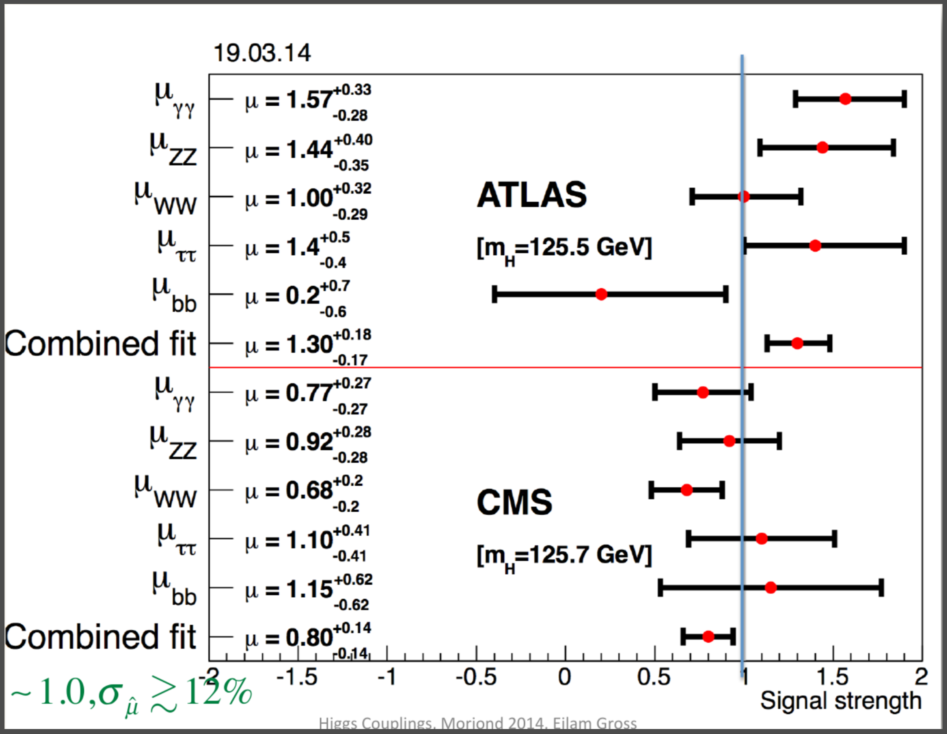
DESY Zeuthen Theory Seminar
15th of May 2014

Higgs Production at N3LO

Franz Herzog (CERN)

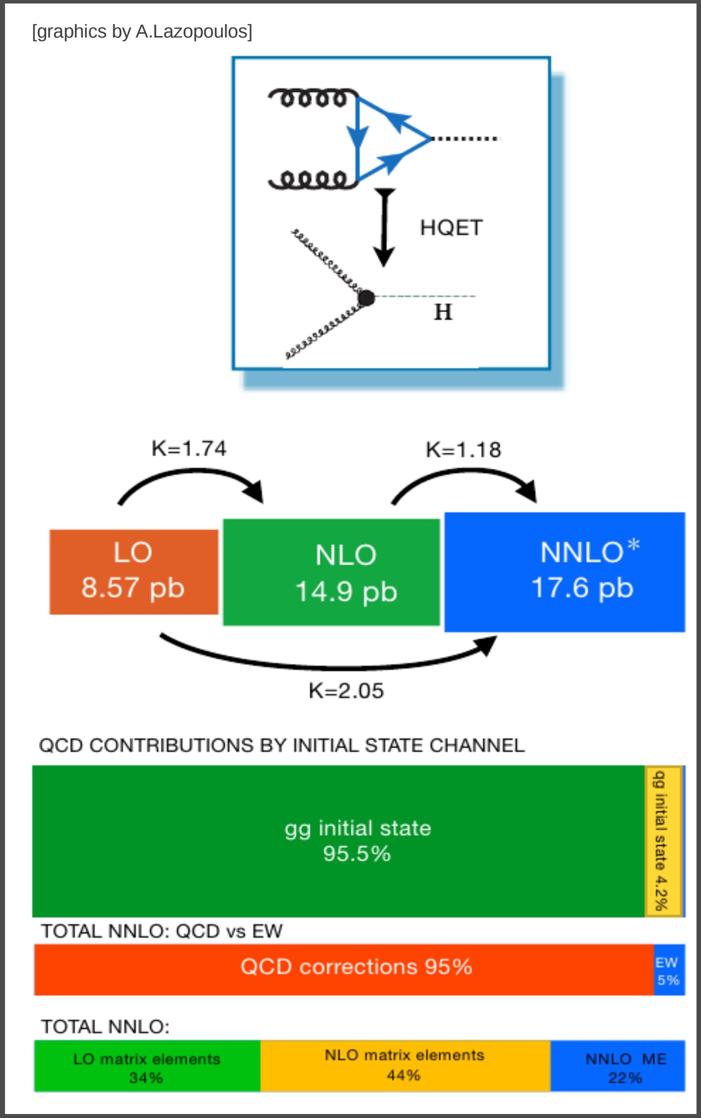
Perhaps the greatest achievement of the LHC



This table requires theoretical input on the size of the Higgs cross-section. Forthcoming LHC data will significantly reduce the experimental uncertainties on Higgs coupling measurements and will beat the current theoretical uncertainties on the Higgs cross section.

We need more precise calculations for the Higgs boson cross section!

Inclusive Hadronic Higgs Production



Perturbative Corrections:

- NLO QCD corrections known exactly (with top-bottom interference) [Graudenz et al 93, Spira et al 95, Harlander et al 05,Anastasiou 06, Aglietti 06]
- NNLO QCD corrections (in HQET) [Harlander et al 02, Anastasiou et al 02, Ravindran et al 03]
- subleading terms in the $\frac{m_H^2}{4m_t^2}$ expansion [Pak et al 09, Harlander et al 09]
- EW corrections known [Actis et al 08+09,Aglietti et al 04,Degrassi et al 04]
- mixed QCD EW corrections [Anastasiou et al 09]

Beyond NNLO:

- Soft gluon NNLL, [Catani et al 03]
- Soft gluon SCET NNLL and π^2 [Ahrens et al 08]
- Approximate N3LO [Moch et al 05, Ball et al 13]
- Soft Virtual Approximation [Anastasiou et al 14]

NNLO Theory Uncertainty

IHixs@8TeV:[Anastasiou, Buehler, FH, Lazopoulos]
(all known perturb. corr.)

m_H(GeV)	MSTW08 $\sigma(pb)$	$\%\delta_{PDF}$	$\%\delta_{\mu_F}$
125	20.69	$^{+7.79}_{-7.53}$	$^{+8.37}_{-9.26}$

Central scale $\mu = \mu_R = \mu_F = \frac{m_H}{2}$

Scale variation $\frac{m_H}{4} \leq \mu \leq m_H$

The pdf uncertainty is computed at 90%CL.

