Exploring tree amplitudes with massive particles

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Simple expressions for maximally helicity violating gauge theory amplitudes known since mid-80s (Parke/Taylor 86, Berends/Giele 88)

- Helicity methods, Recursion relations...

2000s: New methods (initially) for massless QCD amplitudes

- Relation to twistor string theory (Witten 2003)

⇒ New representations of QCD amplitudes

- CSW rules (Cachazo, Svrček, Witten 04)
  - All massless born QCD amplitudes from MHV vertices
- BCFW rules: (Britto, Cachazo, Feng/Witten, 04/05)
  - Construct born amplitudes from on-shell subamplitudes
- Unitarity methods (Britto/Cachazo/Feng 04, Ossola et al. 05, Forde 07,...)

Common ideas:

on-shell amplitudes as building blocks, complex kinematics

SUSY as tool

- SUSY Ward identities to obtain QCD helicity ampliutdes (Parke/Taylor 85; Kunszt 86)
- One loop $\mathcal{N}=4$ SUSY amplitudes from quadruple cuts $\Rightarrow$ unitarity methods for arbitrary QFTs (Britto/Cachazo/Feng 04, Ossola/Papadopoulos/Pittau 05; Forde 07,...)
- Tree $\mathcal{N}=4$ S-matrix known in closed form (Drummond/Henn 08)
  $\Rightarrow$ all tree QCD amplitudes (Dixon/Henn/Plefka/Schuster 10, Bourjaily 10)

Massive particles (top, W, Z, H,...)

- Phenomenology (numerical approach sufficient, analytical results can be useful)
- Rational part in loop diagrams (Bern/Morgan 96; Badger 08; numerical: Giele/Kunst/Melnikov 08, ...)
- IR regulator in $\mathcal{N}=4$ (Alday/Henn/Plefka/Schuster 09)

Plan of the talk

N-Gluon amplitudes

- Spinor helicity, MHV amplitudes
- SUSY Ward Identities as tool
- MHV diagrams (Cachazo, Svrček, Witten 04)

CSW rules for massive scalars (Boels, CS, 07/08)

- Methods of derivation
- One-loop applications

On-shell supersymmetry

- massless $\mathcal{N}=1$ SUSY
- extension to massive particles (Boels, CS, 11)
- example: SQCD with massive matter

CSW rules for massive quarks (CS 08)

Two-component Weyl spinors in braket notation

$$|k+\rangle = k_\alpha = \left(\tfrac{1+\gamma^5}{2}\right) u(k) \quad , \quad |k-\rangle = k^{\dot\alpha} = \left(\tfrac{1-\gamma^5}{2}\right) u(k)$$

- antisymmetric spinor products

$$\langle pk\rangle = \varepsilon^{\alpha\beta} p_\beta k_\alpha = \langle p - |k+\rangle$$

$$[pk] = \epsilon_{\dot\alpha\dot\beta} p^{\dot\alpha} k^{\dot\beta} = \langle p + |k-\rangle$$

- Express momenta in terms of spinors:

$$\langle k + |\gamma^\mu|k+\rangle = k_{\dot\alpha} \bar\sigma^{\dot\alpha\beta} k_\beta = 2k^\mu$$

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Simple expression for MHV amplitudes (Parke-Taylor 1986)

$$A_n(g_1^+, \ldots, g_i^-, \ldots, g_j^-, \ldots g_n^+) = \mathrm{i} 2^{n/2-1} \frac{\langle ij\rangle^4}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle}$$

"Effective Supersymmetry" of QCD: (Parke, Taylor 1985; Kunszt 1986)

Tree level colour-stripped amplitudes for massless quarks are the same as for gluinos in a fictitious, unbroken, SUSY QCD.

SUSY transformations of helicity states of gluons and gluinos with Grassmann-valued spinor θ:

$$\delta_\theta g^{\pm}(k) = \langle \theta \pm | k \mp \rangle\, \lambda^{\pm}(k) \qquad \delta_\theta \lambda^{\pm}(k) = -\langle \theta \mp | k \pm \rangle\, g^{\pm}(k)$$

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SUSY Ward-Identities (Grisaru, Pendleton 1977)

$$0 = \langle 0 | [Q_{\mathrm{SUSY}}, \psi_1 \dots \psi_n] | 0 \rangle = \sum_i \langle 0 | [\psi_1 \dots (\delta_\theta \psi_i) \dots \psi_n] | 0 \rangle)$$

Fermionic MHV amplitudes (set $|\theta+\rangle \propto |j+\rangle$) (Parke, Taylor 1985; Kunszt 1986)

$$A_n(\bar{\lambda}_1^-, g_2^+, \dots g_j^-, \dots, \lambda_n^+) = \frac{\langle nj \rangle}{\langle 1j \rangle} A_n(g_1^-, g_2^+, \dots g_j^-, \dots, g_n^+)$$

$$= \mathrm{i} 2^{n/2-1} \frac{\langle 1j \rangle^3 \langle nj \rangle}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

MHV diagrams (CSW rules): (Cachazo, Svrček, Witten 2004)

All Yang-Mills amplitudes from MHV vertices

$$V_{\mathsf{CSW}}(g_1^+ \dots g_i^- \dots g_j^- \dots g_n^+) = 2^n \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

with off-shell continuation $|k+\rangle \to \not{k}\, |q-\rangle$

Scalar propagators $\frac{\mathrm{i}}{k^2}$ connecting $+$ and $-$ labels

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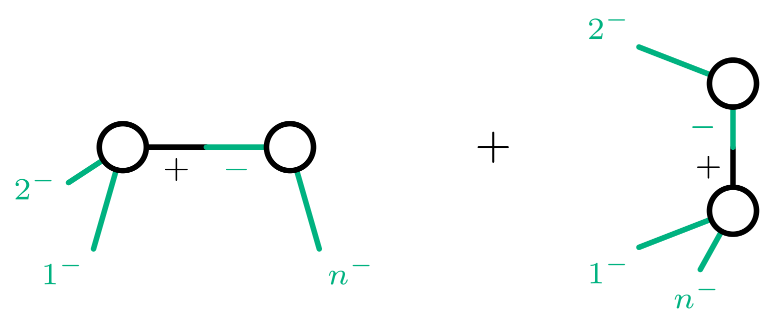
Example: NMHV amplitudes $A(g_1^-, g_2^-, g_3^+, \dots g_n^-)$:

