NLO QCD corrections to $VVjj$ production at the LHC

DESY-HU seminar
Francisco Campanario, Matthias Kerner, LE Duc Ninh, Dieter Zeppenfeld | June 19 2014
Outline

- $VVjj$ production (with leptonic decays) at the LHC: motivation
- $VVjj \ @ \ NLO \ QCD$ : some calculational details
- Phenomenological results: $WZ, W\gamma, ZZ, W^+W^+$
- Summary
VVjj production at the LHC: why?

- Motivation:
  - Sensitive to $VV \rightarrow VV$ scattering, quartic quage-boson couplings.
  - Important background for new physics searches.
**VVjj production at the LHC: why?**

- **Motivation:**
  - Sensitive to $VV \rightarrow VV$ scattering, quartic quage-boson couplings.
  - Important background for new physics searches.

- **Classification at LO: 2 mechanisms**
  - EW mechanism (vector boson fusion, VBF): $\sigma_{EW} \propto \alpha^6$
  - QCD mechanism: $\sigma_{QCD} \propto \alpha_s^2\alpha^4$

- Interference: color and kinematically suppressed.
  - $\sim$ can be neglected for a-few-percent precision measurements at the LHC.
What have been done at NLO QCD?

- EW mechanism (VBF): consider QCD corrections to each quark lines separately
  \[ \rightarrow \] pentagons at most.
  - \( W^+ W^- jj \): [Jager, Oleari, Zeppenfeld, 2006]
  - \( ZZjj \): [Jager, Oleari, Zeppenfeld, 2006]
  - \( W^\pm Zjj \): [Bozzi, Jager, Oleari, Zeppenfeld, 2007]
  - \( W^\pm W^\pm jj \): [Jager, Oleari, Zeppenfeld, 2009], [Denner, Hosekova, Kallweit, 2012]
  - \( W^\pm \gamma jj \): [Campanario, Kaiser, Zeppenfeld, 2013]

- QCD mechanism: two quark lines are not independent
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  - \( W^\pm W^\pm jj \): [Melia, Melnikov, Rontsch, Zanderighi, 2010], [Campanario, Kerner, LDN, Zeppenfeld, 2013]
  - \( W^\pm W^\pm jj \): [Melia, Melnikov, Rontsch, Zanderighi, 2011], [Greiner, Heinrich, Mastrolia, Ossola, et al, 2012]
  - \( W^\pm Zjj \), \( W^\pm \gamma jj \), \( ZZjj \): [Campanario, Kerner, LDN, Zeppenfeld, 2013, 2014]

Almost all processes (notable exception: QCD \( W^+ W^- jj \)) are included in VBFNLO program.

Same-sign \( W^\pm W^\pm jj \) are special: clean signal, small background (no \( t\bar{t} \)), simplest calculation.
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- **QCD mechanism: two quark lines are not independent \( \Rightarrow \) hexagons at most.**
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  - \( W^\pm Zjj, W^\pm \gamma jj, ZZjj \): [Campanario, Kerner, LDN, Zeppenfeld, 2013, 2014]

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- Same-sign \( W^\pm W^{\pm} jj \) are special: clean signal, small background (no \( t\bar{t} \)), simplest calculation.
**Same-sign $W^\pm W^\pm jj$@ATLAS, 2014**

**ATLAS**

20.3 fb\(^{-1}\), $\sqrt{s} = 8$ TeV

$m_{jj} > 500$ GeV

Data 2012

Syst. Uncertainty

<table>
<thead>
<tr>
<th></th>
<th>$W^\pm W^\pm jj$ Electroweak</th>
<th>$W^\pm W^\pm jj$ Strong</th>
<th>Prompt</th>
<th>Conversions</th>
<th>Other non-prompt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prompt</td>
<td>3.0 ± 0.7</td>
<td>6.1 ± 1.3</td>
<td>2.6 ± 0.6</td>
<td>2.2 ± 0.5</td>
<td>4.2 ± 1.0</td>
</tr>
<tr>
<td>Conversions</td>
<td>3.2 ± 0.7</td>
<td>2.4 ± 0.8</td>
<td>−</td>
<td>2.1 ± 0.5</td>
<td>1.9 ± 0.7</td>
</tr>
<tr>
<td>Other non-prompt</td>
<td>0.61 ± 0.30</td>
<td>1.9 ± 0.8</td>
<td>0.41 ± 0.22</td>
<td>0.50 ± 0.26</td>
<td>1.5 ± 0.6</td>
</tr>
<tr>
<td>$W^\pm W^\pm jj$ Strong</td>
<td>0.89 ± 0.15</td>
<td>2.5 ± 0.4</td>
<td>1.42 ± 0.23</td>
<td>0.25 ± 0.06</td>
<td>0.71 ± 0.14</td>
</tr>
<tr>
<td>$W^\pm W^\pm jj$ Electroweak</td>
<td>3.07 ± 0.30</td>
<td>9.0 ± 0.8</td>
<td>4.9 ± 0.5</td>
<td>2.55 ± 0.25</td>
<td>7.3 ± 0.6</td>
</tr>
<tr>
<td>Total background</td>
<td>6.8 ± 1.2</td>
<td>10.3 ± 2.0</td>
<td>3.0 ± 0.6</td>
<td>5.0 ± 0.9</td>
<td>8.3 ± 1.6</td>
</tr>
<tr>
<td>Total predicted</td>
<td>10.7 ± 1.4</td>
<td>21.7 ± 2.6</td>
<td>9.3 ± 1.0</td>
<td>7.6 ± 1.0</td>
<td>15.6 ± 2.0</td>
</tr>
</tbody>
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**Events**

