end{array}\right\}\Pi\Sigma^*\text{-extension}$$

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- $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a (nested) $\Pi\Sigma^*$-extension (resp. Π-extension, Σ^*-extension) of $(\mathbb{F},\sigma)$ if it is a tower of such extensions.

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- **Definition** Such an extension in which the constants remain unchanged ($\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$) is called
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- $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a (nested) $\Pi\Sigma^*$-extension (resp. Π-extension, Σ^*-extension) of $(\mathbb{F},\sigma)$ if it is a tower of such extensions.
- It is a $\Pi\Sigma^*$-field if $\text{const}_\sigma\mathbb{F} = \mathbb{F}$.

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

such that $\mathrm{const}_\sigma \mathbb{F} = \{c \in \mathbb{K} \quad |\sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K}$$

such that $\mathrm{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1) \qquad | \sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\ \sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \end{aligned}$$

such that $\mathrm{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2) \quad | \sigma(c) = c\} = \mathbb{K}$.

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- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= a_3\, t_2 + f_3, \quad a_3 \in \mathbb{K}(t_1, t_2)^*, \qquad f_3 \in \mathbb{K}(t_1)
\end{aligned}$$

such that $\mathrm{const}_\sigma \mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)(t_3) \quad | \sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a difference field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)(t_3)(t_4)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, & f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, & f_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= a_3\, t_2 + f_3, \quad a_3 \in \mathbb{K}(t_1,t_2)^*, & f_3 \in \mathbb{K}(t_1) \\
\sigma(t_4) &= a_4\, t_2 + f_4, \quad a_4 \in \mathbb{K}(t_1,t_2,t_3)^*, & f_4 \in \mathbb{K}(t_1,t_2,t_3)
\end{aligned}$$

such that $\mathrm{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)(t_3)(t_4) \quad |\sigma(c) = c\} = \mathbb{K}$.

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)(t_3)(t_4)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= a_3\, t_2 + f_3, \quad a_3 \in \mathbb{K}(t_1,t_2)^*, && f_3 \in \mathbb{K}(t_1) \\
\sigma(t_4) &= a_4\, t_2 + f_4, \quad a_4 \in \mathbb{K}(t_1,t_2,t_3)^*, && f_4 \in \mathbb{K}(t_1,t_2,t_3) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that $\mathrm{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)(t_3)(t_4)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

SUMMARY Let $(\mathbb{F}(t), \sigma)$ be a DF-extension of $(\mathbb{F}, \sigma)$ with t transcendental over $\mathbb{F}$ and

$$\sigma(t) = t + f \qquad \text{with } f \in \mathbb{F}.$$

Then this is a Σ^*-extension

$$\Leftrightarrow \quad \mathrm{const}_\sigma \mathbb{F}(t) = \mathrm{const}_\sigma \mathbb{F}$$

Theorem (Karr 81) Let $(\mathbb{F}(t), \sigma)$ be a DF-extension of $(\mathbb{F}, \sigma)$ with t transcendental over $\mathbb{F}$ and

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Karr's summation algorithm solves (among others) the telescoping problem:

Given a $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$ and $f\in\mathbb{F}$.
Compute, if possible, a $g\in\mathbb{F}$ such that

$$\sigma(g)=g+f.$$

Start with the $\Pi\Sigma^*$-field $(\mathbb{Q}(x),\sigma)$ over $\mathbb{Q}$ with $\sigma(x)=x+1$.

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\left(\displaystyle\sum_{i=1}^{l}\frac{1}{i}\right)^2+\displaystyle\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\boxed{\displaystyle\sum_{i=1}^{l}\frac{1}{i}}}{l}}{r}$$

1 By Gosper's, Karr's, or Sigma's algorithm:

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$$\nexists g\in\mathbb{Q}(x)(h):\sigma(g)=g+\frac{\sigma(h)}{x+1}.$$

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$$\boxed{\sum_{l=1}^{r+1}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}=\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}+\frac{\sum_{i=1}^{r+1}\frac{1}{i}}{r+1}}$$

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3 Given the telescoping problem

$$\sigma(g)=g+\frac{1}{(x+1)^2},$$

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$$g=2s-h^2\in\mathbb{Q}(x)(h)(s).$$

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Note:

$$\boxed{\sum_{i=1}^{l}\frac{1}{i^2}=2\sum_{i=1}^{l}\frac{\sum_{j=1}^{i}\frac{1}{j}}{i}-\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}$$

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a

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The DF $(\mathbb{Q}(x)(h)(s)(t)(a), \sigma)$ of $(\mathbb{Q}(x)(h)(s)(t), \sigma)$ with