- Distribute negative helicities over $d = n^- - 1 = 2$ MHV vertices



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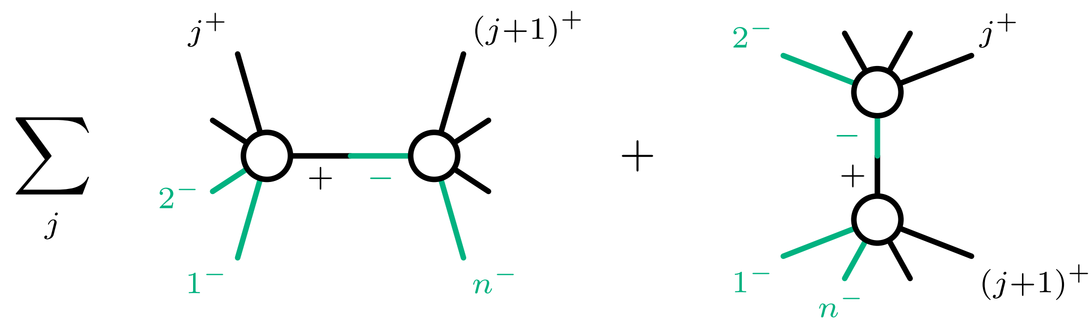
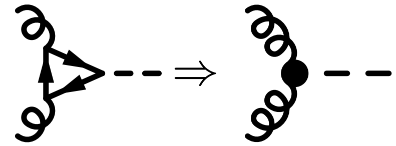


- Distribute positive helicities $\Rightarrow 2(n-3)$ diagrams
- Complexity grows with n^-

External massive particles

Effective Higgs-gluon coupling

$$\mathcal{L}_{\text{eff}} = \frac{\alpha_s}{6\pi v} \int dx^4 H \, \mathrm{tr}[F_{\mu\nu} F^{\mu\nu}]$$



Trick: add effective vertex for pseudoscalar

$$\mathcal{L}_{\text{eff},A} = \frac{\alpha_s}{6\pi v} \int dx^4 \ \mathrm{i}A \ \mathrm{tr}[F_{\mu\nu} F_{\rho\sigma}]\epsilon^{\mu\nu\rho\sigma}$$

MHV vertex for complex scalar $\phi = H + \mathrm{i}A$ (Dixon, Glover, Khoze 04)

$$V_{\text{CSW}}(\phi, g_1^+ \ldots g_i^- \ldots g_j^- \ldots g_n^+) = \mathrm{i}2^{n/2-1} \frac{\alpha_s}{6\pi v} \frac{\langle ij \rangle^4}{\langle 12 \rangle \ldots \langle (n-1)n \rangle \langle n1 \rangle}$$

External W/Z bosons (Bern, Forde, Kosower, Mastrolia 04)

Can CSW rules be extended to massive, coloured particles?

Massive colored Scalar (R.Boels, CS 07/08)

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No obvious extension of CSW rules

- nonvanishing amplitudes with only positive helicity gluons
- explicit expression does not look twistoresque:

(Ferrario, Rodrigo, Talavera 06)

$$A(\bar{\phi}_1^+, g_2^+, \ldots, \phi_n^-) = \frac{i2^{n/2-1} m^2 \langle 2 + | \Pi_{j=3}^{n-2} (y_{1,j} - \not{k}_j \not{k}_{1,j-1}) | (n-1) - \rangle}{y_{1,2} \ldots y_{1,n-2} \langle 23 \rangle \langle 34 \rangle \ldots \langle (n-2)(n-1) \rangle}$$

$$k_{1,j} = \sum_{n=1}^{j} k_n, \qquad y_{1,j} = k_{1,j}^2 - m^2$$

$\Rightarrow$ **need methods to derive CSW rules**

Generalized on-shell recursion

- Complex deformation of g^- momenta (Risager 05)
- Complex deformation of all momenta (Elvang/Freedman/Kiermaier 08)

Field-redefinition in light-cone QCD (Mansfield 05; Ettle/Moris 06)

- Eliminate non-physical degrees of freedom from Lagrangian

$\Rightarrow$ Lagrangian for positive/negative helicity gluons only

- Eliminate non MHV vertices by canonical transformation

$\Rightarrow$ generates tower of MHV vertices

Gauge theory on twistor space (Mason 05, Boels, Mason, Skinner 06/07)

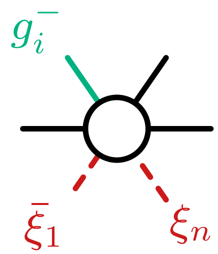
- action for gauge fields on twistor space
- Extended gauge freedom
- special gauges lead to Yang Mills and CSW rules

CSW rules for massive scalars

Derivation using light-cone/twistor methods (R.Boels, CS, 07/08)

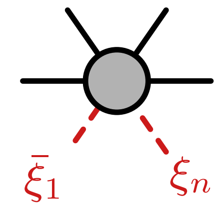
Use same canonical transformation as for massess scalars $\Rightarrow$ new vertex from transformation of mass term

massless MHV vertices



$$= \mathrm{i}2^{n/2-1} \frac{\langle in \rangle^2 \langle 1i \rangle^2}{\langle 12 \rangle \ldots \langle (n-1)n \rangle \langle n1 \rangle}$$

new vertex $\sim m^2$



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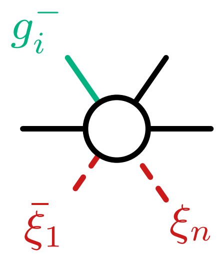

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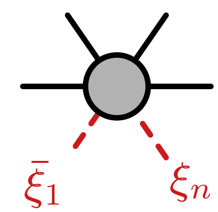