<table>
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<th>VBS Region</th>
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<tr>
<td>Data</td>
<td>12</td>
<td>26</td>
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Total number of events: 50 (INC) and 34 (VBS)

Data vs. background (significance): 4.5σ (INC, signal = QCD + EW), 3.6σ (VBS, signal = EW)
Same-sign $W^{\pm}W^{\pm}jj@ATLAS$, 2014

- Total number of events: 50 (INC) and 34 (VBS)
- Data vs. background (significance): 4.5$\sigma$ (INC, signal = QCD + EW), 3.6$\sigma$ (VBS, signal = EW)

| $|\Delta y_{jj}|$ | Inclusive Region | VBS Region |
|---|---|---|
| | $e^{\pm}e^{\pm}$ | $e^{\pm}\mu^{\pm}$ | $\mu^{\pm}\mu^{\pm}$ | $e^{\pm}e^{\pm}$ | $e^{\pm}\mu^{\pm}$ | $\mu^{\pm}\mu^{\pm}$ |
| Prompt | 3.0 ± 0.7 | 6.1 ± 1.3 | 2.6 ± 0.6 | 2.2 ± 0.5 | 4.2 ± 1.0 | 1.9 ± 0.5 |
| Conversions | 3.2 ± 0.7 | 2.4 ± 0.8 | - | 2.1 ± 0.5 | 1.9 ± 0.7 | - |
| Other non-prompt | 0.61 ± 0.30 | 1.9 ± 0.8 | 0.41 ± 0.22 | 0.50 ± 0.26 | 1.5 ± 0.6 | 0.34 ± 0.19 |
| $W^{\pm}W^{\pm}jj$ Strong | 0.89 ± 0.15 | 2.5 ± 0.4 | 1.42 ± 0.23 | 0.25 ± 0.06 | 0.71 ± 0.14 | 0.38 ± 0.08 |
| $W^{\pm}W^{\pm}jj$ Electroweak | 3.07 ± 0.30 | 9.0 ± 0.8 | 4.9 ± 0.5 | 2.55 ± 0.25 | 7.3 ± 0.6 | 4.0 ± 0.4 |
| Total background | 6.8 ± 1.2 | 10.3 ± 2.0 | 3.0 ± 0.6 | 5.0 ± 0.9 | 8.3 ± 1.6 | 2.6 ± 0.5 |
| Total predicted | 10.7 ± 1.4 | 21.7 ± 2.6 | 9.3 ± 1.0 | 7.6 ± 1.0 | 15.6 ± 2.0 | 6.6 ± 0.8 |
| Data | 12 | 26 | 12 | 6 | 18 | 10 |
VBF vs. QCD background: a Higgs example

LHC $\sqrt{s} = 14$ TeV, $p_{T_j} \geq 20$ GeV, $|\eta_j| < 5$, $\Delta R_{jj} \geq 0.7$, $|\eta_\gamma| < 2.5$, $\Delta R_{j \gamma} \geq 0.7$;

$\eta_{j,\text{min}} + 0.7 < \eta_\gamma < \eta_{j,\text{max}} - 0.7$, $\eta_{j_1} \eta_{j_2} < 0$

The two tagging jets are more separated in VBF than in QCD background!
\[ W^\pm W^\pm jj @ \text{LO} \]

- EW: \( \sigma_{\text{EW}} \propto \alpha^6 \)

- QCD: \( \sigma_{\text{QCD}} \propto \alpha_s^2 \alpha^4 \)

- Interference: \( \sigma_{\text{Int}} \propto \alpha_s \alpha^5 \), maximal for same-sign \( W^\pm W^\pm jj \) due to the absence of gluon-induced processes and only left chiral quarks and leptons involve.
\( W^\pm W^\pm jj@LO: \) distributions

\[ pp \rightarrow e^+\nu_e \mu^+\nu_\mu jj+X \]

- \( p_T,j > 20 \text{ GeV}, \quad |\eta_j| < 4.5, \quad R_{jj}^{\text{anti}-k_t} = 0.4, \quad R_{jl} > 0.4; \)
- \( p_T,l > 20 \text{ GeV}, \quad |\eta_l| < 2.5, \quad R_{ll} > 0.4, \quad p_T > 30 \text{ GeV}. \)

Dynamic scale: \( \mu_F = \mu_R = \mu_0 = \left( \sum_{\text{partons}} p_T,i + \sum_i \sqrt{p^2_{T,W_i} + m^2_{W_i}} \right) / 2, \)

\( p_T,W \) (\( m_W \)) are reconstructed from leptons.