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a

$$A(k)=\boxed{\sum_{r=1}^{k}\frac{\overbrace{\sum_{l=1}^{r}\frac{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{t+s}{x}}}{r}}$$

SUMMARY $A(k)$ is represented by a in the $\Pi\Sigma^*$-field $(\mathbb{Q}(x)(h)(s)(t)(a),\sigma)$

$$\begin{aligned}
\forall k\in\mathbb{Q}\quad \sigma(k)&=k\\
\sigma(x)&=x+1\\
\sigma(h)&=h+\tfrac{1}{x+1}\\
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SUMMARY $A(k)$ is represented by a in the $\Pi\Sigma^*$-field $(\mathbb{Q}(x)(h)(s)(t)(a),\sigma)$

$$\begin{aligned}
\forall k\in\mathbb{Q}\quad \sigma(k)&=k & \forall k\in\mathbb{Q}\quad \delta(k)&=0\\
\sigma(x)&=x+1 & \delta(x)&=1\\
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\sigma(t)&=t+2\tfrac{\sigma(s)}{x+1} & \delta(t)&=4\\
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a

$$A(k)=\boxed{\sum_{r=1}^{k}\frac{\overbrace{\sum_{l=1}^{r}\frac{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{t+s}{x}}}{r}}$$

$$||$$

$$\sum_{r=1}^{n}\frac{\sum_{l=1}^{r}\frac{2\sum_{i=1}^{l}\frac{\sum_{j=1}^{i}\frac{1}{j}}{i}}{l}+\sum_{l=1}^{r}\frac{\sum_{j=1}^{l}\frac{1}{j}}{l}}{r}\qquad \text{depth}=5$$

- For a given $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ the depth function

$$\delta : \mathbb{F} \to \mathbb{N}$$

measures the depth (w.r.t. the shift-operator σ).

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Example. Take our $\Pi\Sigma^*$-field $(\mathbb{Q}(x)(h)(s)(t)(a), \sigma)$

$$\sigma(x) = x + 1 \qquad \delta(x) = 1$$
$$\sigma(h) = h + \frac{1}{x+1} \qquad \delta(h) = 2$$
$$\sigma(s) = s + \frac{\sigma(h)}{x+1} \qquad \delta(s) = 3$$
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$$\sigma(a) = a + \frac{\sigma(s+t)}{x+1} \qquad \delta(a) = 5$$

For any $f \in \mathbb{Q}(k, h, s, t, a)$,

$$\delta(f) = \max(\{\delta(x) | x \in \{k, h, s, t, a\} \text{ occurs in } f\} \cup \{0\})$$

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- For a $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ the extension depth is

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\end{aligned}$$

The extension depth of the Σ^*-extension $(\mathbb{Q}(x)(h)(s)(t)(a),\sigma)$ of $(\mathbb{Q}(x)(h)(s),\sigma)$ is $\max(\delta(t),\delta(a)) = 5$.

Theorem (Karr 81) Let $(\mathbb{F}(t),\sigma)$ be a DF-extension of $(\mathbb{F},\sigma)$ with t transcendental over $\mathbb{F}$ and

$$\sigma(t)=t+f \qquad \text{with } f\in\mathbb{F}.$$

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Sigma solves the following problem:

Given a $\Pi\Sigma^\delta$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}$; $f \in \mathbb{F}$.
Compute, if possible, a Σ^δ-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with extension depth $\leq \delta(f)$ and $g \in \mathbb{E}$ such that

$$\sigma(g) = g + f.$$

Start with the $\Pi\Sigma^*$-field $(\mathbb{Q}(x),\sigma)$ over $\mathbb{Q}$ with $\sigma(x)=x+1$ and $\delta(x)=1$.

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\boxed{\sum_{i=1}^{l}\frac{1}{i}}}{l}}{r}$$

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Note: there is no Σ^*-extension of $(\mathbb{Q}(x),\sigma)$ with extension depth 1.

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Note: there is no Σ^*-extension of $(\mathbb{Q}(x),\sigma)$ with extension depth 1.
Hence the DF-extension $(\mathbb{Q}(x)(h),\sigma)$ of $(\mathbb{Q}(x),\sigma)$ with

$$\sigma(h)=h+\frac{1}{x+1},\quad \delta(h)=2$$

is a Σ^δ-extension.

