A new determination of $\alpha_s$ from $\tau$ decays

Matthias Jamin
For 0.6% precision at $M_Z$ need “only” $\approx 2\%$ at $M_\tau$.
Consider the physical quantity $R_{\tau}$: (Braaten, Narison, Pich 1992)

$$R_{\tau} \equiv \frac{\Gamma(\tau^- \rightarrow \text{hadrons} \, \nu_\tau(\gamma))}{\Gamma(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau(\gamma))} = 3.6280(94) . \text{ (HFAG 2012)}$$

$R_{\tau}$ is related to the QCD correlators $\Pi^{(1,0)}(z)$: ($z \equiv s/M_T^2$)

$$R_{\tau} = 12\pi \int_0^1 dz (1-z)^2 \left[ (1+2z) \text{Im} \Pi^{(1)}(z) + \text{Im} \Pi^{(0)}(z) \right],$$

with the appropriate combinations

$$\Pi^{(J)}(z) = |V_{ud}|^2 \left[ \Pi_{ud}^{V,J} + \Pi_{ud}^{A,J} \right] + |V_{us}|^2 \left[ \Pi_{us}^{V,J} + \Pi_{us}^{A,J} \right].$$
Additional experimental information can be inferred from the moments

\[ R_w^\tau \equiv \frac{1}{0} \int d\tau \ w(\tau) \frac{dR_{\tau}}{d\tau} = R_{\tau,V}^w + R_{\tau,A}^w + R_{\tau,S}^w. \]

Theoretically, \( R_w^\tau \) can be expressed as:

\[ R_w^\tau = N_c S_{EW} \left\{ \left( |V_{ud}|^2 + |V_{us}|^2 \right) \left[ 1 + \delta w^{(0)} \right] \right. \]

\[ + \sum_{D \geq 2} \left[ |V_{ud}|^2 \delta_{ud}^{(D)} + |V_{us}|^2 \delta_{us}^{(D)} \right] \right\}. \]

\( \delta_{ud}^{(D)} \) and \( \delta_{us}^{(D)} \) are corrections in the Operator Product Expansion, the most important ones being \( \sim m_s^2 \) and \( m_s \langle \bar{q}q \rangle \).
OPAL data can be updated with new branching fractions.

**ALEPH data currently miss correlations from unfolding.**

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The perturbative part $\delta^{(0)}$ is related to the Adler function $D(s)$:

$$
D(s) \equiv -s \frac{d}{ds} \Pi_{V}(s) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_{\mu}^n \sum_{k=1}^{n+1} k c_{n,k} \ln^{k-1}\left(\frac{-s}{\mu^2}\right)
$$

where $a_{\mu} \equiv \alpha_s(\mu)/\pi$.

Resumming the Log’s with the scale choice $\mu^2 = -s \equiv Q^2$:

$$
D(Q^2) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} c_{n,1} a^n(Q^2)
$$

As a consequence, only the coefficients $c_{n,1}$ are independent:

$$
c_{0,1} = c_{11} = 1, \quad c_{2,1} = 1.640, \quad c_{3,1} = 6.371,
$$

$$
c_{4,1} = 49.076!! \quad \text{(Baikov, Chetyrkin, Kühn 2008)}
$$
Fixed order perturbation theory amounts to choose $\mu^2 = M^2_\tau$:

$$
\delta^{(0)}_{\text{FO}} = \sum_{n=1}^{\infty} a^n (M^2_\tau) \sum_{k=1}^{n+1} k c_{n,k} J_{k-1} = \sum_{n=1}^{\infty} [c_{n,1} + g_n] a^n (M^2_\tau)
$$

A given perturbative order $n$ depends on all coefficients $c_{m,1}$ with $m \leq n$, and on the coefficients of the QCD $\beta$-function.

Contour improved perturbation theory employs $\mu^2 = -M^2_\tau x$:

(Pivovarov; Le Diberder, Pich 1992)

$$
\delta^{(0)}_{\text{CI}} = \sum_{n=1}^{\infty} c_{n,1} J_n^a (M^2_\tau)
$$

with

$$
J_n^a (M^2_\tau) = \frac{1}{2\pi i} \oint dx \frac{a^n}{x} (1-x)^3 (1+x)
$$

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DESY, Zeuthen 2012
The purely perturbative contribution $\delta^{(0)}$ is plagued by differences for different RG-resummations. (FOPT vs CIPT.)

Using $\alpha_s(M_\tau) = 0.3186$, the numerical analysis results in:

$$\delta^{(0)}_{FO} = 0.101 + 0.054 + 0.027 + 0.013 (+0.006) = 0.196 (0.202)$$

$$\delta^{(0)}_{CI} = 0.137 + 0.026 + 0.010 + 0.007 (+0.003) = 0.181 (0.185)$$

Contour improved PT appears to be better convergent.

The difference between both approaches is $0.015 (0.017)$!