Interference effects: \( \leq 15\% \) in relevant phase space region.
$pp \rightarrow VVjj$: QCD mechanism at NLO QCD
True $VVjj$ processes

Q: What is the NLO QCD correction to this process?

A: Before answering this question, some classifications are needed.
LO: subprocesses

- 2-quark lines [4q]: \( q_1 + q_2 \rightarrow q_3 + q_4 + (WV) \)

- 76 subprocesses (2 generations).
- 12 crossings.

- 1-quark line [2g]: \( q_1 + q_2 \rightarrow g + g + (WV) \)

- 14 subprocesses (2 generations).
- 7 crossings.

- 4 QCD gauge invariant groups: \( 4q(W), 4qWV, 2g(W), 2gWV \).
\[ d\sigma_{NLO} = d\sigma_{2\to N}^{\text{virt}} + d\sigma_{2\to N+1}^{\text{real}}. \]

- Both terms are IR divergent. The sum is finite for IR-safe observables (e.g. jet distributions)
NLO calculation: theory vs. practice

\[ d\sigma_{NLO} = d\sigma_{2\to N}^{\text{virt}} + d\sigma_{2\to N+1}^{\text{real}}. \]

- Both terms are IR divergent. The sum is finite for IR-safe observables (e.g. jet distributions).
- Real: IR divergences can be separated using Catani-Seymour dipole subtraction method.
- Virtual: 1-loop amplitude is, unfortunately, much more complicated than tree-level one. Use Feynman-diagram and tensor reduction methods. The most difficult part.
Complexity overview

- LO: 4840 \([WZjj, \text{two generations}]\)
- NLO real emission: 79784 \([WZjjj, \text{two generations}]\)
- NLO virtual: 116896 (up to 6-point rank 5) \([WZjj, \text{two generations}]\)
- Many subprocesses: most complicated \(ZZjjj\) (real emission) has 275 subprocesses (with b quarks)

- This calculation can be done with:
  - good classifications: effective currents
  - \(V \rightarrow l_1 l_2, V \rightarrow l_1 l_2 l_3 l_4\), building blocks
  - use crossing symmetry (to a minimum extent to reduce computing time):
    - subprocesses are not completely independent (diagrams share common parts).
- Two independent calculations:
  - manual implementation using VBFNLO framework
  - more automated approach using HELAS/MadGraph, FeynArts, FormCalc
- Loop integrals: 2 different codes, in-house LoopInts and VBFNLO implementation.
- Numerical instabilities in the virtual part: difficult \(\Rightarrow\) gauge tests
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This calculation can be done with:

- good classifications: effective currents \(V \rightarrow l_1 l_2, V \rightarrow l_1 l_2 l_3 l_4\), building blocks (hexlines, penlines, \ldots)
- use crossing symmetry (to a minimum extent to reduce computing time): subprocesses are not completely independent (diagrams share common parts).
- two independent calculations (Fact: manual calculations are buggy):
  - manual implementation using VBFNLO framework
  - more automated approach using HELAS/MadGraph, FeynArts, FormCalc
  - loop integrals: 2 different codes, in-house LoopInts and VBFNLO implementation.

- numerical instabilities in the virtual part: difficult \(\sim\) gauge tests
VBFNLO framework

- tree-level amplitudes, similar to MadGraph approach, but better optimization:

\[ A = \langle f | i \rangle, \quad |i\rangle = \Gamma |i_1\rangle, \quad \langle f_1 | \Gamma = \langle f |, \]

where \(|i\rangle\) (or \(<f|\)) are just products of spinors, \(\gamma^\mu\), propagators and couplings. We have to identify the common (occurring in many diagrams and subprocesses) \(|i\rangle\), a part of a Feynman diagram, then calculate and save them at the beginning. This step is difficult to automatize \(\Rightarrow\) manual implementation.

- one-loop amplitudes: built from universal (occurring in many processes, with generic couplings) building blocks, each is a group of Feynman diagrams with a fixed number and a fixed order of external particles. E.g. a penline:

- Effective currents: \(J^\mu, \epsilon^\mu, p^\mu \Rightarrow\) efficient gauge tests.
Adding leptonic decays directly does not work: code for \( j_1 j_2 \rightarrow l_1 l_2 l_3 j_3 j_4 \) cannot be generated (Mathematica+FORM takes forever).

Tricks: generate Fortran code for semi-on-shell \( (p^2 \neq M_V^2) \) \( VVjj \) and either replace \( \epsilon^{\mu} \rightarrow J^{\mu} \) or cut the V propagators.

