Quasi-Shuffle Algebras

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Outline

The Challenge

It is as if QFT were taunting us with our ignorance of the mapping between diagrams and numbers that results from the Feynman rules. A vast quantity of data, collected by painfully inadequate methods, reduces to an amazingly simple answer. We are, physicists and mathematicians alike, stumbling on the edge of a structure that is far more refined than the clumsy methods by which we investigate it.
– David Broadhurst, "Where do the tedious products of ζ's come from?", Nuclear Phys. B (Proc. Suppl.) **116** (2003).

Introduction

Quasi-shuffle products have proven useful in understanding

- the multiple zeta values (MZVs)

$$\zeta(i_1,\dots,i_k)=\sum_{n_1>\cdots>n_k\ge 1}\frac{1}{n_1^{i_1}\cdots n_k^{i_k}},\ \ i_1>1;$$

- the finite multiple harmonic sums

$$S(N;i_1,\dots,i_k)=\sum_{N\ge n_1\ge\cdots\ge n_k\ge 1}\frac{1}{n_1^{i_1}\cdots n_k^{i_k}};$$

- and multiple polylogarithms

$$Li_{i_1,\dots,i_k}(x_1,\dots,x_k)=\sum_{n_1>\cdots>n_k\ge 1}\frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1^{i_1}\cdots n_k^{i_k}},\ \ i_1x_1\neq 1,$$

where, say, the x_i are all rth roots of unity for some r.

Introduction Cont'd

The idea behind the quasi-shuffle algebra is to encode the properties of multiplication of MZVs, sums, polylogarithms, etc. in an appropriate commutative product on the underlying vector space of the free noncommutative polynomial algebra $k\langle A\rangle$, for some appropriate set A of "letters". I constructed such products in [J. Alg. Combin. 2000]. Recently K. Ihara, J. Kajikawa, Y. Ohno, and J. Okuda [J. Algebra 2011] gave a more general version of this construction which is more widely applicable. Although their generalization was not motivated by physics, this more general setting removes significant restrictions of the original construction with no apparent loss in effectiveness.

Introduction Cont'd

Specifically, we start with a set A and a commutative product $\diamond$ on the vector space kA with A as basis. Then we define the product $*$ on $k\langle A\rangle$ by the inductive formula

$$au * bv = a(u * bv) + b(au * v) + (a \diamond b)(u * v), \qquad (1)$$

and this turns out to be commutative as well. Corresponding to the MZVs, we take $A = \{z_1, z_2, \dots\}$ and define $z_i \diamond z_j = z_{i+j}$. Equation (1) then implies, e.g., that

$$z_i * z_j z_k = z_i z_j z_k + z_j z_i z_k + z_j z_k z_i + z_{i+j} z_k + z_j z_{i+k},$$

which the homomorphism $z_{i_1} \cdots z_{i_k} \to \zeta(i_1, \dots, i_k)$ maps to

$$\zeta(i)\zeta(j,k) = \zeta(i,j,k) + \zeta(j,i,k) + \zeta(j,k,i) + \zeta(i+j,k) + \zeta(j,i+k).$$

Introduction Cont'd

In the case $A = \{z_1, z_2, \dots\}$, $z_i \diamond z_j = z_{i+j}$, the quasi-shuffle algebra $(k\langle A\rangle, *)$ turns out to be the algebra QSym of quasisymmetric functions, which contains the very well-known subalgebra Sym of symmetric functions. This means that relations in Sym give rise to relations of MZVs. For example, since

$$m_{2,2,2} = \frac{1}{6}(p_2^3 - 3p_2p_4 + 2p_6)$$

in Sym (where m_λ is a monomial symmetric function and the p_i are power sums), it follows that

$$\zeta(2,2,2) = \frac{1}{6}(\zeta(2)^3 - \zeta(2)\zeta(4) + 2\zeta(6)).$$

Introduction Cont'd

My original construction required that $a \diamond b$ be a letter for $a, b \in A$, and that A have a grading preserved by $\diamond$. The Ihara-Kajikawa-Ohno-Okuda paper (henceforth IKOO) removed these restrictions, but did not use some important features of my original construction, particularly the representation of formal power series by linear maps, and the coalgebra structure. Fortunately Ihara and I were both visiting the Max Planck Institut für Mathematik in Bonn at the beginning of this year, and we began collaborating on extending these features to the more general setting. Today I will be presenting some results from our joint paper "Quasi-shuffle products revisited", available online as MPIM preprint 2012-16.