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2 Given the telescoping problem

$$\sigma(g)-g=\frac{\sigma(h)}{x+1},$$

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\boxed{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\sum_{i=1}^{l}\frac{1}{i}}^{\frac{h}{x}}}{l}}}{r}$$

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Sigma finds the Σ^{δ}-extension h_2 over $\mathbb{Q}(x)(h)$ with

$$\sigma(h_2)=h_2+\frac{1}{(x+1)^2},\quad \delta(h_2)=2$$

$$\frac{1}{2}(h^2 - h_2)$$

$$A(k) = \sum_{r=1}^{k} \frac{\displaystyle\sum_{l=1}^{r} \frac{\overbrace{\Big(\sum_{i=1}^{l} \frac{1}{i}\Big)^2}^{h^2} + \sum_{i=1}^{l} \frac{1}{i^2}}{l} + \boxed{\displaystyle\sum_{l=1}^{r} \frac{\overbrace{\sum_{i=1}^{l} \frac{1}{i}}^{\frac{h}{x}}}{l}}}{r}$$

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$$\sigma(g) - g = \frac{\sigma(h)}{x+1},$$

Sigma finds the Σ^δ-extension h_2 over $\mathbb{Q}(x)(h)$ with

$$\sigma(h_2) = h_2 + \frac{1}{(x+1)^2}, \quad \delta(h_2) = 2$$

together with the solution

$$g = \frac{1}{2}(h^2 + h_2)$$

$$A(k)=\sum_{r=1}^{k}\frac{\displaystyle\sum_{l=1}^{r}\frac{\overbrace{\Big(\sum_{i=1}^{l}\frac{1}{i}\Big)^2}^{h^2}+\boxed{\sum_{i=1}^{l}\frac{1}{i^2}}}{l}+\overbrace{\sum_{l=1}^{r}\frac{\displaystyle\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{1}{2}(h^2+h_2)}}{r}$$

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Sigma computes the Σ^{δ}-extension h_3 over $(\mathbb{Q}(x)(h)(h_2),\sigma)$ with

$$\sigma(h_3)=h_3+\frac{1}{(x+1)^3},\quad \delta(h_3)=2$$

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a

$$A(k)=\boxed{\sum_{r=1}^{k}\frac{\overbrace{\sum_{l=1}^{r}\frac{\left(\sum_{i=1}^{l}\frac{1}{i}\right)^2+\sum_{i=1}^{l}\frac{1}{i^2}}{l}+\sum_{l=1}^{r}\frac{\sum_{i=1}^{l}\frac{1}{i}}{l}}^{\frac{1}{3}(h^3+3hh_2+2h_3)+\frac{1}{2}(h^2+h_2)}}{r}}$$

5 Finally: Sigma finds the Σ^{δ}-extension h_4 over $(\mathbb{Q}(x)(h)(h_2)(h_3),\sigma)$ with

$$\sigma(h_4)=h_4+\tfrac{1}{(x+1)^4},\quad \delta(h_4)=2$$

and represents $A(k)$ by

$$a=\tfrac{1}{12}(h^4+2h^3+6(h+1)h_2h+3h_2^2+(8h+4)h_3+6h_4)$$

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SUMMARY $A(k)$ is represented in the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(x)(h)(h_2)(h_3)(h_4),\sigma)$:

$$\begin{aligned}
\forall k\in\mathbb{Q}\quad \sigma(k)&=k & \forall k\in\mathbb{Q}\quad \delta(k)&=0\\
\sigma(x)&=x+1 & \delta(x)&=1\\
\sigma(h)&=h+\tfrac{1}{x+1} & \delta(h)&=2\\
\sigma(h_2)&=h_2+\tfrac{1}{(x+1)^2} & \delta(h_2)&=2\\
\sigma(h_3)&=h_3+\tfrac{1}{(x+1)^3} & \delta(h_3)&=2\\
\sigma(h_4)&=h_4+\tfrac{1}{(x+1)^4} & \delta(h_4)&=2
\end{aligned}$$

$$A(k) = \frac{1}{12}(h^4 + 2h^3 + 6(h+1)h_2h + 3h_2^2 + (8h+4)h_3 + 6h_4)$$

$$\boxed{\sum_{r=1}^{k} \frac{\displaystyle\sum_{l=1}^{r} \frac{\left(\sum_{i=1}^{l} \frac{1}{i}\right)^2 + \sum_{i=1}^{l} \frac{1}{i^2}}{l} + \sum_{l=1}^{r} \frac{\sum_{i=1}^{l} \frac{1}{i}}{l}}{r}}$$

