Generalization of Risch's Algorithm to Special Functions

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Theory Seminar
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Indefinite integration

Antiderivatives

$$\int f(x)\,dx = g(x)$$

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Examples

$$\int \frac{\mathrm{Li}_3(x) - x\mathrm{Li}_2(x)}{(1-x)^2}dx = \frac{x}{1-x}(\mathrm{Li}_3(x) - \mathrm{Li}_2(x)) + \frac{\ln(1-x)^2}{2}$$

$$\int \mathrm{Ai}'(x)^2\,dx = \tfrac{1}{3}\left(x\mathrm{Ai}'(x)^2 + 2\mathrm{Ai}(x)\mathrm{Ai}'(x) - x^2\mathrm{Ai}(x)^2\right)$$

$$\int \frac{1}{xJ_n(x)Y_n(x)}\,dx = \frac{\pi}{2}\ln\left(\frac{Y_n(x)}{J_n(x)}\right)$$

Definite integration

Integrals depending on parameters

$$\int_a^b f(\vec{y}, x)\, dx = g(\vec{y})$$

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Examples

$$\begin{aligned}
\int_0^\infty \frac{zx}{e^x - z}\,dx &= \mathrm{Li}_2(z)\\
\int_0^\infty e^{-sx}\gamma(a,x)\,dx &= \frac{\Gamma(a)}{s(s+1)^a}\\
\int_0^1 e^{-2n\pi ix}\ln(\sin(\tfrac{\pi}{2}x))\,dx &= -\frac{1}{4n} + \frac{i}{n\pi}\sum_{k=1}^n \frac{1}{2k-1}
\end{aligned}$$

Example: recurrence for gamma function

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In other words, we proved

$$z\Gamma(z) - \Gamma(z+1) = 0$$

Parametric integration

Compute linear relation of integrals

Given $f\ (x)$, find $g(x)$ s.t.

$$f\ (x) \qquad = g'(x)$$

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Given $f_0(x), \dots, f_m(x)$, find $g(x)$ s.t.

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Given $f_0(x), \ldots, f_m(x)$, find $g(x)$ and $c_0, \ldots, c_m$ const. w.r.t. x s.t.

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Transfer this to a relation of corresponding integrals

$$c_0 \int_a^b f_0(x)dx + \cdots + c_m \int_a^b f_m(x)dx = g(b) - g(a)$$

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Choose the f_i

- For obtaining an ODE for $I(y) := \int_a^b f(y,x)\,dx$ compute

$$c_0(y)f(y,x) + \cdots + c_m(y)\frac{\partial^m f}{\partial y^m}(y,x) = \frac{d}{dx}g(y,x)$$

- For obtaining a recurrence for $I(n) := \int_a^b f(n,x)\,dx$ compute

$$c_0(n)f(n,x) + \cdots + c_m(n)f(n+m,x) = \frac{d}{dx}g(n,x)$$

Differential algebra

Differential field

(F, D) such that for any $f, g \in F$

$$D(f+g) = Df + Dg \quad \text{and} \quad D(fg) = (Df)g + f(Dg)$$

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Examples

$$(\mathbb{Q}(x), \tfrac{d}{dx}) \qquad (\mathbb{Q}(e^x), \tfrac{d}{dx}) \qquad (\mathbb{R}(n, x, x^n, \ln(x)), \tfrac{d}{dx})$$

$$(\mathbb{C}(n, x, J_n(x), J_{n+1}(x), Y_n(x), Y_{n+1}(x)), \tfrac{d}{dx})$$

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NB

$$f, g \in F \quad \Rightarrow \quad f+g, f \cdot g, \tfrac{f}{g}, Df \in F,$$

but f^g, $f \circ g$, and $\int f$ in general are not in F

Monomial extensions

Differential field extensions

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- $Dt_i \in F(t_1, \ldots, t_n)$ and
- D can be extended consistently to $F(t_1, \ldots, t_n)$.

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Definition

t is a monomial over (F, D) if

- t is transcendental over F and
- Dt is a polynomial in t with coefficients from F

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Towers of monomial extensions

We consider differential fields $(C(t_1, \dots, t_n), D)$ such that each t_i is a monomial over $(C(t_1, \dots, t_{i-1}), D)$.

Elementary integrals

Elementary extension

Any (E, D) generated from (F, D) by adjoining

- algebraics: $y(x)^m + a_{m-1}(x)y(x)^{m-1} + \cdots + a_0(x) = 0$
- logarithms: $y(x) = \log(a(x))$
- exponentials: $y(x) = \exp(a(x))$

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Definition

We say that $f \in F$ has an elementary integral over (F, D) if there exists an elementary extension (E, D) of (F, D) and $g \in E$ s.t.

$$Dg = f$$

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$$c_0 f_0 + \cdots + c_m f_m$$

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Risch's algorithm

- Risch 1969, Mack 1976: for regular elementary (F, D)
- Singer et al. 1985: for regular Liouvillian (F, D)
- Bronstein 1990, 1997: partial generalizations to (F, D) a tower of monomial extensions