Problems: Very long expression: use FormCalc6, for virtual \( WZjj \), process \( ggqqVV \): hex = 205M, pen = 84M, rest = 26M, tensor reduction not counted.

Takes a lot of time to compile, code is slow, divide into small pieces: \( \sim \) only useful to check at amplitude level.

Bottom line: automated tools is faster in generating the amplitude, but much slower in providing cross section.
Numerical instabilities: gauge test

Passarino-Veltman: \( D_{ijkl} = N / \det(G)^4 \)
\( \sim \) small Gram-det problem.

\[
B^N = T^N_{\mu} \epsilon^\mu(k), \quad \epsilon^\mu \rightarrow k^\mu,
\]

\[
1 \frac{1}{q + k} \frac{1}{q + k} = 1 \frac{1}{q} - \frac{1}{q + k},
\]

\[
T^N_{\mu} k^\mu - A^{N-1} = 0,
\]

\[
Q = 1 - \frac{A^{N-1}}{T^N_{\mu} k^\mu},
\]

if \( Q < \varepsilon \) : accept the point.

if \( Q > \varepsilon \) : use quadruple precision,
\( \sim \) \( Q < \varepsilon \) ? accept or discard points.

\( pp \rightarrow e^+ e^- \mu^+ \mu^- jj+X \)

QCD virtual

Band: \( 2\sigma \) deviation off \( \varepsilon = 10^{-2} \)
Computing time

With an Intel i5-3470 computer with one core and using the compiler Intel-ifort version 12.1.0, to get a statistical error of 1%:

- $W^+ W^+ jj$: 30 minutes.
- $W^+ Zjj$: 2.5 hours.
- $W^+ \gamma jj$: 3 hours. Importance: two integrals in two different regions on-shell $W^+ \rightarrow l^+ \nu l$ and on-shell $W^+ \rightarrow l^+ \nu l \gamma$. Two different Breit-Wigner mappings.
- $ZZjj$: 3.5 hours.
partons $\rightarrow$ jets: anti-$k_t$ algorithm with a cone radius of $R = 0.4$.

Inclusive cuts:

\[ p_T,j > 20 \text{ GeV}, \quad |y_j| < 4.5, \quad R_{jj}^{\text{anti-}k_t} = 0.4, \quad R_{jl} > 0.4; \]
\[ p_T,l > 20 \text{ GeV}, \quad |y_l| < 2.5, \quad R_{ll} > 0.4, \quad M_{l+\gamma} > 15 \text{ GeV}; \]
\[ p_T,\gamma > 30 \text{ GeV}, \quad |y_\gamma| < 2.5, \quad R_{l\gamma} > 0.4, \quad R_{j\gamma} > 0.7, \quad p_T > 30 \text{ GeV}. \]

Final-state real photon: Frixione smooth cone isolation cut. Events are accepted if

\[ \sum_{i \in \text{partons}} p_T,i \theta(R - R_{\gamma i}) \leq p_T,\gamma \frac{1 - \cos R}{1 - \cos \delta_0} \quad \forall R < \delta_0 \]

with $\delta_0 = 0.7 \implies$ events with a soft gluon are accepted (IR safety).
**VVjj setup: scale choice**

\[ \mu_{HT} = \left( \sum_{\text{partons}} p_{T,i} + \sum_{i=1}^{2} E_{T,V_i} \right) / 2 \]  

(1)

\[ \mu'_{HT} = \left( \sum_{\text{jet}} p_{T,i}e^{y_i - y_{12}} + \sum_{i=1}^{2} E_{T,V_i} \right) / 2 \]  

(2)

\[ \mu_{ET} = \left[ E_{T(jj)} + E_{T(VV)} \right] / 2 \]  

(3)

with \( y_{12} = (y_1 + y_2)/2 \) and \( E_{T(jj)} = (m_{jj}^2 + p_{jj}^2)^{1/2} \), and

\[ m_{jj}^2 \approx 2p_{T,j_1}p_{T,j_2} \left[ \cosh(\Delta y_{jj}) - \cos(\Delta \phi_{jj}) \right] \]

Remarks:

- For large \( \Delta y_{jj} \), then \( \sum p_T \ll m_{jj} \Rightarrow \mu_{HT} \) too small.
- \( \mu'_{HT} \) and \( \mu_{ET} \) interpolate between \( \sum p_T \) and \( m_{jj} \), for small and large \( \Delta y_{jj} \), respectively.
$\Delta y_{jj}$ distribution: scale choice

$$\mu_0 = \left( \sum_{\text{partons}} p_{T,i} + \sum_i E_{T,W_i} \right)/2; \quad \mu'_0 = \left( \sum_{\text{jets}} p_{T,i} e^{y_i - y_{12}} + \sum_i E_{T,W_i} \right)/2$$
Overview

Leptonic decays
Off-shell effects
Virtual photon

Leptonic decays
Off-shell effects
Real photon

Leptonic decays
Off-shell effects
Virtual photons

Leptonic decays
Off-shell effects
Real photon

W^+ Z

W^+ γ

ZZ

W^+ W^+
Scale dependence

\( \sigma \) vs. \( \xi \) for different processes:

- \( pp \rightarrow l^+ \nu_l l^+ \nu_l jj+X \) (W+Z)
- \( pp \rightarrow l^+ \nu_l \gamma jj+X \) (W+\gamma)
- \( pp \rightarrow l^+ l^+ l^- l^- jj+X \) (ZZ)
- \( pp \rightarrow l^+ \nu_l l^+ \nu_l jj+X \) (W+W+)
$m_{jj}$ distributions

$pp \rightarrow l^+ l^- j j + X$

$W^+ Z$

$W^+ \gamma$

$W^+ W^+$

$Z Z$

$Z Z$

$Z Z$
Summary

- $VVjj$ is a special class of processes: sensitive to $VV \rightarrow VV$ scatterings, quartic gauge couplings, background for new physics searches, ...
- Two mechanisms: EW (VBF), QCD, interference effects are small (at most 15% for $W^+ W^+ jj$, expected $\leq 5\%$ for the others).
- NLO QCD corrections for all $VVjj$ are under control.
- A good scale choice has to take $m_{jj}$ into account, $p_T$ alone is not enough.
- Virtual amplitude: difficult part, gauge tests are good to deal with numerical instabilities.
- The program VBFNLO 2.7.0: includes $WZjj$, $W\gamma jj$, $W^{\pm} W^{\pm} jj$. $ZZjj$ will be in the next release (or upon request).

Thank you!
$\rho_{T,j_1}$ distributions

$pp \rightarrow l^+\nu\ l^+\nu\ jj + X$

$W^+ Z$

$W^+ \gamma$

$Z Z$

$W^+ W^+$
$m_{VV}$ distributions

$pp \rightarrow l'\nu l'\nu jj + X$

$W^+Z$

$W^+W^+$

$pp \rightarrow l'\nu l'\nu jj + X$

$Z\gamma$
But, large $K$ factor at small $m_{jj}$ always there: new $q_{\text{final}} \rightarrow j_1j_2$ configuration at NLO.
Dipole subtraction method

\[ \int_{N+1} d\sigma_{N+1}^{\text{real}}(p) J^{N+1}(p) = \int_{N+1} \left[ (d\sigma_{N+1}^{\text{real}}(p) J^{N+1}(p) - \sum_{i,j} S_{ij}^{N}(\tilde{p}_{ij}) J_{ij}^{N}(\tilde{p}_{ij}) \right] \]

\[ + \int_{N+1} \sum_{i,j} S_{ij}^{N}(\tilde{p}_{ij}) J_{ij}^{N}(\tilde{p}_{ij}) \]

PK+I

PK = \int_0^1 dx \int_N \sum_{j \neq a} S_{aj}^{N}(x, p) J_{a}^{N}(x, p) + (a \leftrightarrow b) \]

I = \int_N \sum_{i,j} S_{ij}^{N}(p) J^{N}(p).
Dipole subtraction method

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+ \int_{N+1} \sum_{i,j} S_{ij}^{N}(\bar{p}_{ij})J_{ij}^{N}(\bar{p}_{ij}) \]

PK + I

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I = \int_{N} \sum_{i,j} S_{ij}^{N}(p)J^{N}(p). \]

The Jet function: a cut, a histogram, PDF(\(Q\)), \(\alpha_s(\bar{Q})\), ...

- IR safety: \(J^{N} \rightarrow J^{N+1}\) in the IR singular limits.
- An easy mistake: in Eq. (4), set \(\alpha_s^{N}(\bar{Q}) = \alpha_s^{N+1}(Q)\): the result is finite, BUT wrong (almost correct) because the integrated part \(\neq\) the subtraction part.
- If we do \(J^{N} = J^{N+1}\) in Eq. (4), then PK term will get more complicated. [arXiv: 0802.1405]
Tree-level, virtual and real matching

Partonic level:

\[
\begin{align*}
& d\sigma_{\text{soft}}^1 + d\sigma_{\text{soft}}^\text{virt} = 0, \\
& d\sigma_{\text{coll}}^1 + d\sigma_{\text{coll}}^\text{virt} = 0, \\
& d\sigma_{\text{coll}}^{\text{PK}} \neq 0.
\end{align*}
\]
Tree-level, virtual and real matching

- Partonic level:
  \[
  d\sigma_{\text{soft}}^{I} + d\sigma_{\text{soft}}^{\text{virt}} = 0,
  
  d\sigma_{\text{coll}}^{I} + d\sigma_{\text{coll}}^{\text{virt}} = 0,
  
  d\sigma_{\text{coll}}^{\overline{\text{PK}}} \neq 0.
  