De Florian & Grazzini @ 8TeV:
(all known pert. Corr. + NNLL soft resummation)

m_H (GeV)	σ (pb)	scale(%)	PDF+α_S(%)
125.0	19.31	+7.2 -7.8	+7.5 -6.9

Central scale $\mu = \mu_R = \mu_F = m_H$

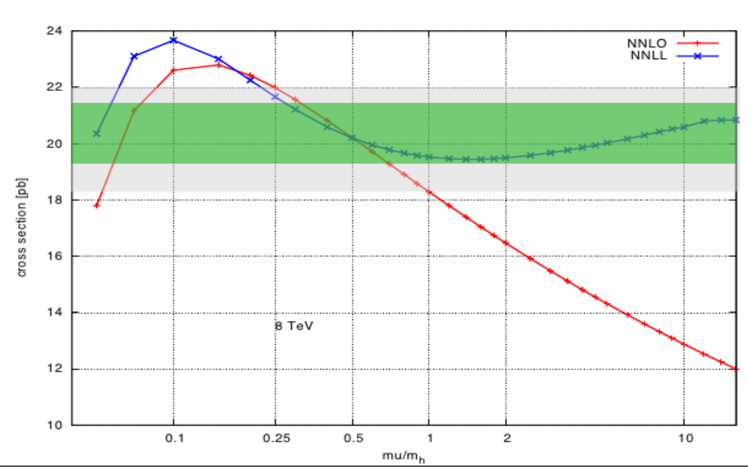
Scale variation $\frac{m_H}{2} \leq \mu_R, \mu_F \leq 2m_H$

The pdf uncertainty is computed at 68%CL. acc. To PDF4LHC

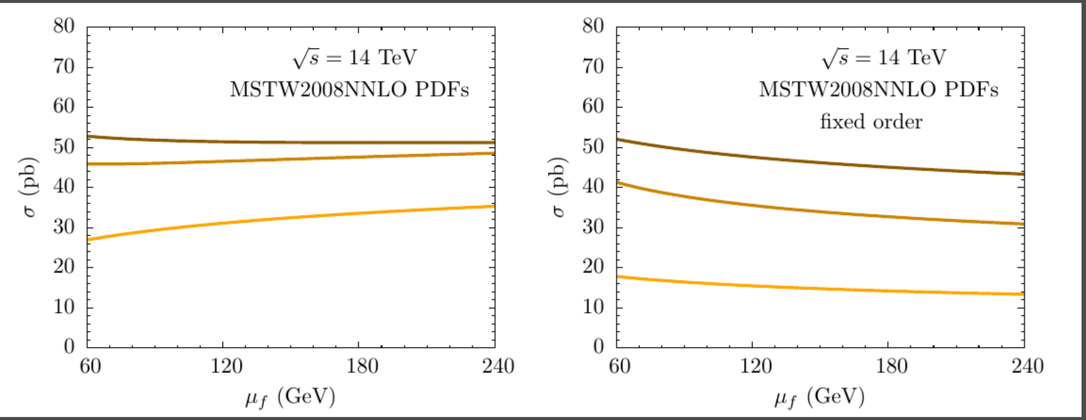
Note: dominant uncertainty is from μ_R variation

How can one reduce perturbative uncertainties?

Resummation may reduce scale uncertainties:



Treshhold resummation
[Grazzini's Online calculator]

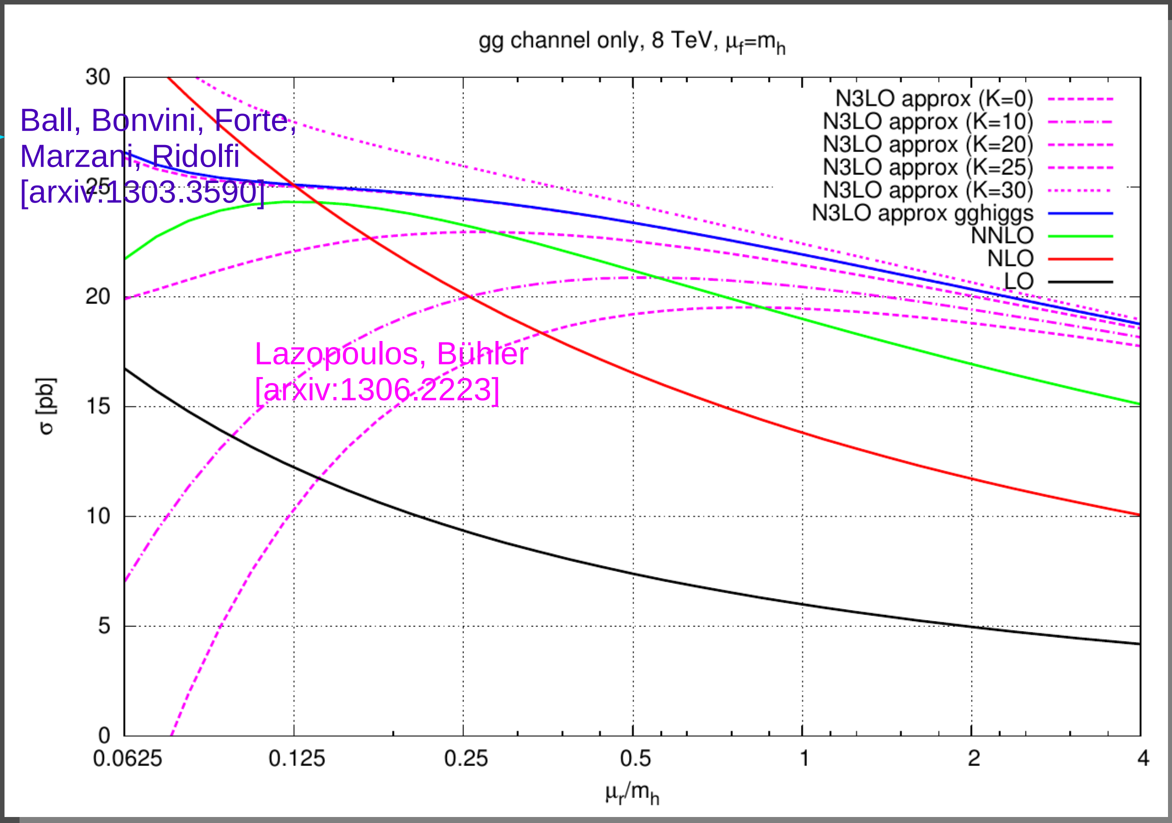


Treshhold with SCET and π² – resummation
[0809.4283]

Another way to reduce uncertainties is to compute the next order:

$$\sigma^{\text{N3LO}}_{PP\to H+X} = \alpha_s(\mu_R)^5 \left[K + f(\sigma^{\text{lower orders}}_{PP\to H+X}, \log\mu_R) \right]$$

N3LO approx
Soft + BFKL



The ultimate precision at N3LO

Order	Cross section [pb]	$\sigma/\sigma_{\mathrm{NNLO}}$	$\sigma/\sigma_{\mathrm{LO}}$
LO	$10.31\,^{+26.9\%}_{-16.6\%}$	0.51	1.00
NLO	$17.41\,^{+20.8\%}_{-12.7\%}$	0.86	1.69
NNLO	$20.27\,^{+8.3\%}_{-7.1\%}$	1.00	1.97
$\mathrm{N^3LO}$ (K=0)	$18.53\,^{+1.2\%}_{-7.9\%}$	0.91	1.80
$\mathrm{N^3LO}$ (K=5)	$19.23\,^{+0.3\%}_{-5.1\%}$	0.95	1.87
$\mathrm{N^3LO}$ (K=10)	$19.92\,^{+0.0\%}_{-2.6\%}$	0.98	1.93
$\mathrm{N^3LO}$ (K=15)	$20.62\,^{+0.4\%}_{-2.2\%}$	1.02	2.00
$\mathrm{N^3LO}$ (K=20)	$21.31\,^{+2.0\%}_{-3.1\%}$	1.05	2.07
$\mathrm{N^3LO}$ (K=30)	$22.70\,^{+6.0\%}_{-4.9\%}$	1.12	2.20
$\mathrm{N^3LO}$ (K=40)	$24.09\,^{+9.6\%}_{-6.5\%}$	1.19	2.34

Towards exact N3LO in HQET

Pros:

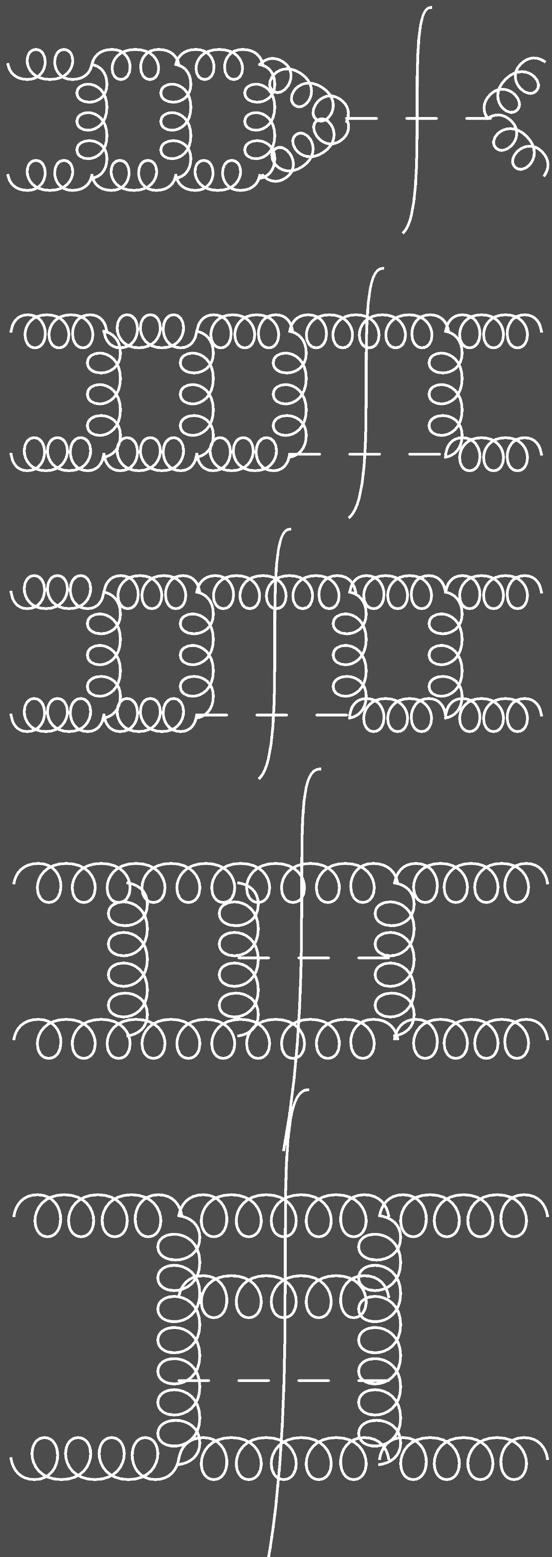
- Triple virtual was the first contribution to have been computed, so not the problem
- Integrals only depend on a single parameter,

$$z = \frac{m_H^2}{\hat{s}}$$

- Integrals can most likely be written in terms of harmonic polylogs.

Cons:

- There is still a huge number of 3-loop diagrams to compute ~100 000
- Problem of infra-red divergences even more pronounced! Most singular limits unknown! IR poles up to $\frac{1}{\epsilon^6}$.
- Phase space integrals completely unknown from other processes



Can we tackle this giant with existing technology ?

Reverse Unitarity, IBPs and Differential Equations

Used for NNLO

Write cut-propagators as a difference of Feynman propagators

$$2\pi i\delta^{+}(q^2) \to \left(\frac{1}{q^2}\right)_c = \frac{i}{q^2+i0} - \frac{i}{q^2-i0}$$

to establish differentiation properties

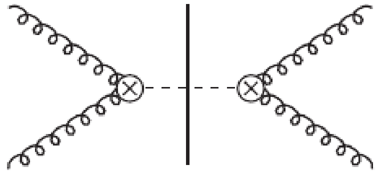
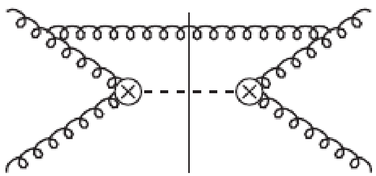
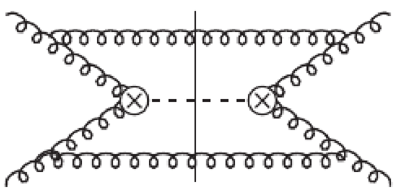
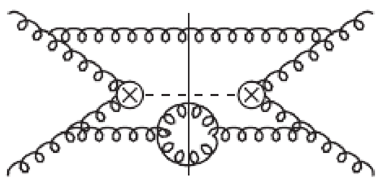
$$\frac{\partial}{\partial q^\mu}\left(\frac{1}{q^2}\right)_c = 2q^\mu \left(\frac{1}{q^2}\right)_c^2$$

This “trick” allows us derive IBP identities to find all linear relations among the Master Integrals. This then also allows us to set up a system of differential equations:

$$\frac{\partial}{\partial z}M_i(z,\epsilon) = \sum_j C_{ij}(z,\epsilon)M_j(z,\epsilon)$$

In principle (if we can triangulate the system), then all we need is a boundary condition.

Can we hope to use this tools for N3LO?

LO		1 diagram	1 integral
NLO		10 diagrams	1 integral
NNLO		381 diagrams	18 integrals
N3LO		26565 diagrams	~500 integrals

Is there an easier path?

The tripple real may be the worst of the N3LO corrections. But the number and difficulty of master integrals represents a serious challenge!

We should check if there is an easier way to reliably estimate the N3LO correction.

What z-range dominates the hadronic cross section?

$$\sigma_{PP\to H+X} = \int \frac{\mathrm{d}z}{z} \mathcal{L}_{ij}(\tau/z)\sigma_{ij\to H+X}(z)$$

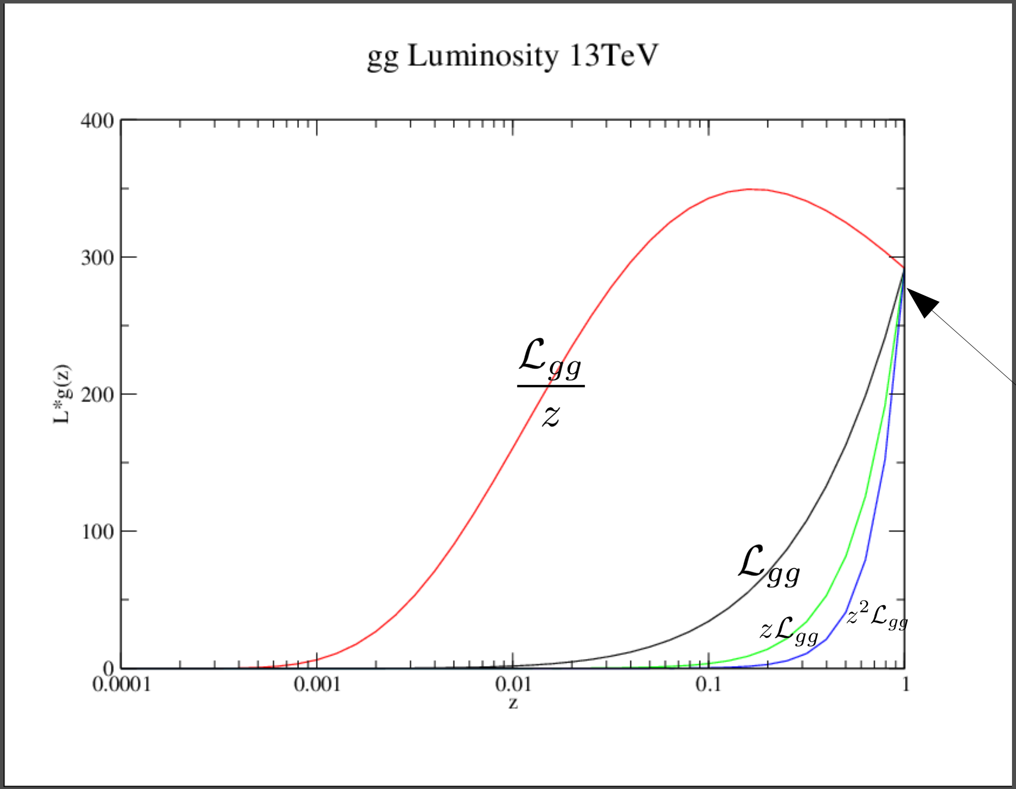
Luminosity

$$\frac{\mathcal{L}_{ij}(y)}{y} = \int_y^1 \frac{\mathrm{d}x}{x} f_i(x) f_j(y/x)$$