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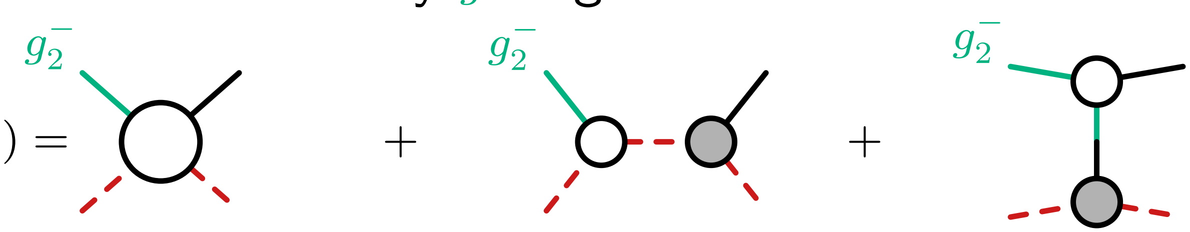
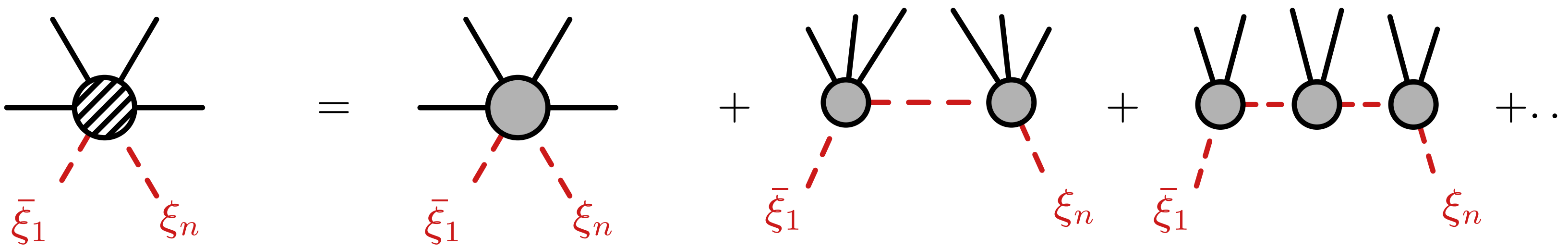
Comments:

- Not an "on-shell" method
- Number of vertices not fixed by g^- legs:

$A_4(\bar{\xi}_1, g_2^-, g_3^+, \xi_4) =$



Scalar amplitudes with positive helicity gluons:



Leading piece for $m \to 0$ from single vertex:

$$A_n(\bar{\xi}_1, g_2^+ \ldots, , \xi_n) = \mathrm{i}2^{n/2-1} \frac{-m^2 \langle 1n \rangle}{\langle 12 \rangle \ldots \langle (n-1)n \rangle} + \mathcal{O}(m^2)$$

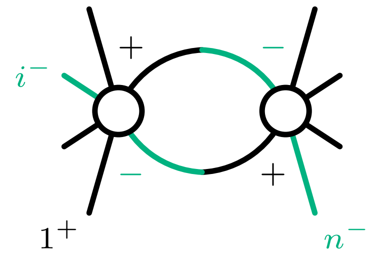
$$= \mathrm{i}2^{n/2-1} \frac{m^2 \langle 2 + |\not{k}_1 \not{k}_n|(n-1)- \rangle}{2(k_1 \cdot k_2) 2(k_n \cdot k_{n-1}) \langle 23 \rangle \ldots \langle (n-2)(n-1) \rangle} + \mathcal{O}(m^2)$$

(agrees with Bern, Dixon, Dunbar, Kosower 96)

All-n expression from CSW rules: (Kiermaier 11)

$$A_n(\bar{\xi}_1, g_2^+ \ldots, , \xi_n) = \frac{-\mathrm{i}2^{n/2-1} m^2}{\langle 12 \rangle \ldots \langle (n-1)n \rangle} \langle 1- | \prod_{j=2}^{n-2} \left(1 - \frac{m^2 |k_{1,j}+\rangle \langle j(j-1) \rangle \langle -(k_{1,j})|}{y_{1,j} \langle k_{1,j}\, j \rangle \langle (j+1) k_{1,j} \rangle} \right) |n+\rangle$$

MHV rules for one-loop amplitudes



- "cut constructable" part of amplitudes

  (Bedford, Brandhuber, Spence, Travaglini; Quigly, Rozali 04)

- Where is all-plus amplitude $A(g_1^+, \ldots g_2^+)$?

Proposals for rational part of one-loop amplitudes

- 4-D regulator in light-cone QCD (Brandhuber et.al. 07)
- D-dim MHV vertices + careful application of LSZ (Ettle et.al 07)

SUSY decomposition of gluon one-loop amplitudes:

$$A^{\mathcal{N}=0}_{\text{gluon}} = A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1}_{\text{chiral}} + A^{\mathcal{N}=0}_{\text{scalar}}$$

no rational part in SUSY pieces ⇒ need scalar in $4-2\epsilon$ dim.

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no rational part in SUSY pieces $\Rightarrow$ need scalar in $4-2\epsilon$ dim.

$$\ell_D^2 = \ell^2 + \ell_{-2\epsilon}^2 \equiv \ell^2 - \mu^2$$

$$\int \frac{d^D\ell}{(2\pi)^D} f(\ell_D^2) = \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{d^4\ell}{(2\pi)^4} f(\ell^2 - \mu^2)$$ (Bern, Morgan 95)

$\Rightarrow$ use CSW rules for massive scalar

- Simplest example: $A(g_1^+, g_2^+, g_3^+, g_4^+)$ works (Boels, CS 08)
- combination with unitarity cuts: (Glover, Williams 08)
  five points, negative helicity gluons
- *integrands* of n-point all-plus and one-minus amplitudes
  (Elvang/Freedman/Kiermaier 11)

Anticommutator of SUSY generators:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2\sigma^\mu_{\alpha\dot{\alpha}} k_\mu = -2k_\alpha k_{\dot{\alpha}}$$

Massless case: solved in arbitrary frame by

$$Q_\alpha = k_\alpha Q_- \;, \quad \bar{Q}_{\dot{\alpha}} = k_{\dot{\alpha}} \bar{Q}_+ \qquad \text{with} \quad \{Q_+, \bar{Q}_-\} = -2$$

Representations: Multiplets (ψ_+, ψ_-):

$$\frac{1}{\sqrt{2}}\bar{Q}_+\psi_+ = 0 \qquad \frac{1}{\sqrt{2}}Q_-\psi_+ = \psi_-$$

$$\frac{1}{\sqrt{2}}\bar{Q}_+\psi_- = -\psi_+ \qquad \frac{1}{\sqrt{2}}Q_-\psi_- = 0$$

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SUSY transformation:

$$Q(\theta) \equiv \theta^\alpha Q_\alpha + \theta_{\dot\alpha}\bar{Q}^{\dot\alpha} = \langle\theta k\rangle\, Q_- + [\theta k]\bar{Q}_+$$

$\Rightarrow$ Transformation of states

$$\tfrac{1}{\sqrt{2}}Q(\theta)\psi_+ = \langle\theta k\rangle\,\psi_- \ , \qquad \tfrac{1}{\sqrt{2}}Q(\theta)\psi_- = -[\theta k]\psi_+$$

Define coherent states ($\mathcal{N}=4$: Arkani-Hamed et.al.08)

$$\Psi(\eta) = e^{-\frac{1}{\sqrt{2}}\eta Q_-}\psi_+ = \psi_+ - \eta\,\psi_-, \qquad \Psi(\bar\eta) = e^{-\frac{1}{\sqrt{2}}\bar\eta \bar Q_+}\psi_- = \psi_- + \bar\eta\,\psi_+.$$