This problematic entails a $\approx 6\%$ difference for $\alpha_s(M_\tau)$. 
To further investigate the difference between CI and FOPT, let us consider the Borel-transformed Adler function.

\[ 4\pi^2 D(s) \equiv 1 + \hat{D}(s) \equiv 1 + \sum_{n=0}^{\infty} r_n \alpha_s (s)^{n+1}, \]

where \( r_n = c_{n+1,1}/\pi^{n+1} \). The Borel-transform reads:

\[ \hat{D}(\alpha_s) = \int_0^\infty dt \, e^{-t/\alpha_s} B[\hat{D}](t); \quad B[\hat{D}](t) = \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}. \]

Generally, the Borel-transform \( B[\hat{D}] \) develops poles and cuts at integer values \( p \) of \( u \equiv \beta_1 t/(2\pi) \). (Except at \( u=1 \).)

The poles at negative \( p \) are called UV renormalon poles and the ones at positive \( p \) IR renormalons.
To proceed, realistic model $B[\hat{D}](u)$: (Beneke, MJ 2008)

$$B[\hat{D}](u) = B[\hat{D}_1^{\text{UV}}](u) + B[\hat{D}_2^\text{IR}](u) + B[\hat{D}_3^\text{IR}](u) + d_0^\text{PO} + d_1^\text{PO} \, u,$$

where

$$B[\hat{D}_p](u) = \frac{d_p}{(p\pm u)^{1+\gamma}} \left[ 1 + b_1(p\pm u) + b_2(p\pm u)^2 \right].$$

☞ Our main model incorporates the leading UV pole ($u = -1$), as well as the two leading IR renormalons ($u = 2, 3$).

☞ It should reproduce the exactly known $c_{n,1}$, $n \leq 4$.

☞ For both UV and IR, the residues $d_p$ are free while $\gamma$, $b_{1,2}$ depend on anomalous dimensions and $\beta$-coefficients.
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$c_{5,1} = 283$, \quad $\alpha_s(M_\tau) = 0.3186$.  (Beneke, MJ 2008)
In the OPE, close to the Minkowskian axis ($s > 0$), so-called Duality Violations (DV’s) can appear.

They can be studied on the basis of a toy-model:

\[ \Pi_V(s) = -\psi\left(\frac{M_V^2 + u(s)}{\Lambda^2}\right) + \text{const.} \]

where

\[ u(s) = \Lambda^2 \left(\frac{-s}{\Lambda^2}\right) \zeta \quad \text{and} \quad \zeta = 1 - \frac{a}{\pi N_c}. \]

The model is based on large-$N_c$ QCD and Regge-theory.

\[ M_V = 770 \text{ MeV}, \quad \Lambda = 1.2 \text{ GeV}, \quad a = 0.4. \]

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The OPE corresponds to the asymptotic expansion of the $\psi$-function for large $s$ (large $u$).

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n z^{2n}}, \quad \text{Re}z > 0.$$ 

In the Minkowskian region, an additional term arises:

$$- \pi \left[ \cot (\pi z) \pm i \right], \quad \text{Re}z < 0, \text{Im}z \geq 0.$$ 

Formally, this term is exponentially suppressed, but it is enhanced by the poles of the $\psi$-function.
\[ z = 1.5 \cdot \exp(i\varphi) \]
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$\psi$-function moment for $w(z) = 1$. 
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In fits to experimental data, a model for DV’s should be included.

The $\psi$-function model suggests an oscillating, decaying exponential, which can be chosen of the form:

$$\rho_{DV/V/A}^V(s) = \kappa_{V/A} e^{\gamma_{V/A}s} \sin \left( \alpha_{V/A} + \beta_{V/A}s \right).$$

The fit quantities are the $w$-moments of the exp spectra.

$$R_{\tau,V/A}^w(s_0) \equiv \int_{0}^{s_0} ds \ w(s) \ \rho_{V/A}^V(s).$$

The cleanest moment turns out to be $w(s) = 1$.

Fitting combinations of several moments is complicated by very strong correlations.

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A new determination of $\alpha_s$ from $\tau'$s

(Boito, Golterman, MJ, Mahdavi, Maltman, Osborne, Peris 2012)
(Boito, Golterman, MJ, Mahdavi, Maltman, Osborne, Peris 2012)
Presently, the most reliable value of $\alpha_s$ from $\tau$’s including DV’s comes from the trivial moment $w(s) = 1$.

$\Rightarrow \alpha_s(M_\tau) = 0.325 \pm 0.016 \pm 0.007$ (FOPT)

$\Rightarrow \alpha_s(M_\tau) = 0.347 \pm 0.024 \pm 0.005$ (CIPT)

These values should be compared to the World Average (Bethke 2009): $\alpha_s(M_\tau) = 0.3186 \pm 0.0058$.

Better data on exclusive and inclusive $\tau$ decay spectra would be very helpful to resolve theoretical issues.
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Thank You for Your attention!

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