Partonic Cross section

$$\sigma_{gg\to H+X}(z) = A\delta(1-z) + \sum_n B_n \left[\frac{\log^n(1-z)}{1-z}\right]_+ + C(z)$$

$$\tau = \frac{m_H^2}{S}$$



Threshold

$$z = \frac{m_H^2}{\hat{s}}$$

Suggests to expand around threshold!

However there is an ambiguity!

$$\sigma_{PP\to H+X} = \int \frac{\mathrm{d}z}{z} \mathcal{L}_{ij}(\tau/z)\sigma_{ij\to H+X}(z)$$

Do we expand $\sigma_{ij}(z)$ or $\frac{\sigma_{ij}(z)}{z}$ around z=1?

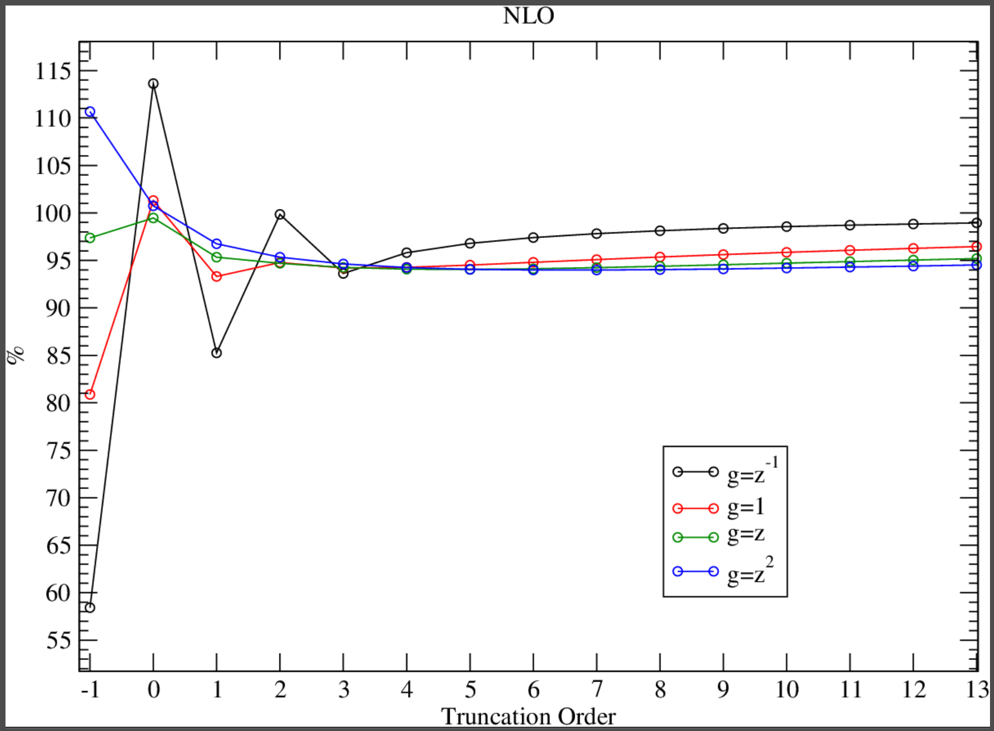
To parameterise this ambiguity we introduce a function $g(z)$ which satisfies $\lim_{z\to 1} g(z) = 1$

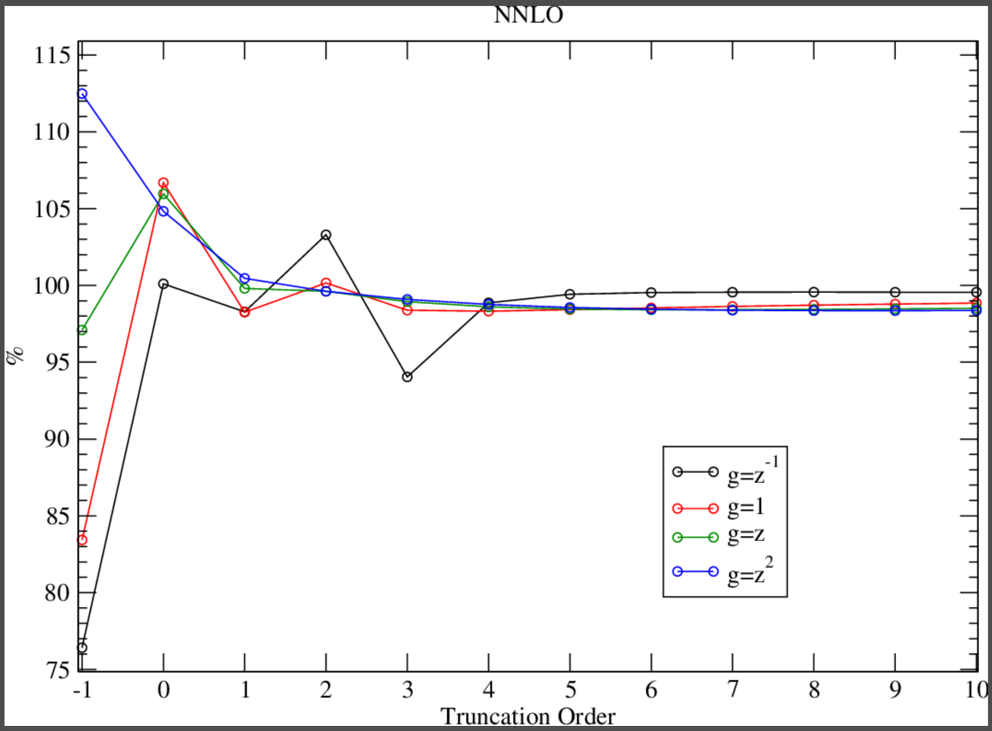
$$\sigma_{PP\to H+X} = \int \mathrm{d}z \; g(z)\mathcal{L}_{ij}(\tau/z)\left[\frac{\sigma_{ij\to H+X}(z)}{zg(z)}\right]$$

Don't expand Expand around z=1

Varying the function g(z) we can gauge the quality of the approximation

How well does this work at NLO and NNLO?