- Hadronc level:
  \[
  d\sigma^{\text{NLO}} = d\sigma^{\text{tree}} + d\sigma^{\text{virt}} + d\sigma^{\text{real}},
  
  \text{PDF}^{\text{NLO}} \otimes d\sigma^{\text{NLO}} = \int dx_1 \int dx_2 f_a(x_1, Q)f_b(x_2, Q)d\sigma^{\text{NLO}} + \delta_{\text{PDF}}(d\sigma^{\text{tree}}, 1/\epsilon),
  
  d\sigma^{\overline{\text{PK}}} \equiv d\sigma^{\overline{\text{PK}}}(1/\epsilon) + \delta_{\text{PDF}}(d\sigma^{\text{tree}}, 1/\epsilon) : \text{ finite}
Tree-level, virtual and real matching

- Partonic level:

\[ \begin{align*}
    d\sigma_{\text{soft}}^{l} + d\sigma_{\text{soft}}^{\text{virt}} &= 0, \\
    d\sigma_{\text{coll}}^{l} + d\sigma_{\text{coll}}^{\text{virt}} &= 0, \\
    d\sigma_{\text{coll}}^{PK} &\neq 0.
\end{align*} \]

- Hadronic level:

\[ \begin{align*}
    d\sigma_{\text{NLO}} &= d\sigma_{\text{tree}} + d\sigma_{\text{virt}} + d\sigma_{\text{real}}, \\
    \text{PDF}^{\text{NLO}} \otimes d\sigma_{\text{NLO}} &= \int dx_1 \int dx_2 f_a(x_1, Q)f_b(x_2, Q)d\sigma_{\text{NLO}}^{\text{NLO}} + \delta_{\text{PDF}}(d\sigma_{\text{tree}}^{\text{tree}}, 1/\epsilon), \\
    d\sigma_{\text{PK}} &\equiv d\sigma_{\text{PK}}^{\text{PK}}(1/\epsilon) + \delta_{\text{PDF}}(d\sigma_{\text{tree}}^{\text{tree}}, 1/\epsilon) : \text{ finite}
\end{align*} \]

- Default matching: PDF and \( d\sigma_{\text{real}} \) are defined in the conventional dimensional-regularization scheme (CDR), AND \( \overline{\text{MS}} \) scheme.
Tree-level, virtual and real matching

- Partonic level:

\[
\begin{align*}
    d\sigma^I_{\text{soft}} + d\sigma^{\text{virt}}_{\text{soft}} &= 0, \\
    d\sigma^I_{\text{coll}} + d\sigma^{\text{virt}}_{\text{coll}} &= 0, \\
    d\sigma^{PK}_{\text{coll}} &\neq 0.
\end{align*}
\]

- Hadronc level:

\[
\begin{align*}
    d\sigma^\text{NLO} &= d\sigma^\text{tree} + d\sigma^{\text{virt}} + d\sigma^{\text{real}}, \\
    \text{PDF}^\text{NLO} \otimes d\sigma^\text{NLO} &= \int dx_1 \int dx_2 f_a(x_1, Q)f_b(x_2, Q)d\sigma^\text{NLO} + \delta_{\text{PDF}}(d\sigma^\text{tree}, 1/\epsilon), \\
    d\sigma^{PK} &\equiv d\sigma^{PK}(1/\epsilon) + \delta_{\text{PDF}}(d\sigma^\text{tree}, 1/\epsilon): \text{ finite}
\end{align*}
\]

- Default matching: PDF and \(d\sigma^{\text{real}}\) are defined in the conventional dimensional-regularization scheme (CDR), AND \(\overline{\text{MS}}\) scheme.

- Virtual: use ’t Hooft-Veltman (HV) scheme (external momenta in 4D)

\[
    d\sigma^{\text{virt}}_{\text{HV}} = d\sigma^{\text{virt}}_{\text{CDR}}, \quad \alpha^\text{HV}_s = \alpha^\text{CDR}_s.
\]

- Dimensional regularization scheme (DRS) independence [Catani, Seymour, Trócsányi 1997]:

\[
    d\sigma^\text{tree} + d\sigma^{\text{virt}} + d\sigma^I: \neq \text{DRS at partonic level!}
\]

- 4 flavors (no b/t) or 5 flavors (b and t loop): \(\alpha_s\), PDF, tree, virtual, real.
Subtraction term: implementation for $2 \to N + 1$

$$\sigma^{N+1} = \sum_{\text{subproc}} \int_{N+1} \left[ d\sigma^{\text{real}}_{N+1}(p) - \sum_{i,j,k} S^N_{ijk}(\tilde{p}_{ijk}) \right]$$

- Do for one subprocess, then use crossing: 27 crossings, caching (M. Kerner).
Subtraction term: implementation for $2 \rightarrow N + 1$

\[ \sigma^{N+1} = \sum_{\text{subproc}} \int_{N+1} \left[ \frac{d\sigma^\text{real}_{N+1}(p)}{d\sigma^\text{real}_{N+1}(p)} - \sum_{i,j,k} S^N_{ijk}(\tilde{p}_{ijk}) \right] \]

Do for one subprocess, then use crossing: 27 crossings, caching (M. Kerner).