Shuffle Products

Let k be a field, A an "alphabet", and $k\langle A\rangle$ be the k-vector space of noncommutative monomials in elements of A. A basis for $k\langle A\rangle$ is the set of "words" (monomials) in elements of A, including the empty word 1. Define $\ell(w)$ to be the number of letters in the word w.

Besides the noncommutative product given by juxtaposition of words, $k\langle A\rangle$ admits the commutative shuffle product $\sqcup\!\sqcup$: for example, if $a, b, c \in A$, then

$$a \sqcup\!\sqcup bc = abc + bac + bca.$$

More precisely, the shuffle product can be defined by $v \sqcup\!\sqcup 1 = 1 \sqcup\!\sqcup v = v$ for all words w, together with the inductive rule

$$av \sqcup\!\sqcup bw = a(v \sqcup\!\sqcup bw) + b(av \sqcup\!\sqcup w)$$

for $a, b \in A$ and words v, w.

Quasi-Shuffle Products

Now suppose there is a product $\diamond$ on kA (the vector space with A as basis) which is commutative and associative. We define the quasi-shuffle product $*$ by $v * 1 = 1 * v = 1$ for any word v, and

$$av * bw = a(v * bw) + b(av * w) + (a \diamond b)(v * w)$$

for $a, b \in A$ and words v, w. By induction on $\ell(v) + \ell(w)$ one can show that the product so defined is commutative and associative.
We shall denote by $\star$ the quasi-shuffle product obtained by replacing $\diamond$ with its negative $\bar{\diamond}$ (i.e., $a \bar{\diamond} b = -a \diamond b$). It is often useful to treat $*$ and $\star$ in parallel.

Quasi-Shuffle Products Cont'd

For example, if $a, b, c, d \in A$,

$$a * b = ab + ba + a \diamond b$$
$$a \star b = ab + ba - a \diamond b$$

$$a * bc = abc + bac + bca + a \diamond bc + ab \diamond c$$
$$a \star bc = abc + bac + bca - a \diamond bc - ab \diamond c$$

$$ab \star cd = abcd + acbd + acdb + cabd + cadb + cdab - a \diamond cbd - a \diamond cdb - ca \diamond db - acb \diamond d - cab \diamond d - ab \diamond cd + a \diamond cb \diamond d,$$

and $ab * cd$ looks the same without minus signs.

Quasi-Shuffle Products Cont'd

To summarize, we have at this point four products on the vector space $k\langle A\rangle$:

1. The original noncommutative product (juxtaposition);
2. The commutative shuffle product $ш$;
3. The commutative quasi-shuffle product $*$; and
4. The commutative quasi-shuffle product $\star$.

Examples

We have already seen that MZVs can be described by taking $A = \{z_1, z_2, \dots\}$ and $z_i \diamond z_j = z_{i+j}$, with the homomorphism $z_{i_1} \cdots z_{i_k} \to \zeta(i_1, \dots, i_k)$. Actually, since $\zeta(i_1, \dots, i_k)$ doesn't converge unless $i_1 > 1$, this is a homomorphism from $(\mathfrak{H}^0, *)$ to $\mathbb{R}$, where $\mathfrak{H}^0 \subset k\langle A\rangle$ is the subalgebra consisiting of those monomials that don't start with z_1.
For the finite sums

$$S(N; i_1, \dots, i_k) = \sum_{N \ge n_1 \ge \dots \ge n_k \ge 1} \frac{1}{n_1 \cdots n_k}$$

we can use the same alphabet A: this time we have a homomorphism $z_{i_1} \cdots z_{i_k} \to S(N; i_1, \dots, i_k)$ that goes from $(k\langle A\rangle, \star)$ to $\mathbb{R}$.

Examples Cont'd

For multiple polylogarithms

$$Li_{i_1,\dots,i_k}(x_1,\dots,x_k) = \sum_{n_1>\cdots>n_k\geq 1} \frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1^{i_1}\cdots n_k^{i_k}},$$

where the x_i are in some set P closed under multiplication, we have to expand the alphabet to $A=\{z_{i,u}|i=1,2,\dots,\ u\in P\}$. We put $z_{i_1,u_1}\diamond z_{i_2,u_2}=z_{i_1+i_2,u_1u_2}$. Then there is a homomorphism $(\mathfrak{E}^0,*)\to\mathbb{C}$ given by

$$z_{i_1,u_1}\cdots z_{i_k,u_k}\to Li_{i_1,\dots,i_k}(u_1,\dots,u_k)$$

where $\mathfrak{E}^0\subset k\langle A\rangle$ is the subalgebra generated by words that don't begin with $z_{1,1}$.