Diagonalize half of the SUSY generators:

$$\frac{1}{\sqrt{2}}\bar Q_+ \Psi(\eta) = -\eta\,\Psi(\eta), \qquad \frac{1}{\sqrt{2}} Q_- \Psi(\bar\eta) = -\bar\eta\,\Psi(\bar\eta).$$

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SUSY invariance of Superamplitudes $A(\{k_i,\eta_i\},\{k_j\bar\eta_j\})$:

$$A(\{\eta_i\},\{\bar\eta_j\}) = e^{\sum_i \eta_i\langle\theta k_i\rangle} A(\{\eta_i\},\{\bar\eta_j - \langle\theta k\rangle\}),$$

(Conventions are for outgoing states: transformations of $\eta \leftrightarrow \bar\eta$ reversed)

Manifest SUSY of "pure" η amplitudes ($\delta^2(X_\alpha) = \frac{1}{2}X^\alpha X_\alpha$)

$$A(\{\eta_i\}) = e^{\sum_i \eta_i\langle\theta k_i\rangle} A(\{\eta_i\}) \qquad \Rightarrow A(\{\eta_i\}) = \tilde A(\{\eta_i\})\,\delta^2\left(\sum_i \eta_i k_{i,\alpha}\right)$$

Example: $\mathcal{N}=1$ SYM: Two coherent-state representations

$$G^- = \lambda^- - \eta g^- \ , \qquad G^+ = g^+ - \eta\lambda^+$$

MHV amplitude: (Elvang/Huang/Peng 11)

$$A_{\mathsf{MHV}}(G_1^+,\dots G_i^-,\dots G_j^-,\dots,G_n^+) = \frac{\langle ij\rangle^3}{\langle 12\rangle\dots\langle n1\rangle}\underbrace{\delta^2\left(\sum_\ell \eta_\ell k_{\ell,\alpha}\right)}_{\frac{1}{2}\sum_{\ell m}\eta_\ell\eta_m\langle \ell m\rangle}$$

$\mathcal{N}=4$ SYM: all states in one supermultiplet (Nair 88)

$$\Omega = g^+ + \eta^a\lambda_a^+ + \frac{1}{2}\eta^a\eta^b\phi_{ab} + \frac{1}{3!}\epsilon^{abcd}\eta^a\eta^b\eta^c\lambda_d^- + \frac{1}{4!}\epsilon^{abcd}\eta^a\eta^b\eta^c\eta^d\, g^-$$

MHV amplitude:

$$A_{\mathsf{MHV}} = \frac{1}{\langle 12\rangle\dots\langle n1\rangle}\underbrace{\delta^8\left(\sum_\ell \eta_\ell^a k_{\ell,\alpha}\right)}_{\frac{1}{2^4}\prod_a\sum_{\ell m}\eta_\ell^a\eta_m^a\langle \ell m\rangle}$$

Define light cone projection of massive momenta:

$$k^\flat = k - \frac{m^2}{2k\cdot q} q$$

Basis for two-spinor space: $\left\{k^\flat_\alpha \,,\ \tilde{q}_\alpha = \frac{m}{\langle kq\rangle} q_\alpha\right\}$, $\left\{k^\flat_{\dot\alpha} \,,\ \tilde{q}_{\dot\alpha} \equiv \frac{m}{[qk]} q_{\dot\alpha}\right\}$.

Spinors for massive quarks (Kleiss, Stirling 85;. . . ; CS, S.Weinzierl 05)

$$u(\pm, q) = \frac{1}{\langle k^\flat \pm | q\mp\rangle} (\not{k} + m)\, |q\mp\rangle = |k^\flat \pm\rangle \mp |\tilde{q}\mp\rangle$$

Eigenstates of projectors $(1 \pm \not{s}\gamma^5)$ with spin vector

$$s^\mu = \frac{p^\mu}{m} - \frac{m}{(p\cdot q)} q^\mu$$

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Massive gauge boson polarization vectors (Kleiss/Stirling 85, Dittmaier 98)

$$\epsilon^+_{\alpha\dot\alpha} = \sqrt{2}\frac{q_\alpha k^\flat_{\dot\alpha}}{\langle q k^\flat\rangle}\,, \qquad \epsilon^-_{\alpha\dot\alpha} = \sqrt{2}\frac{k^\flat_\alpha q_{\dot\alpha}}{[k^\flat q]}\,, \qquad \epsilon^0_{\alpha\dot\alpha} = \frac{1}{m}\left(k^\flat_\alpha k^\flat_{\dot\alpha} - \frac{m^2}{2q\cdot k} q_\alpha q_{\dot\alpha}\right)$$

SUSY algebra:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = -2\sigma^\mu_{\alpha\dot\alpha} k_\mu = -2(k^\flat_\alpha k^\flat_{\dot\alpha} + \tilde{q}_\alpha \tilde{q}_{\dot\alpha})$$

Solve by

$$Q_\alpha = k^\flat_\alpha Q_- + \tilde{q}_\alpha Q_+, \qquad \bar{Q}_{\dot\alpha} = k^\flat_{\dot\alpha} \bar{Q}_+ + \tilde{q}_{\dot\alpha} \bar{Q}_- \quad \Rightarrow \{Q_\pm, \bar{Q}_\mp\} \quad = -2$$

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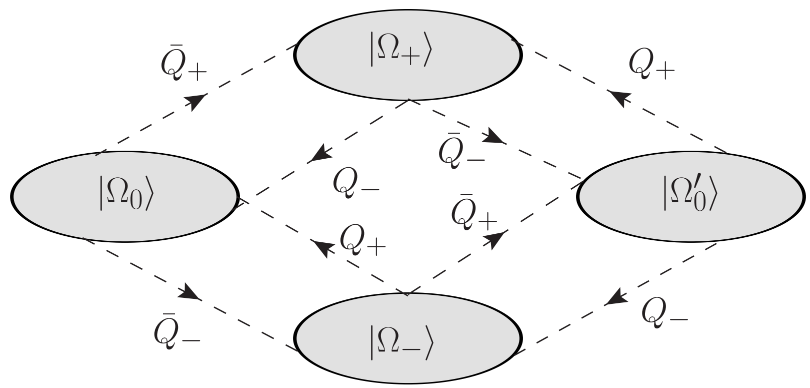
Multiplets

$$Q_\pm \psi_0 = 0$$

$$\frac{1}{\sqrt{2}} \bar{Q}_\pm \psi_0 = \psi_\pm$$

$$\frac{1}{2} \bar{Q}_- \bar{Q}_+ \psi_0 = \psi_0'$$



SUSY transformation:

$$Q(\theta) = \theta^\alpha Q_\alpha + \theta_{\dot\alpha} \bar{Q}^{\dot\alpha} = \langle \theta k^\flat \rangle\, Q_- + m \frac{\langle \theta q \rangle}{\langle k^\flat q \rangle} Q_+ + [\theta k^\flat] \bar{Q}_+ + m \frac{[\theta q]}{[q k^\flat]} \bar{Q}_- \,.$$