\[ \sigma^{N+1} = \int_{N+1} \left[ \sum_{\text{subproc}} d\sigma^\text{real}_{N+1}(p) - \sum_{i,j,k} \sum_{\text{subproc}} S^N_{ijk}(\tilde{p}_{ijk}) \right] \]
Subtraction term: implementation for $2 \rightarrow N + 1$

$$\sigma^{N+1} = \sum_{\text{subproc}} \int_{N+1} \left[ d\sigma_{N+1}^{\text{real}}(p) - \sum_{i,j,k} S_{ijk}^N(\tilde{p}_{ijk}) \right]$$

- Do for one subprocess, then use crossing: 27 crossings, caching (M. Kerner).

$$\sigma^{N+1} = \int_{N+1} \left[ \sum_{\text{subproc}} d\sigma_{N+1}^{\text{real}}(p) - \sum_{i,j,k \text{ subproc}} \sum S_{ijk}^N(\tilde{p}_{ijk}) \right]$$

- Universal kinematic mapping $M(i, j; k)$: emitter ($p_i$), unresolved ($p_j$), spectator ($p_k$) with $i, k \in \{1, N + 1\}; j \in \{3, N + 1\}$.
  - final-final ($i, k > 2$): $M_{ff}(i, j; k) = M_{ff}(j, i; k) \neq M_{ff}(k, j; i)$. $\leadsto N + 1$
  - final-initial ($i > 2, a \leq 2$): $M_{fi}(i, j; a) = M_{fi}(j, i; a)$. $\leadsto N(N + 1)$
  - initial-final ($a \leq 2, k > 2$): $M_{if}(a, j; k) = M_{if}(k, j; a)$. $\leadsto 0$
  - initial-initial ($a \leq 2, b \leq 2$): $M_{ii}(a, j; b) \neq M_{ii}(b, j; a)$. $\leadsto 2(N + 1)$
  - Number of independent mappings: $(N + 1)(N + 3)$.

Subtraction term: implementation for $2 \rightarrow N + 1$

$$\sigma^{N+1} = \sum_{\text{subproc}} \int_{N+1} d\sigma^{\text{real}}_{N+1}(p) - \sum_{i,j,k} S^N_{ijk}(\tilde{p}_{ijk})$$

- Do for one subprocess, then use crossing: 27 crossings, caching (M. Kerner).

$$\sigma^{N+1} = \int_{N+1} \left[ \sum_{\text{subproc}} d\sigma^{\text{real}}_{N+1}(p) - \sum_{i,j,k} S^N_{ijk}(\tilde{p}_{ijk}) \right]$$

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  - final-final ($i, k > 2$): $M_{ff}(i, j; k) = M_{ff}(j, i; k) = M_{ff}(k, j; i). \sim N + 1$
  - final-initial ($i > 2, a \leq 2$): $M_{fi}(i, j; a) = M_{fi}(j, i; a). \sim N(N + 1)$
  - initial-final ($a \leq 2, k > 2$): $M_{if}(a, j; k) = M_{if}(k, j; a). \sim 0$
  - initial-initial ($a \leq 2, b \leq 2$): $M_{ii}(a, j; b) \neq M_{ii}(b, j; a). \sim 2(N + 1)$
  - Number of independent mappings: $(N + 1)(N + 3)$.

- Each QCD dipole: kinematics, splitting (vertex, propagator), Born amplitude matrix (mapped kinematics). A generic Fortran code!

- Input: Born amplitude matrix (helicity, color), color-correlated matrices. Sum over subprocesses is optimized here.

- Check: singular limits (necessary BUT not enough).
PK term: implementation

\[ \sigma_{PK}^N = \int_0^1 dy \int_0^1 dx_1 \int_0^1 dx_2 f_a(x_1)f_b(x_2) \]
\[ \times \left\{ \sum_{a_1} \int_N [PK^{a_1} d\sigma_{a_1 b}(yp_a, \rho_b)] + \sum_{b_1} \int_N [PK^{b_1} d\sigma_{a_1 b}(\rho_a, yp_b)] \right\}, \]
\[ \text{PK}(y) = A(y) + \delta(1 - y)B(y) + [C(y)]_+ D(y), \]
\[ \int_0^1 [C(y)]_+ E(y) dy \equiv \int_0^1 C(y)[E(y) - E(1)] dy, \quad E(y) = D(y)d\sigma(y). \]

- Straight-forward coding: calculate Born amplitudes twice \( d\sigma(y) \) and \( d\sigma(1) \).
PK term: implementation

\[\sigma_{PK}^N = \int_0^1 dy \int_0^1 dx_1 \int_0^1 dx_2 f_a(x_1)f_b(x_2) \]
\[\times \left\{ \sum_{a_1} \int_{N} [PK^{a_1} \sigma_{a_1 b}(yp_a, \rho_p)] + \sum_{b_1} \int_{N} [PK^{b_1} \sigma_{ab_1}(\rho_a, yp_b)] \right\},\]

\[PK(y) = A(y) + \delta(1-y)B(y) + [C(y)]_+ D(y),\]

\[\int_0^1 [C(y)]_+ E(y) dy \equiv \int_0^1 C(y)[E(y) - E(1)] dy, \quad E(y) = D(y)d\sigma(y).\]