Linear Maps Induced by Formal Power Series

Let

$$f = c_1 t + c_2 t^2 + \cdots$$

be a formal power series with $c_1 \neq 0$. The set $\mathfrak{P}$ of such formal power series evidently forms a group under the composition operation: i.e., if

$$g = d_1 t + d_2 t^2 + \dots$$

is also in $\mathfrak{P}$, then

$$\begin{aligned} g \circ f &= d_1 f + d_2 f^2 + \dots \\ &= d_1 c_1 t + (d_1 c_2 + d_2 c_1^2) t^2 + \cdots \end{aligned}$$

Now define a linear map Ψ_f from $k\langle A\rangle$ to itself as follows.

Induced Linear Maps Cont'd

For words $w = a_1 a_2 \cdots a_n$ in $k\langle A\rangle$, let

$$\Psi_f(w) = \sum_{I \in \mathcal{C}(n)} c_I I[w]$$

where the sum is over all compositions $I = (i_1, \ldots, i_k)$ of n (finite sequences of positive integers with $i_1 + \cdots + i_k = n$), where

$$c_{(i_1,\ldots,i_k)} = c_{i_1} \cdots c_{i_k}$$

and

$$I[w] = I[a_1 \ldots a_n] = \\ a_1 \diamond \cdots \diamond a_{i_1} a_{i_1+1} \diamond \cdots \diamond a_{i_1+i_2} \cdots a_{i_1+\cdots+i_{k-1}+1} \diamond \cdots \diamond a_n .$$

Examples of Induced Linear Maps

Some important examples of power series in $\mathfrak{P}$ are

$$e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$$
$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots$$
$$\frac{t}{1-t} = t + t^2 + t^3 + \cdots$$

leading to

$$\exp := \Psi_{e^t - 1}, \quad \log := \Psi_{\log(1+t)}, \quad \Sigma := \Psi_{\frac{t}{1-t}}$$

We also define $T := \Psi_{-t}$.

What is the Significance of Σ?

In the paper of S. Moch, P. Uwer and S. Weinzierl [J. Math. Phys. 2002], "S-sums" and "Z-sums" are given by

$$S(N; i_1, \dots, i_k) = \sum_{N \ge n_1 \ge \dots \ge n_k \ge 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}$$

$$Z(N; i_1, \dots, i_k) = \sum_{N \ge n_1 > \dots > n_k \ge 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}.$$

For the quasi-shuffle algebra $(k\langle A\rangle, *)$ with $A = \{z_1, z_2, \dots\}$ and $z_i \diamond z_j = z_{i+j}$, there is a homomorphism $Z : (k\langle A\rangle, *) \to \mathbb{R}$ sending $z_{i_1} \cdots z_{i_k}$ to $Z(N; i_1, \dots, i_k)$; then $Z\Sigma : (k\langle A\rangle, \star) \to \mathbb{R}$ sends $z_{i_1} \cdots z_{i_k}$ to $S(N; i_1, \dots, i_k)$.

Significance of Σ Cont'd

Evidently

$$\zeta(i_1,\ldots,i_k)=\lim_{N\to\infty} Z(N;i_1,\ldots,i_k)$$

when $i_1>1$. The corresponding limit of the S-sums are usually called "multiple zeta-star values" in the literature. So if $\zeta:(\mathfrak{H}^0,*)\to\mathbb{R}$ (where $\mathfrak{H}^0$ is the subalgebra of $k\langle A\rangle$ consisting of words that don't begin with z_1) is defined by $\zeta(z_{i_1}\cdots z_{i_k})=\zeta(i_1,\ldots,i_k)$, then $\zeta^\star(i_1,\ldots i_k)=\zeta\Sigma(z_{i_1}\cdots z_{i_k})$. The study of multiple zeta-star values was a principal objective of IKOO.

Induced Maps of Composed Functions

The key result about induced maps is the following.