Coherent states defined with two parameters

$$\Psi_0(\bar{\eta},\iota) = e^{-\frac{1}{\sqrt{2}}(\bar{\eta}\bar{Q}_+ + \iota\bar{Q}_-)}\psi_0, \qquad \Psi'_0(\eta,\bar{\iota}) = -e^{-\frac{1}{\sqrt{2}}(\eta Q_- + \bar{\iota}Q_+)}\psi'_0$$

SUSY invariance of Superamplitudes:

$$A(\{\eta_k,\bar{\iota}_k\},\{\bar{\eta}_l,\iota_l\}) = e^{\sum_k \eta_k\langle\theta k_k^\flat\rangle + \bar{\iota}_k m_k \frac{\langle\theta q_k\rangle}{\langle k_k^\flat q_k\rangle}} A(\{\eta_l,\bar{\iota}_l\},\{\bar{\eta}'_k,\iota'_k\})$$

with $\bar{\eta}' = \bar{\eta} + \langle\theta k^\flat\rangle$, $\iota' = \iota + m\frac{\langle\theta q\rangle}{\langle k^\flat q\rangle}$

Coherent states defined with two parameters

$$\Psi_0(\bar\eta,\iota) = e^{-\frac{1}{\sqrt{2}}(\bar\eta\bar Q_+ + \iota\bar Q_-)}\psi_0, \qquad \Psi_0'(\eta,\bar\iota) = -e^{-\frac{1}{\sqrt{2}}(\eta Q_- + \bar\iota Q_+)}\psi_0'$$

SUSY invariance of Superamplitudes:

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with $\bar\eta' = \bar\eta + \langle\theta k^\flat\rangle$, $\iota' = \iota + m\frac{\langle\theta q\rangle}{\langle k^\flat q\rangle}$

Manifest SUSY of "pure" $\eta\bar\iota$ amplitudes

$$A(\{\eta_i,\bar\iota_i\}) = e^{\theta^\alpha \mathcal{Q}_\alpha} A(\{\eta_i,\bar\iota_i\}) \qquad \Rightarrow A(\{\eta_i,\bar\iota_i\}) = \tilde A(\{\eta_i,\bar\iota_i\})\,\delta^2(\mathcal{Q}_\alpha)$$

with $\mathcal{Q}_\alpha = \sum_k \left(\eta_k k_{k,\alpha}^\flat + \bar\iota_k m_k \frac{q_{k,\alpha}}{\langle k_k^\flat q_k\rangle}\right)$

Invariance under remaining SUSY:

$$\bar{\mathcal{Q}}^{\dot\alpha}\tilde A \equiv \sum_i \left(-k_i^{\flat,\dot\alpha}\frac{\partial}{\partial\eta_i} + q_i^{\dot\alpha}\frac{m_i}{[k_i^\flat q_i]}\frac{\partial}{\partial\bar\iota_i}\right)\tilde A = 0$$

Example: SQCD with massive matter

Multiplets:

$$\Phi(\eta,\bar\iota) = -\phi^- + \eta Q^- - \bar\iota Q^+ + \eta\bar\iota\,\phi^+, \quad G^-(\eta) = \Lambda^- + \eta g^-, \quad G^+(\eta) = g^+ + \eta\Lambda^+.$$

SUSY transformations (reproduces CS, Weinzierl 06)

$$\delta_\theta \phi^+ = Q^+ \langle\theta k^\flat\rangle + Q^- m\frac{\langle\theta q\rangle}{\langle k^\flat q\rangle}, \qquad \delta_\theta \phi^- = -Q^-[\theta k^\flat] - Q^+ m\frac{[\theta q]}{[k^\flat q]},$$

$$\delta_\theta Q^- = \phi^- \langle\theta k^\flat\rangle + \phi^+ m\frac{[\theta q]}{[k^\flat q]}, \qquad \delta_\theta Q^+ = -\phi^+[\theta k^\flat] - \phi^- m\frac{\langle\theta q\rangle}{\langle k^\flat q\rangle}.$$

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Three-point supervertices:

$$A_3(\bar{\Phi}_1, G_2^+, \Phi_3) = \delta^2(\mathcal{Q}_\alpha) m \frac{\langle 31\rangle}{\langle 12\rangle\langle 23\rangle},$$

$$A_3(\bar{\Phi}_1, G_2^-, \Phi_3) = \delta^2(\mathcal{Q}_\alpha) \frac{\bar{\iota}_1\langle 23\rangle - \bar{\iota}_3\langle 12\rangle}{\langle 31\rangle}$$

with $\delta^2(\mathcal{Q}_\alpha) = \sum_{i,j}\left(\frac{1}{2}\langle ij\rangle\,\eta_i\eta_j + m\frac{\langle iq\rangle}{\langle jq\rangle}\,\eta_i\bar{\iota}_j\right)$

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N-point superamplitude

$$A_3(\bar{\Phi}_1, G_2^+, \ldots \Phi_n) = \delta^2(\mathcal{Q}_\alpha)\frac{i2^{n/2-1} m \langle 2 + |\Pi_{j=3}^{n-2}\,(y_{1,j} - \not{k}_j \not{K}_{1,j-1})\,|(n-1)-\rangle}{y_{1,2}\ldots y_{1,n-2}\,\langle 23\rangle\,\langle 34\rangle\ldots\langle (n-2)(n-1)\rangle}$$

Explicit SUSY relations of massive quark and to scalar amplitudes:

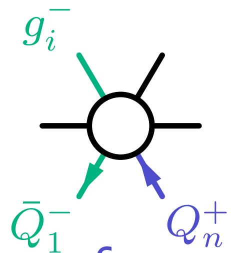
(checked in Ferrario, Rodrigo, Talavera 06)

$$A(\bar{Q}_1^+, 2^+, \ldots, Q_n^-) = \frac{\langle nq\rangle}{\langle q1\rangle} A(\bar{\phi}_1^+, 2^+, \ldots, \phi_n^-)$$

$$A(\bar{Q}_1^-, 2^+, \ldots, Q_n^-) = \frac{\langle 1n\rangle}{m} A(\bar{\phi}_1^+, 2^+, \ldots, \phi_n^-)$$