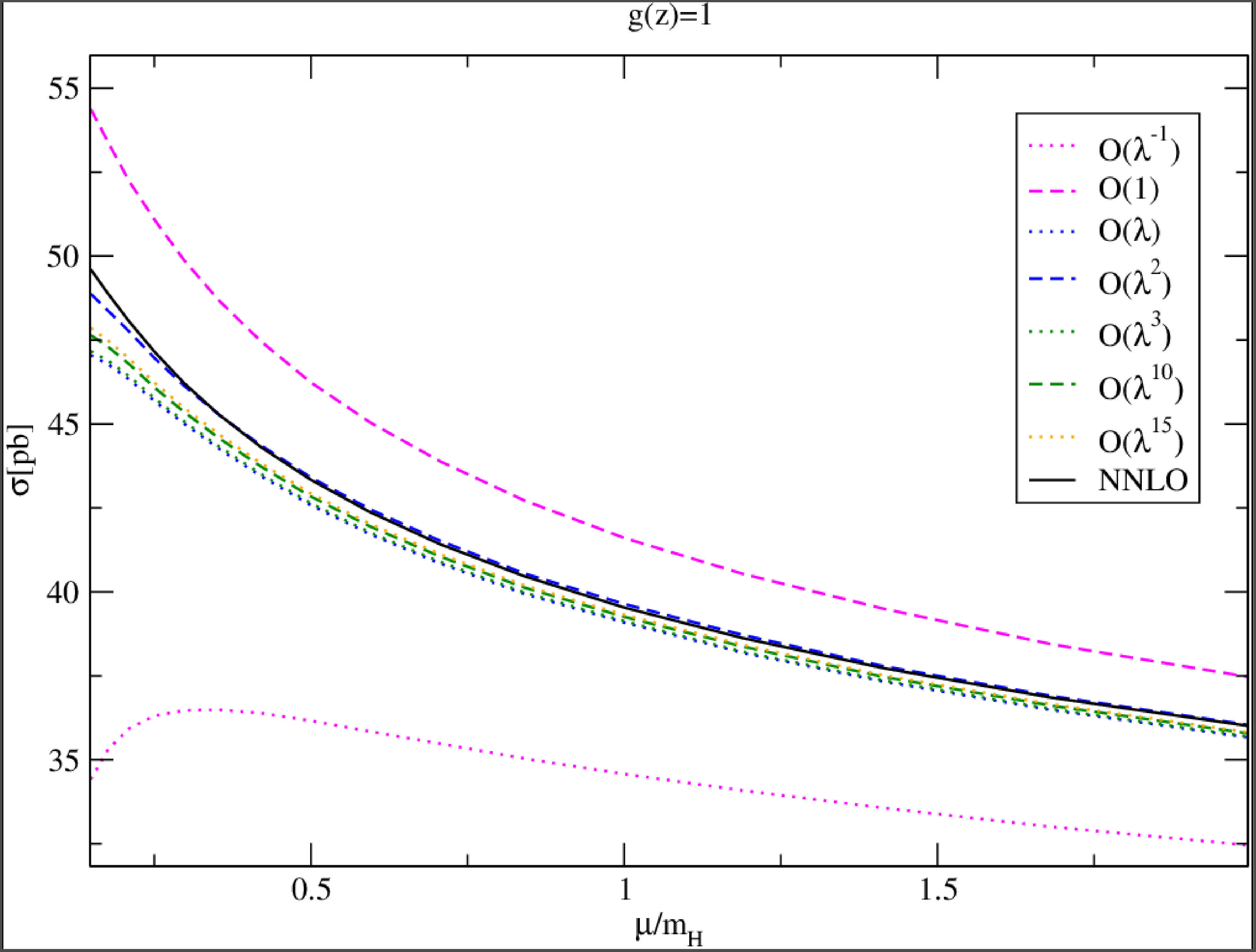


Here include full scale dependence and full dependence on the Wilson Coefficient.
This is at 13TeV keeping

$$\mu = \mu_F = \mu_R = \frac{m_H}{2}$$

Observations: -only g=1/z converges fully but is also slowest.
-g=z converges fastest.

Also works for arbitrary μ at NNLO



At NLO and NNLO the convergence of the threshold expansion is extremely good.

But:

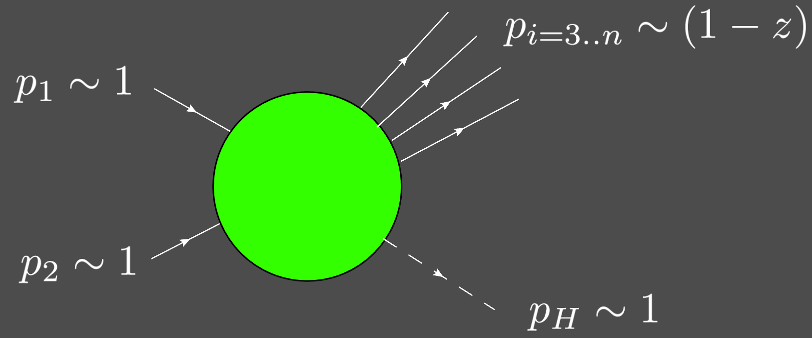
- Is developing a threshold expansion truly simpler than doing the full calculation?
- Can we identify master integrals and how many are there?

Threshold Expansions for Phase Space Integrals

- Soft kinematics <=> Higgs at Threshold

$$p_1 + p_2 = \sum_{i=3..n} p_i + p_H \qquad \rightarrow \qquad p_1 + p_2 = p_H + \mathcal{O}(1-z)$$

$$z = \frac{m_H^2}{s_{12}}$$

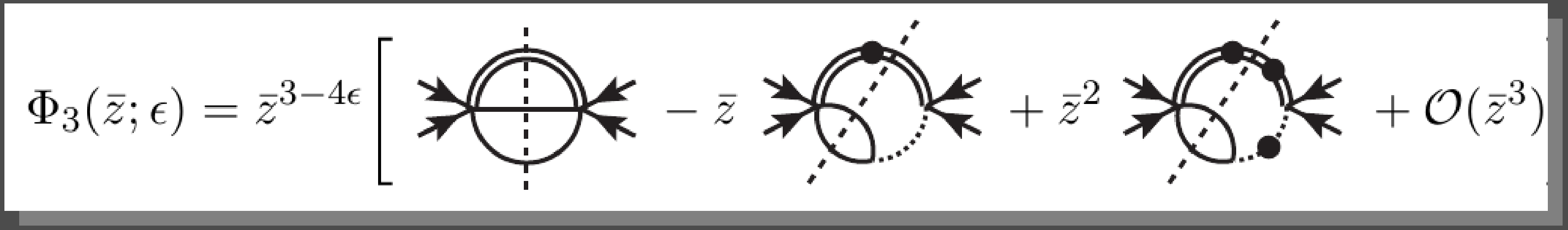


- Knowing the scalings of momenta, the scalings of all propagators can be found and we can expand around z=1:

For Phase space integrals it is straight forward to develop a threshold expansion:

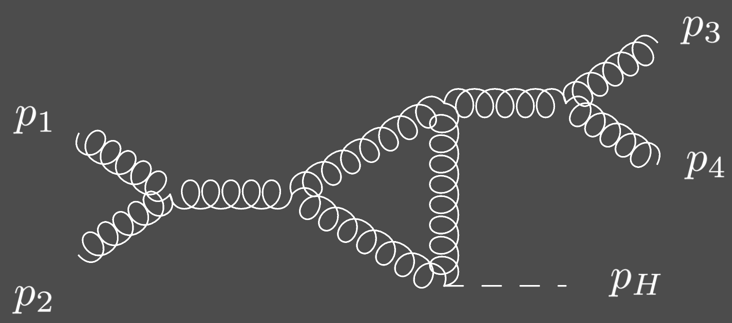
- Rescale momenta with (1-z)
- Taylor expand integrand
- IBPs can be derived using the tricks of reverse unitarity

e.g. the soft expansion of the double real phase space volume



But how does it work with loop integrals or combined loop phase space integrals?