Theorem

Let $f, g \in \mathfrak{P}$. Then $\Psi_{f \circ g} = \Psi_f \Psi_g$.

Corollary

If $f \circ g = t$, then $\Psi_f \Psi_g = \mathrm{id}$.

In particular, $\exp^{-1} = \log$, $T^{-1} = T$, and $\Sigma^{-1} = \Psi_{\frac{t}{1+t}}$.

Corollary

$T \Sigma T = \Sigma^{-1}$.

Corollary

$\Sigma = \exp T \log T$.

Algebraic Properties of Induced Maps

In general the induced maps Ψ_f are not homomorphisms of the quasi-shuffle products. But we do have several results for special cases.

Theorem (Hoffman, J. Alg. Combin. 2000)

For all words v, w, $\exp(v \sqcup\!\sqcup w) = \exp(v) * \exp(w)$.

In fact, exp is an algebra isomorphism from $(k\langle A\rangle, \sqcup\!\sqcup)$ to $(k\langle A\rangle, *)$, and $\exp^{-1} = \log$ is an isomorphism from $(k\langle A\rangle, *)$ to $(k\langle A\rangle, \sqcup\!\sqcup)$.
We note that $T(w) = (-1)^{\ell(w)} w$. Using this fact, it is easy to prove the following.

Proposition

For any words v, w, $T(v * w) = T(v) \star T(w)$ and $T(v \star w) = T(v) * T(w)$.

Algebraic Properties Cont'd

Since $\Sigma = \exp T \log T$, we have the following result (which is proved in IKOO with considerably more effort).

Proposition

For words v, w, $\Sigma(v \star w) = \Sigma(v) * \Sigma(w)$ (so $\Sigma^{-1}(v * w) = \Sigma^{-1}(v) \star \Sigma^{-1}(w)$).

Corollary

ΣT is a homomorphism from $(k\langle A\rangle, *)$ to itself.

Note that in fact ΣT is an involution (i.e., a self-inverse homomorphism) of the algebra $(k\langle A\rangle, *)$ since $\Sigma T \Sigma T = \Sigma\Sigma^{-1} = \text{id}$. Similarly, $T\Sigma$ is an involution of $(k\langle A\rangle, \star)$.

Algebraic Properties Cont'd

A family of induced maps is given by

$$H_p = \exp \Psi_{pt} \log = \Psi_{(1+t)^p - 1}$$

(Here p can be any nonzero element of k). Since Ψ_{pt} is easily seen to be a homomorphism of $(k\langle A\rangle, ⧢)$, we have the following result.

Proposition

For any $p \neq 0$, H_p is a homomorphism of $(k\langle A\rangle, *)$.

Evidently $H_1 = \text{id}$ and $H_p H_q = H_{pq}$, so H_p has inverse $H_{\frac{1}{p}}$. Note that $H_{-1} = \Sigma T$.

The Coalgebra Structure

There is a coalgebra structure on $k\langle A\rangle$ given by having the counit ϵ send 1 to $1 \in k$ and all nonempty words to 0, and coproduct Δ given by the "deconcatenation"

$$\Delta(w) = \sum_{uv=w} u \otimes v,$$

where the sum is over all decompositions of w into subwords u and v (including the cases $u = 1$, $v = w$ and $u = w$, $v = 1$). This coproduct defines a convolution product $\odot$ on the set $\mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ of linear maps from $k\langle A\rangle$ to itself: for $L_1, L_2 \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$, let

$$L_1 \odot L_2(w) = \sum_{uv=w} L_1(u)L_2(v).$$

Inverse Pairs

The identity element of the convolution ring is the element $\eta\epsilon \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ that sends 1 to 1 and all words of positive length to zero. Any $L \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ with $L(1) = 1$ has a convolutional inverse $L^{\odot(-1)}$.

We call $C \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ a contraction if $C(1) = 0$ and $C(w)$ is primitive (i.e., $\Delta C(w) = C(w) \otimes 1 + 1 \otimes C(w)$) for all words w, and $E \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ an expansion if $E(1) = 1$ and E is a coalgebra map (i.e., $E(\Delta(w)) = \Delta(E(w))$).