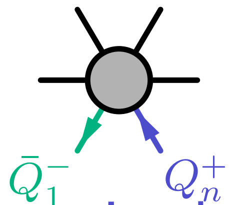
Vertices from canonical transformation (Ettle, Morris, Xiao 08; CS 08)

MHV-vertex: (+ 4-quark MHV vertex)



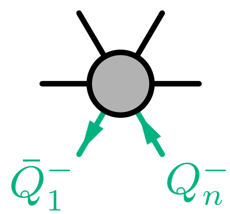
$$= \mathrm{i}2^{n/2-1} \frac{\langle 1i\rangle^3 \langle in\rangle}{\langle 12\rangle \ldots \langle (n-1)n\rangle \langle n1\rangle}$$

Vertex from transformation of mass term $\sim m^2$:

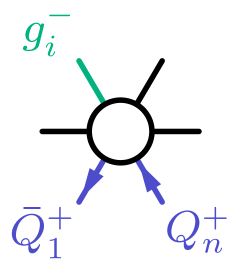


$$= \mathrm{i}2^{n/2-1} m^2 \frac{\langle q1\rangle \langle 1n\rangle}{\langle 12\rangle \ldots \langle (n-1)n\rangle \langle nq\rangle}$$

'Helicity flip vertices' $\sim m$: (+ 4-quark flip vertex)



$$= -\mathrm{i}2^{n/2-1} m \frac{\langle 1n\rangle^2}{\langle 12\rangle \ldots \langle (n-1)n\rangle},$$



$$= -\mathrm{i}2^{n/2-1} m \frac{\langle 1i\rangle \langle in\rangle}{\langle 12\rangle \ldots \langle (n-1)n\rangle} \frac{\langle qi\rangle^2}{\langle q1\rangle \langle qn\rangle},$$

CSW rules

- spontaneously broken $SU(2)$ (Buchta/Weinzierl 10)
- $\mathcal{N}=4$ SYM with scalar vevs (Elvang/Freedman/Kiermaier 11)

Supermultiplets in spontaneously broken gauge theory with massless (A) and massive (W) vectors:

$$\mathcal{A}^+(\eta) = A^+ + \eta\Lambda^+\,, \quad \mathcal{W}^+(\eta,\bar{\iota}) = -\chi^+ + \eta\frac{1}{\sqrt{2}}(W^0+H) - \bar{\iota}\,W^+ + \eta\bar{\iota}\,\tilde{\chi}^+,$$

$$\mathcal{A}^-(\eta) = \Lambda^- + \eta A^-, \quad \mathcal{W}^-(\eta,\bar{\iota}) = -\chi^- + \eta W^- - \bar{\iota}\,\frac{1}{\sqrt{2}}(W^0-H) + \eta\bar{\iota}\,\tilde{\chi}^-$$

Three-point supervertices:

$$A_3(\overline{\mathcal{W}}_1^+, \mathcal{A}_2^-, \mathcal{W}_3^-) = \delta^2(\mathcal{Q}_\alpha)\,(\bar{\iota}_1\langle 23\rangle - \bar{\iota}_3\langle 12\rangle)\,\frac{\langle 23\rangle}{\langle 12\rangle\langle 31\rangle}$$

$$A_3(\overline{\mathcal{W}}_1{}^-, \mathcal{A}_2^+, \mathcal{W}_3^-) = \delta^2(\mathcal{Q}_\alpha)\frac{\langle 13\rangle^2}{\langle 12\rangle\langle 23\rangle}$$

Methods for Yang-Mills calculations:

- spinor helicity, SUSY
- closed expression for MHV amplitudes
- CSW diagrams: MHV amplitudes as vertices
- On-shell SUSY: coherent states, Superamplitudes

Extension to massive particles

- new CSW vertices for massive scalars and quarks
- Superamplitudes for massive particles
- Can be useful for rational part of QCD amplitudes

Outlook:

- On-shell SUSY formalism also applies to spin $3/2$, 2
  $\Rightarrow$ string theory applications (Boels 12; Feng/Lüst/Schlotterer 12)
- Supervertex CSW rules for massive QCD?
- SUSY extension of massive vector boson CSW rules?

So far: All-multiplicity solution for MHV amplitudes

Useful for all amplitudes?

Earlier attempts MHV amplitudes from 2-D field theory (Nair 88),

relation to self-dual Yang-Mills (Bardeen; Cangemi; Chalmers, Siegel 96)

Insights from Twistor space (Witten 2003)

Index notation for spinors: $|p+\rangle = \pi_A \quad , \quad \langle p+| = \bar{\pi}_{\dot{A}}$

Twistor: $Z = (\pi_A, \mu^{\dot{A}}) \quad , \quad Z \sim \lambda Z$

Relation momentum ⇔ spinors:

$$p_\mu \sigma^\mu_{A\dot{A}} = p_{A\dot{A}} = \pi_A \bar{\pi}_{\dot{A}} \quad \text{up to} \quad \pi_A \sim \lambda \pi_A \,,\, \bar{\pi}_{\dot{A}} \sim \lambda^{-1} \bar{\pi}_{\dot{A}}$$

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Relation momentum $\Leftrightarrow$ spinors:

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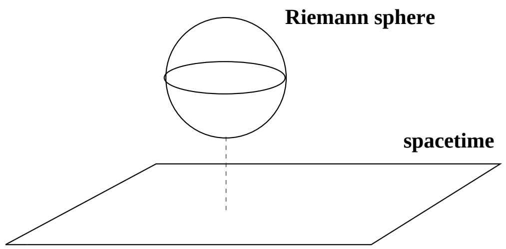
Embedding of space-time in twistor space:

$$x \longrightarrow Z = (\pi_A, x^{\dot{A}A} \pi_A)$$



Space-time point $x^\mu \leftrightarrow$ "line" $\mu^{\dot{A}} = x^{\dot{A}A} \pi_A$ in twistor space

Transformation to Twistor space $(\pi_A, \bar\pi^{\dot A}) \to (\pi_A, \mu^{\dot A})$ (Witten 2003)

$$\bar\pi^{\dot A} \to \mathrm{i}\frac{\partial}{\partial \mu_{\dot A}} \quad , \quad \frac{\partial}{\partial \bar\pi_{\dot A}} \to \mathrm{i}\mu^{\dot A}$$