Threshold Expansion for Loop Integrals



$$\sim \int \mathrm{d}^3 x \delta(1 - \sum_i x_i) \frac{(x_1 + x_2 + x_3)^{-1+2\epsilon}}{(x_1 x_2 \hat{s}_{34} \bar{z}^2 + x_2 x_3 z s_{12} + x_3 x_1 s_{12})^{1+\epsilon}}$$

$$\bar{z} = 1 - z$$

Naive Taylor expansion of integrand around z=1 misses a potentially non-analytic contribution

$$F_0(\bar{z}) + \bar{z}^{-2\epsilon} F_1(\bar{z})$$

$$F_k(\bar{z}) = \sum_n \frac{\partial^n F(0)}{\partial \bar{z}^n} \frac{\bar{z}^n}{n!}$$

Analytic at z=1

Non-analytic at z=1

The singular terms are in one to one correspondence with Landau Singularities on the integrand which contain

$$\bar{z} = 0$$

Landau's condition for singularity:

$$F = x_1 x_2 \hat{s}_{34} \bar{z}^2 + x_2 x_3 z s_{12} + x_3 x_1 s_{12} = 0$$

$$0 \qquad 0 \qquad 0$$

For the present example there is only one singularity which contains $\bar{z} = 0$:

$$\{x_3 = 0, \bar{z} = 0\}$$

Expansion by Regions:

Apply a *projective* scaling to factorize the singularity, here:

$$x_3 \mapsto x_3 \bar{z}^2$$

The “non-analytic” contribution is then given by the Taylor expansion of the “new integrand”, the non-analytic part is factored out.

$$\bar{z}^{-2\epsilon} \int \mathrm{d}^3 x \delta(1 - \sum_i x_i) \frac{(x_1 + x_2 + x_3 \bar{z}^2)^{-1+2\epsilon}}{(x_1 x_2 \hat{s}_{34} + x_2 x_3 z s_{12} + x_3 x_1 s_{12})^{1+\epsilon}}$$

Taylor expand after rescaling

Full result given by $F_0(\bar{z}) + \bar{z}^{-2\epsilon} F_1(\bar{z})$

The expansion by region reproduces the full result and can be applied to “arbitrary” hypergeometric-type integrals.

Naive Taylor Expansion

$$\sim \int \mathrm{d}^3 x \delta(1 - \sum_i x_i) \frac{(x_1 + x_2 + x_3)^{1+2\epsilon}}{(x_1 x_2 \hat{s}_{34} \bar{z}^2 + x_2 x_3 z s_{12} + x_3 x_1 s_{12})^{1+\epsilon}}$$

To find IBPs we need to be able to expand in momentum space

$$\int \frac{d^D k}{k^2(k+p_{12})^2(k-p_{34})^2}$$

The same regions can be found in momentum space by expanding around soft and collinear loop momentum configurations.

$$k^\mu = \alpha p_1^\mu + \beta p_2^\mu + k_\perp^\mu$$

One can show that for the threshold expansion there are in general only 4 regions for each loop momentum in Higgs production:

I) Hard: $k^\mu \sim 1$

2) Collinear 1: $\beta \sim \bar{z} \quad k_\perp^2 \sim \bar{z}$

3) Collinear 2: $\alpha \sim \bar{z} \quad k_\perp^2 \sim \bar{z}$

4) Soft: $k^\mu \sim \bar{z}$

Strategy towards N3LO

- Treat different contributions (real and virtual) seperately.

  a)
  If the basis of Master Integrals is sufficiently small attempt to solve the differential equations. Use expansion by regions to compute the boundary condition in the soft limit.

  b)
  If the basis of master integrals is too large use expansin by region to compute sufficiently many terms in the soft expansion.

- Both a) and b) require the soft limit, hence this is an ideal first goal.

N3LO Status

VVV:

- Known [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]

RVV:

- 2-loop amplitude known up to O(ε) [Gehrmann, Jaquier, Glover, Koukoutsakis]
- One loop soft current known [Duhr, Gehrmann; Li, Zhu]
- Soft limit known
- Full calculation in progress

(RV)^2:

- Known [Anastasiou, Duhr, Dulat, FH, Mistlberger; Kilgore]

RRV:

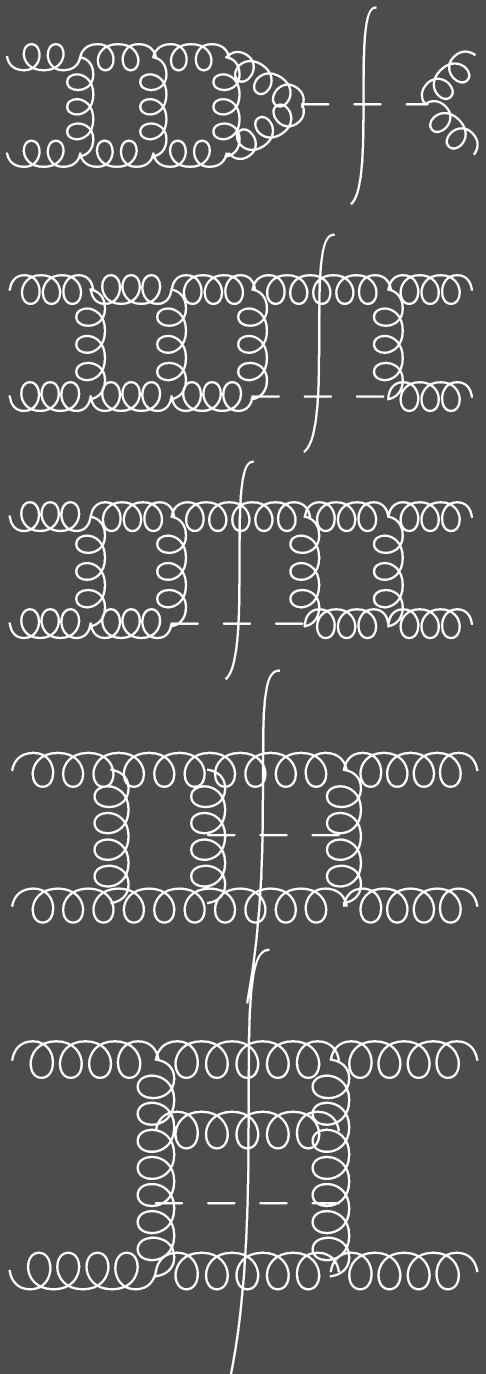
- Soft limit known [Anastasiou, Duhr, Dulat, FH, Mistlberger,Furlan;Li, Mantueffel, Schabinger, Zhu]
- Next term / Full Calculation in progress

RRR:

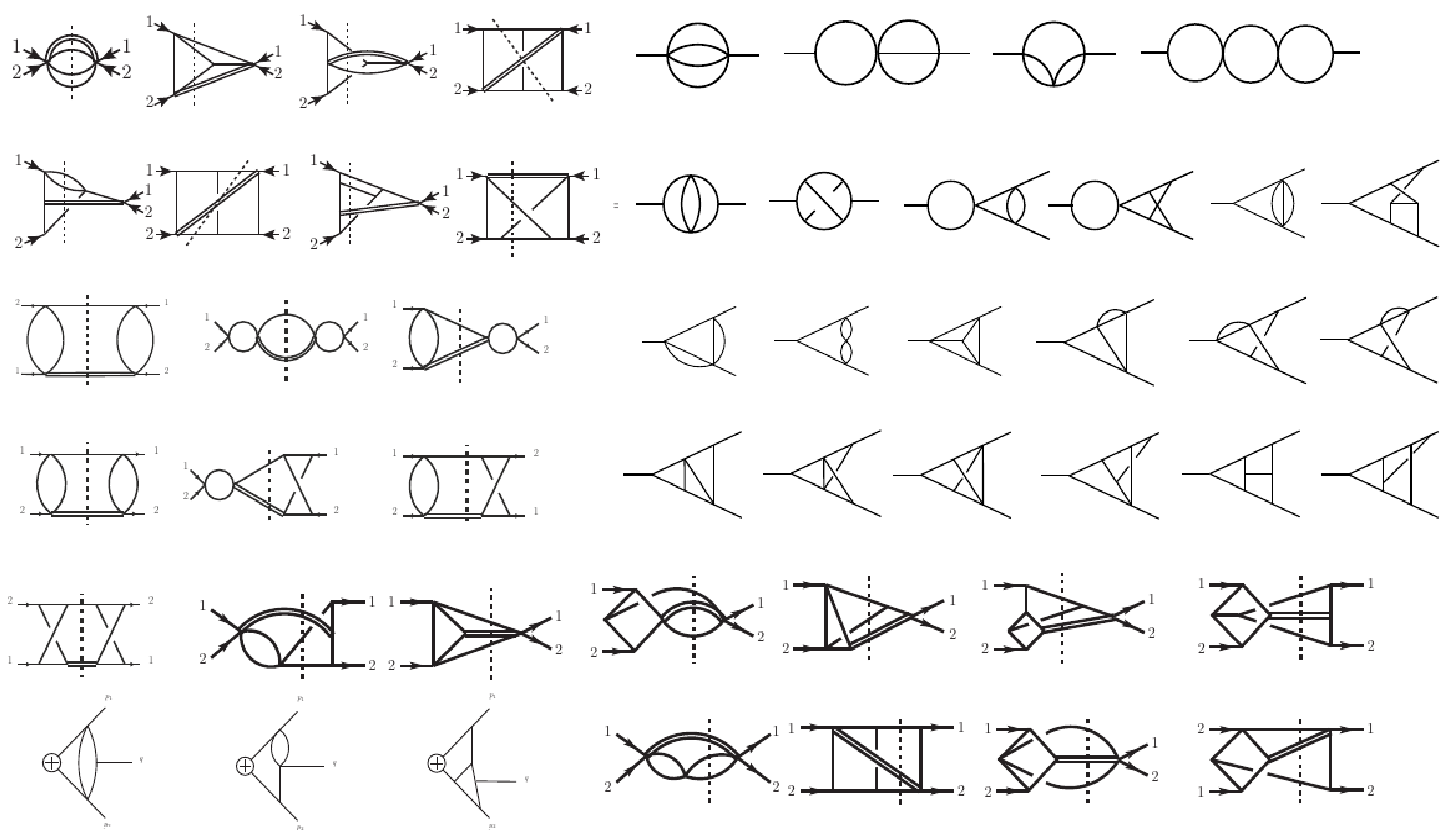
- Know first two terms in soft expansion [Anastasiou, Duhr, Dulat, Mistlberger]

Collinear/UV counterterms:

- known [Pak, Rogal, Steinhauser; Anastasiou, Buehler, Duhr, FH; Höschele, Hoff, Pak, Steinhauser, Ueda; Buehler, Lazopoulos]



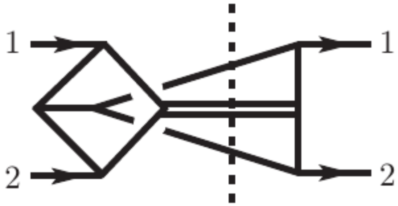
All integrals necessary for the threshold have now been computed!



Artist: C.Duhr

Computation the most difficult Master Integrals for the soft limit of the Double Real Virtual

$$\mathcal{M}_{23} =$$



$$= \int d\Phi_3 \frac{(s_{134}s_{234})^{-1-\epsilon}}{s_{13}s_{24}s_{34}} {}_2F_1\left(1+\epsilon, 1+\epsilon; 2+\epsilon; 1-\frac{s_{12}s_{34}}{s_{134}s_{234}}\right)$$

(Very) clever Phase Space parameterisation trivializes the argument of the 2F1:

$$\mathcal{M}_{23} = \mathcal{N}_2 \int dx_1 (x_1\bar{x}_1)^{-1-3\epsilon} \int d\Omega_2^{(D-1)} d\Omega_3^{(D-1)} \frac{{}_2F_1(1+\epsilon, 1+\epsilon, 2+\epsilon, \bar{y}_{12})}{\bar{y}_{13}y_{23}y_{12}}$$

Then use Van Neerven's Trick to perform integration over one of the angles

$$y_{ij} = \frac{1-\cos\theta_{ij}}{2}, \qquad \bar{y}_{ij} = \frac{1+\cos\theta_{ij}}{2} = 1 - y_{ij}$$

$$\int \frac{d\Omega_3^{(D-1)}}{\bar{y}_{13}y_{23}} = (4\pi)^{1-\epsilon} \frac{\Gamma(-\epsilon)}{-\epsilon\Gamma(-2\epsilon)} {}_2F_1(1,1;1-\epsilon; y_{12})$$

to get

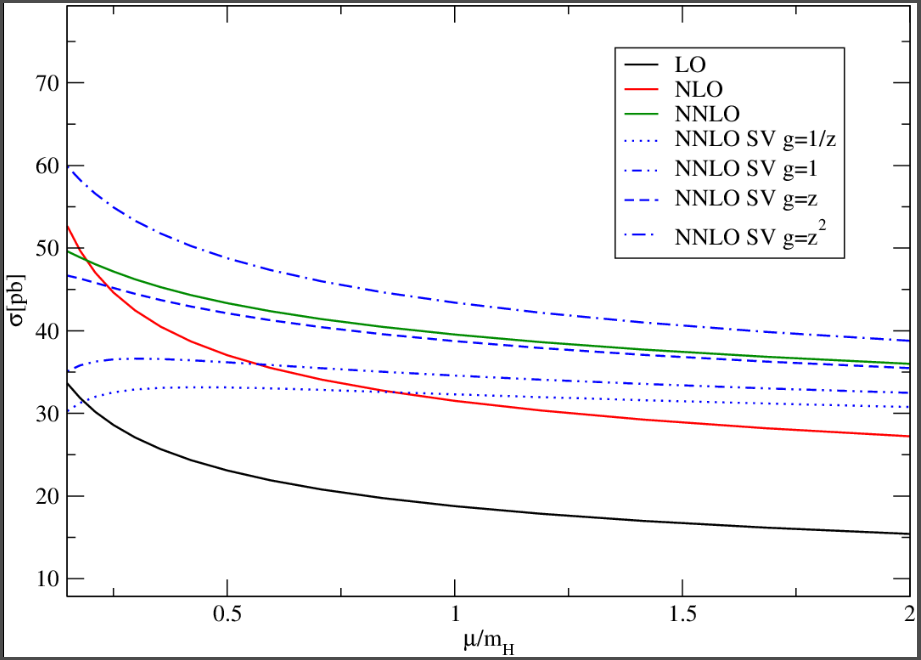
$$\begin{aligned}
&= \int_0^1 dx x^{-1-\epsilon}\bar{x}^{-\epsilon} {}_2F_1(1,1;1-\epsilon;x) {}_2F_1(1+\epsilon, 1+\epsilon, 2+\epsilon, \bar{x}) \\
&= \sum_{n,m=0}^{\infty} \frac{(1)_n^2}{(1-\epsilon)_n} \frac{(1+\epsilon)_m^2}{(2+\epsilon)_m} \frac{1}{n!m!} \int_0^1 dx\, x^{n-1-\epsilon}\bar{x}^{m-\epsilon} \\
&= \sum_{n,m=0}^{\infty} \frac{(1)_n^2}{(1-\epsilon)_n} \frac{(1+\epsilon)_m^2}{(2+\epsilon)_m} \frac{1}{n!m!} B(n-\epsilon, m+1-\epsilon) \\
&= B(-\epsilon, 1-\epsilon) \sum_{n,m=0}^{\infty} \frac{(1)_n^2}{(1-\epsilon)_n} \frac{(1+\epsilon)_m^2}{(2+\epsilon)_m} \frac{(-\epsilon)_n(1-\epsilon)_m}{(1-2\epsilon)_{n+m}} \frac{1}{n!m!}
\end{aligned}$$