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- Risch 1969, Mack 1976: for regular elementary (F, D)
- Singer et al. 1985: for regular Liouvillian (F, D)
- Bronstein 1990, 1997: partial generalizations to (F, D) a tower of monomial extensions
- CGR 2012: for (F, D) a tower of monomial extensions subject to some technical conditions

Liouvillian functions

Characterization

Functions constructed from rational functions by

- basic arithmetic operations $+, -, *, /$
- taking solutions of algebraic equations
  $y(x)^m + a_{m-1}(x)y(x)^{m-1} + \cdots + a_0(x) = 0$
- taking antiderivatives $y'(x) = a(x)$
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Examples

elementary functions, exponential integrals, polylogarithms, error functions, Fresnel integrals, incomplete gamma function, ...

$$\mathrm{Ei}(2\ln(x)) \qquad \mathrm{Li}_2(e^x) \qquad e^{-x^2}\left(\tfrac{\pi}{2}\mathrm{erfi}(x) - \tfrac{1}{2}\mathrm{Ei}(x^2)\right)$$

$$\int_{-\infty}^{x} \cos(\tfrac{\pi}{2}u^2)(C(u) + \tfrac{1}{2})(S(u) - \tfrac{1}{2})\, du$$

Generalization to 2-dim systems

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- taking solutions of 2-dimensional differential systems

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix}$$

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Examples

Liouvillian functions, orthogonal polynomials, associated Legendre functions, complete elliptic integrals, Airy/Scorer functions, Bessel/Struve/Anger/Weber/Lommel/Kelvin functions, Whittaker functions, hypergeometric functions, Heun functions, Mathieu functions, ...

Generalization to 2-dim systems (cont.)

Examples

- Bessel functions

$$\begin{pmatrix} J_n(x) \\ J_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & -1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} J_n(x) \\ J_{n+1}(x) \end{pmatrix}$$

- Legendre polynomials

$$\begin{pmatrix} P_n(x) \\ P_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{(n+1)x}{1-x^2} & -\frac{n+1}{1-x^2} \\ \frac{n+1}{1-x^2} & -\frac{(n+1)x}{1-x^2} \end{pmatrix} \begin{pmatrix} P_n(x) \\ P_{n+1}(x) \end{pmatrix}$$

- Complete elliptic integrals

$$\begin{pmatrix} K(x) \\ E(x) \end{pmatrix}' = \begin{pmatrix} -\frac{1}{x} & \frac{1}{x(1-x^2)} \\ -\frac{1}{x} & \frac{1}{x} \end{pmatrix} \begin{pmatrix} K(x) \\ E(x) \end{pmatrix}$$

Algebraic representation

Special case: $a_{12} \neq 0,\ b_1 = b_2 = 0$

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Riccati-type equation

with $v(x) := \frac{y_2(x)}{y_1(x)}$ the system can be rewritten as

$$v'(x) \quad = \quad -a_{12}(x)v(x)^2 + (a_{22}(x) - a_{11}(x))v(x) + a_{21}(x)$$

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Monomial extensions

For $a_{11}, a_{12}, a_{21}, a_{22} \in K$ construct $(K(t_1, t_2), D)$ with

$$\begin{aligned} Dt_1 &= -a_{12}t_1^2 + (a_{22} - a_{11})t_1 + a_{21} \quad && \in K[t_1] \\ Dt_2 &= (a_{12}t_1 + a_{11})t_2 && \in K(t_1)[t_2] \end{aligned}$$

Algebraic representation (cont.)

Fundamental matrix

$$\Phi(x)' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \Phi(x) \quad \text{where} \quad \Phi(x) = \begin{pmatrix} y_1(x) & \tilde{y}_1(x) \\ y_2(x) & \tilde{y}_2(x) \end{pmatrix}$$

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Liouville's formula

with $w(x) = \det \Phi(x)$ and $\tilde{v}(x) = \frac{\tilde{y}_1(x)}{y_1(x)}$ we have

$$\begin{aligned} w'(x) &= (a_{11}(x) + a_{22}(x))w(x) \\ \tilde{v}'(x) &= \frac{a_{12}(x)w(x)}{y_1(x)^2} \end{aligned}$$

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Monomial extensions

Extend $(K(t_1, t_2), D)$ to $(K(t_1, t_2, t_3, t_4), D)$ with

$$\begin{aligned} Dt_3 &= (a_{11} + a_{22})t_3 & \in K(t_1, t_2)[t_3] \\ Dt_4 &= \frac{a_{12}t_3}{t_2^2} & \in K(t_1, t_2, t_3)[t_4] \end{aligned}$$

Example: Bessel functions

Coupled system of two ODEs

$$\begin{pmatrix} J_n(x) \\ J_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & -1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} J_n(x) \\ J_{n+1}(x) \end{pmatrix}$$

Differential field

- Start from $(K, D) = (C(x), \frac{d}{dx})$ with $C = \mathbb{Q}(n)$

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Differential field

- Start from $(K, D) = (C(x), \frac{d}{dx})$ with $C = \mathbb{Q}(n)$
- Construct extension $(K(t_1, t_2), D)$ with

$$\begin{aligned} Dt_1 &= t_1^2 - \tfrac{2n+1}{x} t_1 + 1 \\ Dt_2 &= (-t_1 + \tfrac{n}{x}) t_2 \end{aligned}$$

- Then t_1 represents $\frac{J_{n+1}(x)}{J_n(x)}$ and t_2 represents $J_n(x)$

Example: Bessel functions (cont.)