- Straight-forward coding: calculate Born amplitudes twice \(d\sigma(y)\) and \(d\sigma(1)\).
- Trick: \(z_1 = x_1 y, z_2 = x_2\) if \(a\) is the emitter; \(z_1 = x_1, z_2 = x_2 y\) if \(b\) is the emitter:

\[\sigma_{PK}^N = \int_0^1 dy \int_0^1 dz_1 \int_0^1 dz_2 \]
\[\times \left\{ \sum_{a_1} \int_{N} d\sigma_{a_1 b}(z_1 z_2) [PK^{a_1} f_a(z_1/y)f_b(z_2)\theta(y - z_1)] \right\} \]
\[+ \sum_{b_1} \int_{N} d\sigma_{ab_1}(z_1 z_2) [PK^{b_1} f_a(z_1)f_b(z_2/y)\theta(y - z_2)] \right\}.\]

- A generic Fortran code. Input: \(\sum_{\text{subproc}} |A_0|^2(c_1, c_2)\) used for subtraction term.
I term: implementation

\[ \sigma_i^N = \int_N \sum_{i\neq j} l_{ij} \otimes d\sigma_{\text{tree}}. \]

- A generic Fortran code. Input: color-correlated squared amplitudes.
- Good check for PKI: \( \alpha \) parameter [Nagy and Trocsanyi, M. Kerner in VBFNLO].
Virtual amplitude

- Include $V$ decays:

$$\epsilon^\mu(k, \lambda) \rightarrow \frac{J^\mu_{\text{eff}}}{(k^2 - M_V^2 + iM_V\Gamma_V)}, \quad \text{Or}$$

\[
g^{\mu\nu} = - \sum_{\lambda=-1,0,1} \epsilon^{\mu}(k, \lambda)\epsilon^{*\nu}(k, \lambda) + \frac{k^\mu k^\nu}{k^2},
\]

\[
k^\mu \bar{F}_1 \gamma_\mu (a + b\gamma_5) F_2 = 0, \quad \text{since} : m_1 = m_2 = 0 \ (\text{no Goldstones}).
\]
Virtual amplitude

- Include $V$ decays:

$$
\epsilon^{\mu}(k, \lambda) \rightarrow J_{\text{eff}}^{\mu}/(k^2 - M_V^2 + iM_V \Gamma_V), \quad \text{Or}
$$

$$
g_{\mu\nu} = - \sum_{\lambda=-1,0,1} \epsilon^{\mu}(k, \lambda) \epsilon^{*\nu}(k, \lambda) + \frac{k^\mu k^\nu}{k^2},
$$

$$
k^\mu \bar{F}_1 \gamma_{\mu}(a + b \gamma_5) F_2 = 0, \quad \text{since} : m_1 = m_2 = 0 \ (\text{no Goldstones}).
$$

- Fermion loop with $\gamma_5$: anomaly free (need both $b$ and $t$)

$$
\Rightarrow 2.5 \text{ generations not OK (we cannot decouple the top quark)}. 
$$
Fermion loops

- Include $V$ decays:

$$
\epsilon^\mu(k, \lambda) \rightarrow J^\mu_{\text{eff}}/(k^2 - M_V^2 + iM_V \Gamma_V), \quad \text{Or}
$$

\begin{align*}
\sum_{\lambda = -1, 0, 1} \epsilon^\mu(k, \lambda) \epsilon^{\nu*}(k, \lambda) + \frac{k^\mu k^\nu}{k^2} &= 0,
\end{align*}

$$
k^\mu \tilde{F}_1 \gamma_\mu (a + b \gamma_5) F_2 = 0, \quad \text{since} \ : m_1 = m_2 = 0 \quad \text{(no Goldstones)}.
$$
Fermion loops

- Include $V$ decays:

$$\epsilon^{\mu}(k, \lambda) \rightarrow J^{\mu}_{\text{eff}}/(k^2 - M^2_V + iM_V \Gamma_V), \quad \text{Or}$$

$$g^{\mu\nu} = - \sum_{\lambda=-1,0,1} \epsilon^{\mu}(k, \lambda)\epsilon^{*\nu}(k, \lambda) + \frac{k^{\mu}k^{\nu}}{k^2},$$

$$k^{\mu}\bar{F}_1\gamma^{\mu}(a + b\gamma_5)F_2 = 0, \quad \text{since: } m_1 = m_2 = 0 \text{ (no Goldstones).}$$

- Fermion loop with $\gamma_5$: anomaly free (need both $b$ and $t$)

$$\leadsto 2.5 \text{ generations not OK (we cannot decouple the top quark).}$$