MHV amplitudes supported on line in twistor space $(\pi_A, \mu_{\dot A})$

$$\int \left(\prod_i \frac{d^2\bar\pi_i}{(2\pi)^2} e^{\mathrm{i}\bar\pi_i \mu_i} \right) \delta^4(\sum_i k_i) A_{\mathsf{MHV}}(g_1, \dots g_n)$$

$$= \int \prod_i \frac{d^2\bar\pi_i}{(2\pi)^2} \frac{d^4x}{(2\pi)^4} e^{\mathrm{i}\sum_i \bar\pi_i^{\dot A}\mu_{i,\dot A} + \pi_i^A x_{A\dot A}\bar\pi^{\dot A}} A_{\mathsf{MHV}}(g_1, \dots g_n)$$

$$= \int \frac{d^4x}{(2\pi)^4} \prod_i \delta^4(x_{A\dot A}\pi_i^A + \mu_{i,\dot A}) A_{\mathsf{MHV}}(g_1, \dots g_n)$$

Conjectures:

- All Yang-Mills ($\mathcal{N} = 4 \dots$) amplitudes lie on curves in twistor space, determined by $\#$ of negative helicities and loops
- Can be computed in string theory on twistor space

Lagrangian for physical polarizations of gluons

$$\mathcal{L}^{(2)} + \mathcal{L}^{(3)}_{++-} + \mathcal{L}^{(3)}_{+--} + \mathcal{L}^{(4)}_{++--}$$

$\Rightarrow$ all interactions but $\mathcal{L}^{(3)}_{++-}$ of MHV type

Canonical transformation to eliminate $\mathcal{L}^{(3)}_{++-}$. $\Rightarrow$ generates MHV-type vertices: (Mansfield 05)

$$\mathcal{L}^{(3)}_{++-} + \mathcal{L}^{(3)}_{+--} + \mathcal{L}^{(4)}_{++--} \Rightarrow \sum_n \mathcal{L}^{(n)}_{+\cdots+--}$$

Explicit solution (Ettle,Morris 06)

$$A_z(p) = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \widetilde{dk_i} \, \frac{(g\sqrt{2})^{n-1} \langle qp \rangle^2}{\langle q1 \rangle \langle 12 \rangle \ldots \langle (n-1)n \rangle \langle qn \rangle} B(k_1) \ldots B(k_n)$$

Similar solution for $\partial_+ A_{\bar{z}} \sim \sum_n B_1 \ldots \partial_+ \bar{B} \ldots B_n$

Application to massive scalars (R. Boels, CS 07)

- Lagrangian in light-cone gauge

$$\mathcal{L}^{(2)}(\bar{\phi}\phi) + \mathcal{L}^{(3)}(\bar{\phi}A_z\phi) + \mathcal{L}^{(3)}(\bar{\phi}A_{\bar{z}}\phi) + \mathcal{L}^{(4)}(\bar{\phi}A_z A_{\bar{z}}\phi) + \mathcal{L}^{(4)}(\bar{\phi}\phi\bar{\phi}\phi)$$

- eliminate $\mathcal{L}^{(3)}(\bar{\phi}A_z\phi)$ by transformation for massless scalars

$$\phi(p) = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \widetilde{dk_i}\, \frac{(g\sqrt{2})^{n-1}\langle qn\rangle}{\langle q1\rangle\langle 12\rangle\dots\langle (n-1)n\rangle} B(k_1)\dots B(k_{n-1})\xi(k_n)$$

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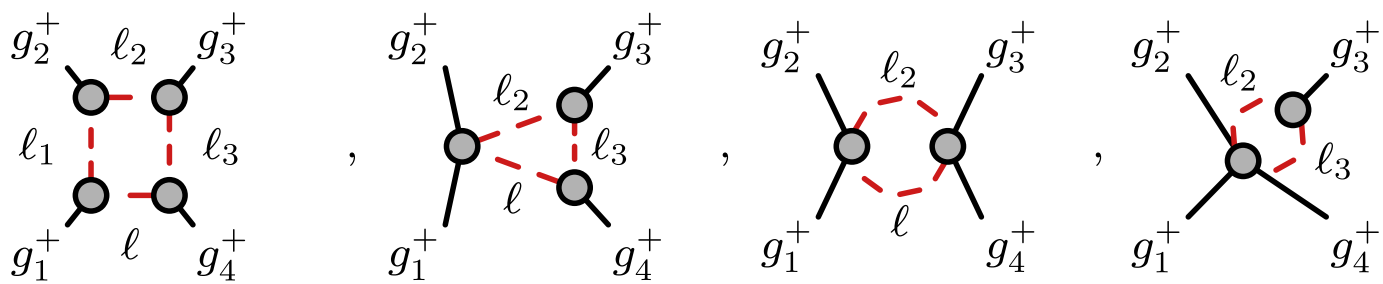
- but mass term not invariant:

$$-m^2 \bar{\phi}(p)\phi(-p) = \sum_{n=2}^{\infty} \int \prod_{i=1}^{n} \widetilde{dp_i} \mathcal{V}_{1,\ldots,n} \, \bar{\xi}(k_1) B(k_2) \ldots B(k_{n-1}) \xi(k_n)$$

$\Rightarrow$ new CSW-vertex $\qquad \mathcal{V}_{1,\ldots,n} = (g\sqrt{2})^{n-2} \dfrac{-m^2 \langle 1n \rangle}{\langle 12 \rangle \ldots \langle (n-1)n \rangle}$

Same result using Twistor Yang-Mills approach

Topologies for $A(g_1^+, g_2^+, g_3^+, g_4^+)$ in massive CSW rules:



Three-point vertex for g_1^+ vanishes for $|\eta-\rangle = |1-\rangle \Rightarrow$ box vanishes

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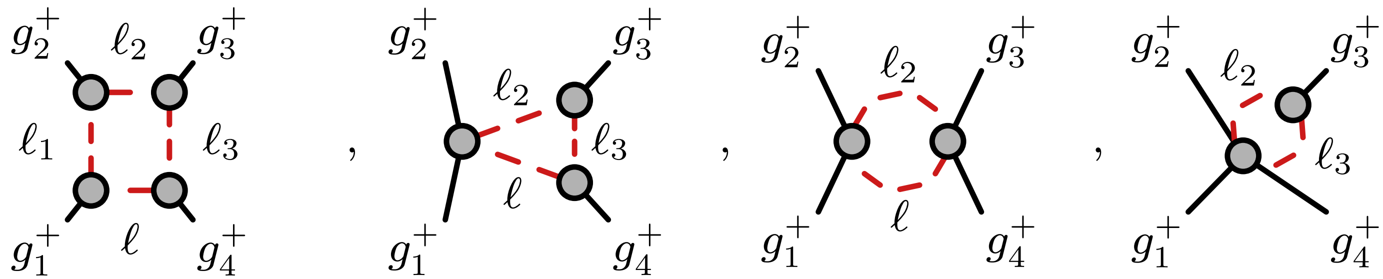