Include second solution

$$\begin{pmatrix} Y_n(x) \\ Y_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & -1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} Y_n(x) \\ Y_{n+1}(x) \end{pmatrix}$$

$$\det \begin{pmatrix} J_n(x) & Y_n(x) \\ J_{n+1}(x) & Y_{n+1}(x) \end{pmatrix} = -\frac{2}{\pi x}$$

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Differential field

- Use $C = \mathbb{Q}(\pi, n)$ instead
- Extend to $(K(t_1, t_2, t_3), D)$ with Dt_1, Dt_2 as before and

$$Dt_3 = \frac{2}{\pi x t_2^2}$$

- Then t_3 represents $\frac{Y_n(x)}{J_n(x)}$

Admissible differential fields

Definition

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1. t_i is a Liouvillian monomial over F_{i-1}, i.e., either
    1. $Dt_i \in F_{i-1}$ (primitive), or
    2. $\frac{Dt_i}{t_i} \in F_{i-1}$ (hyperexponential); or

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    2. $\frac{Dt_i}{t_i} \in F_{i-1}$ (hyperexponential); or
2. there is a $q \in F_{i-1}[t_i]$ with $\deg(q) \geq 2$ such that
    1. $Dt_i = q(t_i)$ and
    2. $Dy = q(y)$ does not have a solution $y \in \overline{F_{i-1}}$.

Decision procedure

Recall

- Given a differential field (F, D) and $f_0, \ldots, f_m \in F$
- Find all $c_0, \ldots, c_m \in \mathrm{Const}(F)$ s.t.

$$c_0 f_0 + \cdots + c_m f_m = Dg$$

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4. proceed recursively with the smaller tower

Decision procedure (cont.)

At each level

1. Hermite Reduction for reducing denominator

$$\int \frac{a}{u \cdot v^m} = \int \frac{b \cdot (1-m) Dv}{v^m} + \int \frac{c}{u \cdot v^{m-1}} = \frac{b}{v^{m-1}} + \int \frac{c - uDb}{u \cdot v^{m-1}}$$

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$$\sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \log(S_i(\alpha, t_n))$$

where $Q_i \in C[z]$ and $S_i \in K[z, t_n]$

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4. remaining integrands are from K, reduce elementary integration over $K(t_n)$ to elementary integration over K

Example: Gradshteyn & Ryzhik 6.539.3

$$\int \frac{1}{xJ_n(x)Y_n(x)}\,dx =$$

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$$\int \frac{1}{t_1 t_3^2 t_4} = \frac{\pi}{2}\ln(t_4)$$ by Residue criterion

Examples: indefinite integrals

- Using $F = \mathbb{Q}(x, \ln(1-x), \mathrm{Li}_2(x), \mathrm{Li}_3(x))$ we find

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$$f(n+1,x) - \tfrac{n}{n+1} f(n,x) = \\ \tfrac{d}{dx} \tfrac{e^{-2(n+1)\pi ix}}{2(n+1)\pi i} \left(\tfrac{1}{4(n+1)} + \tfrac{e^{\pi ix}}{2n+1} + \tfrac{e^{2\pi ix}}{4n} + (e^{2\pi ix}-1) \ln(\sin(\tfrac{\pi}{2}x)) \right)$$

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Solution:

$$I(n) = -\frac{1}{4n} + \frac{i}{n\pi} \sum_{k=1}^{n} \frac{1}{2k-1}$$

Example: Binet-like integrals

$$B_n(\sigma) := \int_0^1 \left(\frac{1}{\ln(x)} + \frac{1}{1-x}\right)^n x^\sigma \, dx \quad \text{for } n \in \mathbb{N}, \sigma > -1$$

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Some additional calculations show

$$\begin{aligned} B_n(\sigma) &= \int_\sigma^\infty \frac{s-n+2}{n-1} B'_{n-1}(s) - \frac{2n-3}{n-1} B_{n-1}(s) + B_{n-2}(s) \, ds \\ B_0(\sigma) &= \frac{1}{\sigma+1} \\ B_1(\sigma) &= \ln(\sigma+1) - \psi(\sigma+1) \end{aligned}$$

Example: connection coefficients

$$c_{m,n} = \int_{-1}^{1} C_m^{\mu}(x) C_n^{\nu}(x)(1-x^2)^{\nu-\frac{1}{2}}\,dx \quad \text{for } m, m \in \mathbb{N},\ \mu, \nu > -\frac{1}{2}$$

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Solution:

$$c_{m,n} = \begin{cases} B(\frac{1}{2}, \nu+\frac{1}{2}) \frac{(\mu)_k (\mu-\nu)_{k-n} (2\nu)_n}{n!(k-n)!(\nu+1)_k} & \text{if } m+n = 2k \\ 0 & \text{if } m+n = 2k+1 \end{cases}$$