If E is an expansion and C is a contraction, (E, C) is called an inverse pair if

$$E = (\eta\epsilon - C)^{\odot(-1)} = \eta\epsilon + C + C \odot C + \cdots$$

or equivalently

$$C = \eta\epsilon - E^{\odot(-1)}.$$

Inverse Pairs Cont'd

Proposition

Suppose $E, C \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$. If C is a contraction and $E = (\eta\epsilon - C)^{\odot(-1)}$, then (E, C) is an inverse pair. If E is an expansion and $C = \eta\epsilon - E^{\odot(-1)}$, then (E, C) is an inverse pair.

For $f = c_1 t + c_2 t + \cdots \in \mathfrak{P}$, let $C_f \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ be defined by $C_f(1) = 0$ and $C_f(a_1 a_2 \cdots a_n) = c_n a_1 \diamond a_2 \diamond \cdots \diamond a_n$ for all nonempty words $a_1 \cdots a_n$. It is easy to check that C is a contraction and $E = (\eta\epsilon - C)^{\odot(-1)}$, so the proposition above implies

Proposition

For any $f \in \mathfrak{P}$, (Ψ_f, C_f) is an inverse pair.

Algebraic Formulas

For $f = c_1 t + c_2 t^2 + c_3 t^3 + \cdots \in \mathfrak{P}$, define

$$f_\bullet(\lambda z) = c_1 \lambda z + c_2 \lambda^2 z \bullet z + c_3 \lambda^3 z \bullet z \bullet z + \cdots \qquad (2)$$

where $z \in kA$, λ is a formal parameter, and $\bullet$ is any of the symbols $*$, $\star$, $\sqcup\!\sqcup$, or $\diamond$. (The expression (2) is in the formal power series ring $kA[[\lambda]]$.)
We define $\exp_\bullet(\lambda z)$ to be $1 + g_\bullet(\lambda z)$ and $\log_\bullet(1 + \lambda z)$ to be $f_\bullet(\lambda z)$, where

$$g = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots = e^t - 1 \in \mathfrak{P}$$
$$f = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots = \log(1 + t) \in \mathfrak{P}$$

The Master Theorem

Our "master theorem" here is

Theorem

For any $f \in \mathfrak{P}$, $z \in kA[[\lambda]]$,

$$\Psi_f\left(\frac{1}{1-\lambda z}\right) = \frac{1}{1 - f_\diamond(\lambda z)},$$

where Ψ_f is extended to $k\langle A\rangle[[\lambda]]$ by $\Psi_f(\lambda^n w) = \lambda^n \Psi_f(w)$.

which can be deduced by taking $(E, C) = (\Psi_f, C_f)$ in

Proposition

Let (E, C) be an inverse pair. Then for $z \in kA[[\lambda]]$,

$$E\left(\frac{1}{1-\lambda z}\right) = \frac{1}{1 - C(\lambda z + \lambda^2 z^2 + \cdots)}.$$

Master Theorem Cont'd

As an example of the use of our "master theorem", take $f = \log(1+t)$. This gives

$$\log\left(\frac{1}{1-\lambda z}\right) = \frac{1}{1-\log_{\diamond}(1+\lambda z)}$$

or

$$\frac{1}{1-\lambda z} = \exp\left(\frac{1}{1-\log_{\diamond}(1+\lambda z)}\right)$$

for $z \in kA[[\lambda]]$. On the other hand, since $a^{\sqcup\!\sqcup n} = n!a$ for any letter a, it follows that

$$\exp_{\sqcup\!\sqcup}(\lambda u) = \frac{1}{1-\lambda u}$$

for all $u \in kA[[\lambda]]$.

Exponential Formula

Further, since exp is a homomorphism from $(k\langle A\rangle, \sqcup\!\sqcup)$ to $(k\langle A\rangle, *)$ we have

$$\exp_*(\lambda u) = \exp(\exp_{\sqcup\!\sqcup}(\lambda u)) = \exp\left(\frac{1}{1-\lambda u}\right).$$

Set $\lambda u = \log_\diamond(1+\lambda z)$ and use the previous slide to get

Theorem

For $z \in kA[[\lambda]]$,

$$\frac{1}{1-\lambda z} = \exp_*(\log_\diamond(1+\lambda z)).$$

This was proved in the MZV setting by K. Ihara, M. Kaneko, and D. Zagier [Compos. Math. 2006].