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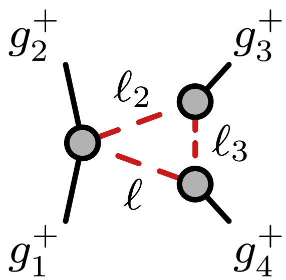
Example for triangle: $d(\ell_i) = \ell_i^2 - \mu^2$



$$= \int \frac{d^D \ell}{(2\pi)^D} \frac{4\mu^6 \langle \ell \ell_2 \rangle \langle \ell_2 \ell_3 \rangle \langle \ell_3 \ell \rangle}{d(\ell) d(\ell_2) d(\ell_3) \langle \ell 1 \rangle \langle 12 \rangle \langle 2 \ell_2 \rangle \langle \ell_2 3 \rangle \langle 3 \ell_3 \rangle \langle \ell_3 4 \rangle \langle 4 \ell \rangle}$$

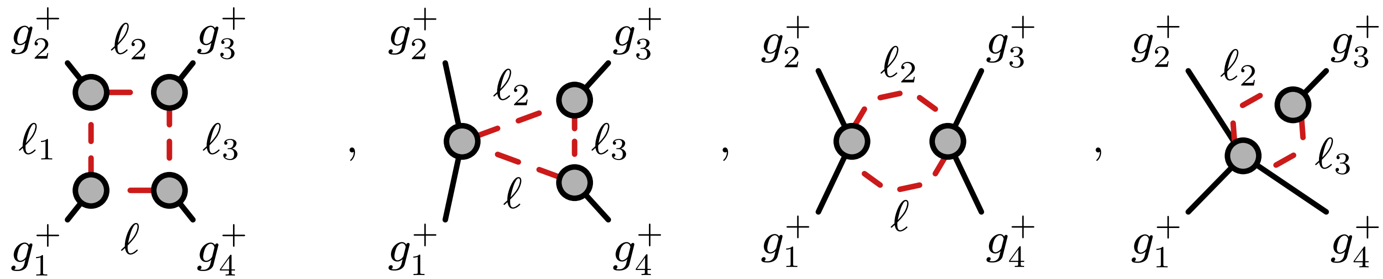
$$= \frac{4[12]}{\langle 12 \rangle \langle 34 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mu^4}{d(\ell) d(\ell_2) d(\ell_3) 2(\ell \cdot k_1)} \frac{-\mu^2 \langle 1 - | \not{k}_3 \not{k}_4 | 1- \rangle}{\langle 3 - | \not{\ell}_3 | 1 - \rangle \langle 4 - | \not{\ell} | 1 - \rangle}$$

Cancel spurious poles using (Brandhuber, Spence, Travaglini 06)

$$\mu^2 \langle 1 + | \not{k}_3 \not{k}_4 | 1 - \rangle = [34] \langle 1 + | \not{\ell}_3 | 3 + \rangle \langle 4 - | \not{\ell}_4 | 1 - \rangle + \sim d(\ell_{i-1}), d(\ell_i), d(\ell_{i+1})$$

CSW rules for massive scalars

Topologies for $A(g_1^+, g_2^+, g_3^+, g_4^+)$ in massive CSW rules:



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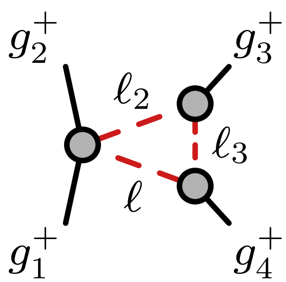

$$= \frac{-4[12][34]}{\langle 12\rangle \langle 34\rangle} \int \frac{d^D\ell}{(2\pi)^D} \frac{\mu^4}{d(\ell)d(\ell_2)d(\ell_3)2(\ell\cdot k_1)} + \text{bubbles}$$

CSW rules for massive scalars

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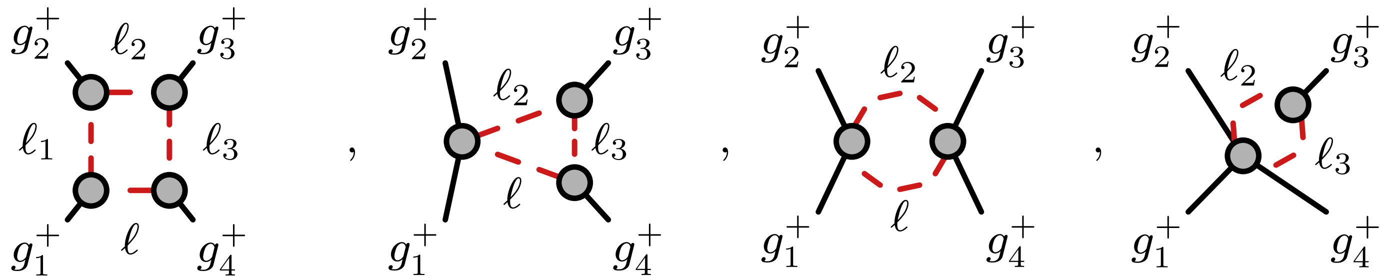

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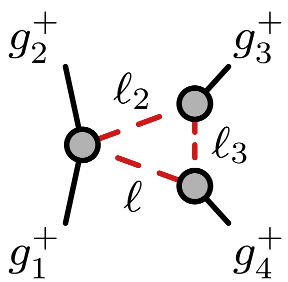

$$= \frac{-4[12][34]}{\langle 12\rangle\langle 34\rangle}\int\frac{d^D\ell}{(2\pi)^D}\frac{\mu^4}{d(\ell)d(\ell_2)d(\ell_3)2(\ell\cdot k_1)} + \text{bubbles}$$

Adding other diagrams: bubbles cancel $\Rightarrow$ known result:

$$A_4(g_1^+, g_2^+, g_3^+, g_4^+) = \frac{4[12][34]}{\langle 12\rangle\langle 34\rangle}\int\frac{d^4\ell}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\frac{\mu^4}{(\ell^2-\mu^2)(\ell_2^2-\mu^2)(\ell_3^2-\mu^2)(\ell_4^2-\mu^2)}$$

Similar pattern for higher point amplitudes, g^-! (Glover, Williams 08)