$$||$$

$$\mathrm{ev}(B,k)$$

with

$$B = \frac{1}{12}\Big(\mathsf{Sum}(1,\tfrac{1}{x})^4 + 2\mathsf{Sum}(1,\tfrac{1}{x})^3 + 6(\mathsf{Sum}(1,\tfrac{1}{x}) + 1)\mathsf{Sum}(1,\tfrac{1}{x^2})\mathsf{Sum}(1,\tfrac{1}{x}) + 3\mathsf{Sum}(1,\tfrac{1}{x^2})^2 + (8\mathsf{Sum}(1,\tfrac{1}{x}) + 4)\mathsf{Sum}(1,\tfrac{1}{x^3}) + 6\mathsf{Sum}(1,\tfrac{1}{x^4})\Big)$$

B has the **optimal depth** 2

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    - $\dots$

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Problem: Depth Optimal Simplification.

Given $A \in \mathsf{Sum}(X)$;
find $B \in \mathsf{Sum}(X)$ and $\lambda \in \mathbb{N}$ such that

$$\mathrm{ev}(A,k) = \mathrm{ev}(B,k) \text{ for all } k \geq \lambda$$

and such that $\mathfrak{d}(B)$ is the optimal depth of A.

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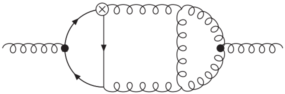
Automatization

Example: All n-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)
(Nuclear Physics B, 2012; arXiv:1206.2252v1)

In total around 50 diagrams (for this class) have been calculated, like e.g.



(containing three massive fermion propagators)

$\Downarrow$

Around 1000 sums have to be calculated for this diagram

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\|$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)$$

$$||$$

$$\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

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$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$

$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$ Example.

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

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$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$
$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(n)=\sum_{i=1}^{n}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

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$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$

$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;n)+S_{2,1,1,1}(1,\frac{1}{2},1,2;n)$$

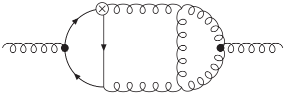
$$+\dots$$

where, e.g.,

145 S–sums occur

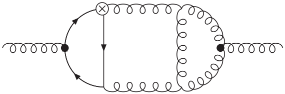
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;n)=\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



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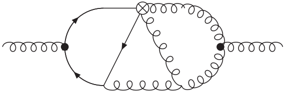
Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(n), S_{-3}(n), S_{-2}(n), S_1(n), S_2(n), S_3(n), S_4(n), S_{-3,1}(n),$$
$$S_{-2,1}(n), S_{2,-2}(n), S_{2,1}(n), S_{3,1}(n), S_{-2,1,1}(n), S_{2,1,1}(n)$$

So far, the most complicated 3–loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

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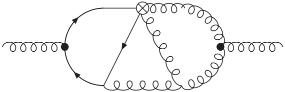

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

$$\|$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-\left(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s)\right)$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\big)S_1(n)^2$$
$$+ \big(-\frac{4(13n+5)}{n^2(n+1)^2} + \big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\big)S_2(n) + \big(\frac{29}{3} - (-1)^n\big)S_3(n)$$
$$+ \big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\big)S_1(n) + \big(\frac{3}{4} + (-1)^n\big)S_2(n)^2$$
$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\big)$$
$$+ \big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\big)S_2(n) + S_{-2}(n)\big(10S_1(n)^2 + \big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+ \frac{4(3n-1)}{n(n+1)}\big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\big)$$
$$+ \big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\big)S_3(n) + \big(\frac{19}{2} - 2(-1)^n\big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+ \big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$- \frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

New Strategies

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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multivariate
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n + d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
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ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
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Holonomic/difference field Approach
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$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad= h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

given

Ansatz (for power series)

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&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

If $F_0(n)$ (with required initial values) is not expressible in terms of indefinite nested sums and products: **game over**

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+& a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+& \\
& \vdots\\
+& a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
& \qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_d(0,n)F_0(n+d)=h_0(n)}
$$

$$
\begin{aligned}
&\quad a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=h'_0(n)+h'_1(n)\varepsilon+h'_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad = \underbrace{h_0'(n)}_{=\mathbf{0}}+h_1'(n)\varepsilon+h_2'(n)\varepsilon^2+\dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h'_1(n)+h'_2(n)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(n), F_2(n), \dots$ Example

Remark: Works the same for Laurent series.

(see J. Blümlein, S. Klein, CS, F. Stan. J. Symbolic Comput. 47, 2012; arXiv:1011.2656v2
J. Ablinger, J. Blümlein, M. Round, CS, LL2012, 2012; arXiv:???)