Exponential Formula Cont'd

This result allows one to express MZVs, finite multiple harmonic sums, multiple polylogarithms, etc. with repeated values in terms of the non-multiple version. For example, for multiple polylogarithms it implies

$$Li_{\underbrace{i,\dots,i}_{k}}(\underbrace{x,\dots,x}_{k}) =$$

$$\text{coefficient of } \lambda^k \text{ in } \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}\lambda^j}{j} Li_{ij}(x^j)\right).$$

A result on H_p . . .

Proposition

For $p \neq 0$, $z \in kA[[\lambda]]$,

$$H_p\left(\frac{1}{1-\lambda z}\right) = \left(\frac{1}{1-\lambda z}\right)^{*p}$$

Proof.

From the preceding result, $H_p((1-\lambda z)^{-1})$ is

$$\begin{aligned} H_p(\exp_*(\log_\diamond(1+\lambda z)) &= \exp_*(H_p(\log_\diamond(1+\lambda z)) \\ &= \exp_*(p\log_\diamond(1+\lambda z)) \end{aligned}$$

and the conclusion follows. □

… and its application

Theorem

If $(1+f(-t))(1+g(t))=1$, *i.e.,* $g=\frac{t}{1-t}\circ(-t)\circ f\circ(-t)$, *then*

$$\Psi_f\left(\frac{1}{1-\lambda z}\right)*\Psi_g\left(\frac{1}{1+\lambda z}\right)=1.$$

for all $z\in kA[[\lambda]]$.

Proof.

Since $\Psi_g=H_{-1}\Psi_f T$, the conclusion is $\xi * H_{-1}(\xi)=1$ for $\xi=\Psi_f((1-\lambda z)^{-1})$. The "master theorem" says that, if $f=c_1t+c_2t^2+\cdots$, then $\xi=(1-\lambda u)^{-1}$ for

$$u=c_1z+\lambda c_2 z^{\diamond 2}+\lambda^2 c_3 z^{\diamond 3}+\cdots\in kA[[\lambda]].$$

Now put $p=-1$ in the preceding result. □

Application Cont'd

In particular, taking $f = t/(1-pt)$ gives

$$\Sigma^p\left(\frac{1}{1-\lambda z}\right) * \Sigma^{1-p}\left(\frac{1}{1+\lambda z}\right) = 1 \qquad (3)$$

for any $p \in k$, $z \in kA[[\lambda]]$. This generalizes a result of IKOO, which gave the result for $p = 1$:

$$\Sigma\left(\frac{1}{1-\lambda z}\right) * \frac{1}{1+\lambda z} = 1.$$

We haven't yet found a nice use for the additional generality of equation (3), but recent work by S. Yamamoto (`arXiv 1203.1118[NT]`) indicates that having results for arbitrary iterates of Σ could be useful.

One Last Formula

The next result, which appears in IKOO, is considerably harder to prove (though I think the proof in Ihara's and my preprint is easier to understand than the proof in IKOO).

Theorem

For $a, b \in A$,

$$\Sigma\left(\frac{1}{1-\lambda ab}\right) = \frac{1}{1-\lambda ab} * \Sigma\left(\frac{1}{1-\lambda a \diamond b}\right)$$

The point of this formula is to treat MZVs, finite multiple harmonic sums, etc. where there is a repeated pattern of length two.

Last Formula Cont'd

Translated into MZVs, this says, for example, that

$$\sum_{n=0}^{\infty} \lambda^n \zeta^{\star}((z_3 z_1)^n) = \left(\sum_{i=0}^{\infty} \lambda^i \zeta((z_3 z_1)^i)\right) \left(\sum_{j=0}^{\infty} \lambda^j \zeta^{*}(z_4^j)\right),$$

which in view of the Zagier-Broadhurst result

$$\zeta((z_3 z_1)^n) = 4^{-n} \zeta((z_4)^n)$$

makes it clear that

$$\zeta^{\star}((z_3 z_1)^n) = \zeta^{\star}(\underbrace{3, 1, \ldots, 3, 1}_{n \text{ repetitions}})$$

is a rational multiple of π^{4n}.

Conclusions

- MZVs, finite multiple harmonic sums, and multiple polylogarithms all appear in pQFT calculations.
- Quasi-shuffle products have been useful in understanding the structure of these numbers.
- From a mathematical and physical point of view, it is desirable to have a construction of quasi-shuffle products with as few assumptions as possible.
- Work is in progress . . . much remains to be done!