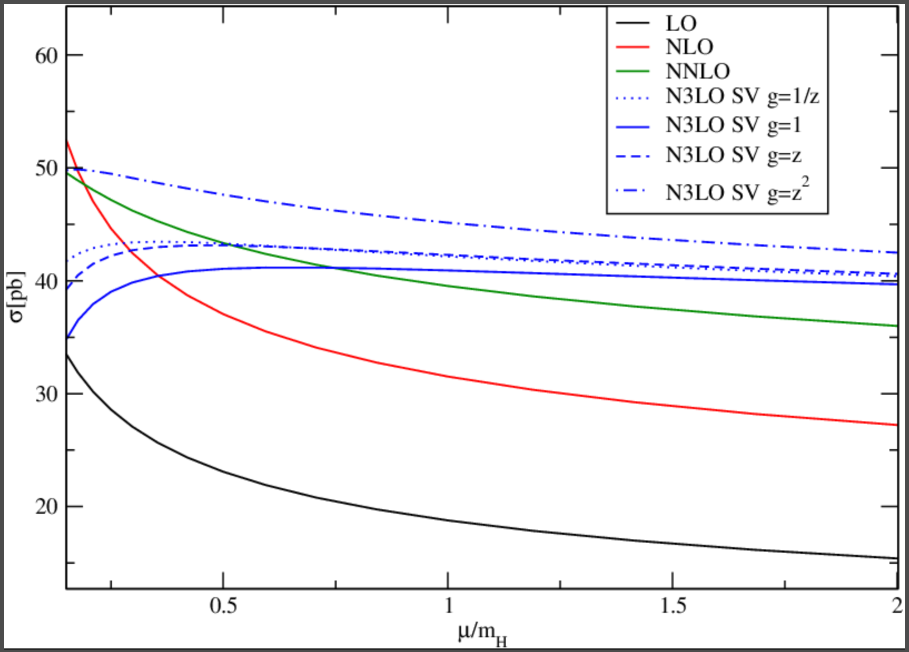
Final double series can be identified as
a Kampe de Feriet function
of the form F(,,,;1,1)

Higss Production at Threshold at N3LO

$$
\begin{aligned}
\hat{\eta}^{(3)}(z) = \delta(1-z) \Bigg\{ & C_A^3 \left(-\frac{2003}{48}\zeta_6 + \frac{413}{6}\zeta_3^2 - \frac{7579}{144}\zeta_5 + \frac{979}{24}\zeta_2\zeta_3 - \frac{15257}{864}\zeta_4 - \frac{819}{16}\zeta_3 + \frac{16151}{1296}\zeta_2 + \frac{215131}{5184} \right) \quad (1 \\
& + N_F \Bigg[C_A^2 \left(\frac{869}{72}\zeta_5 - \frac{125}{12}\zeta_3\zeta_2 + \frac{2629}{432}\zeta_4 + \frac{1231}{216}\zeta_3 - \frac{70}{81}\zeta_2 - \frac{98059}{5184} \right) \\
& \quad + C_A C_F \left(\frac{5}{2}\zeta_5 + 3\zeta_3\zeta_2 + \frac{11}{72}\zeta_4 + \frac{13}{2}\zeta_3 - \frac{71}{36}\zeta_2 - \frac{63991}{5184} \right) + C_F^2 \left(-5\zeta_5 + \frac{37}{12}\zeta_3 + \frac{19}{18} \right) \Bigg] \\
& + N_F^2 \Bigg[C_A \left(-\frac{19}{36}\zeta_4 + \frac{43}{108}\zeta_3 - \frac{133}{324}\zeta_2 + \frac{2515}{1728} \right) + C_F \left(-\frac{1}{36}\zeta_4 - \frac{7}{6}\zeta_3 - \frac{23}{72}\zeta_2 + \frac{4481}{2592} \right) \Bigg] \Bigg\} \\
+ \left[\frac{1}{1-z} \right]_+ \Bigg\{ & C_A^3 \left(186\zeta_5 - \frac{725}{6}\zeta_3\zeta_2 + \frac{253}{24}\zeta_4 + \frac{8941}{108}\zeta_3 + \frac{8563}{324}\zeta_2 - \frac{297029}{23328} \right) + N_F^2 C_A \left(\frac{5}{27}\zeta_3 + \frac{10}{27}\zeta_2 - \frac{58}{729} \right) \\
& + N_F \Bigg[C_A^2 \left(-\frac{17}{12}\zeta_4 - \frac{475}{36}\zeta_3 - \frac{2173}{324}\zeta_2 + \frac{31313}{11664} \right) + C_A C_F \left(-\frac{1}{2}\zeta_4 - \frac{19}{18}\zeta_3 - \frac{1}{2}\zeta_2 + \frac{1711}{864} \right) \Bigg] \Bigg\} \\
+ \left[\frac{\log(1-z)}{1-z} \right]_+ \Bigg\{ & C_A^3 \left(-77\zeta_4 - \frac{352}{3}\zeta_3 - \frac{152}{3}\zeta_2 + \frac{30569}{648} \right) + N_F^2 C_A \left(-\frac{4}{9}\zeta_2 + \frac{25}{81} \right) \\
& + N_F \Bigg[C_A^2 \left(\frac{46}{3}\zeta_3 + \frac{94}{9}\zeta_2 - \frac{4211}{324} \right) + C_A C_F \left(6\zeta_3 - \frac{63}{8} \right) \Bigg] \Bigg\} \\
+ \left[\frac{\log^2(1-z)}{1-z} \right]_+ \Bigg\{ & C_A^3 \left(181\zeta_3 + \frac{187}{3}\zeta_2 - \frac{1051}{27} \right) + N_F \Bigg[C_A^2 \left(-\frac{34}{3}\zeta_2 + \frac{457}{54} \right) + \frac{1}{2} C_A C_F \Bigg] - \frac{10}{27} N_F^2 C_A \Bigg\} \\
+ \left[\frac{\log^3(1-z)}{1-z} \right]_+ \Bigg\{ & C_A^3 \left(-56\zeta_2 + \frac{925}{27} \right) - \frac{164}{27} N_F C_A^2 + \frac{4}{27} N_F^2 C_A \Bigg\} \\
+ \left[\frac{\log^4(1-z)}{1-z} \right]_+ & \left(\frac{20}{9} N_F C_A^2 - \frac{110}{9} C_A^3 \right) + \left[\frac{\log^5(1-z)}{1-z} \right]_+ 8 C_A^3 .
\end{aligned}
$$

The Soft Virtual Approximation
NNLO vs N3LO





Besides Soft N3LO we also include full kinematic wilson coefficient and scale dependent N3LO contributions in all partonic channels. Scale variation is done using $\mu = \mu_R = \mu_F$
This is at 13TeV for a 125GeV Higgs boson.

Conclusions

- We need to improve our understanding of the theoretical uncertainty on the gluon fusion cross section.
- Have presented the analytic result of the N3LO cross section in the soft limit, the first calculation done at N3LO for hadron colliders.
- Further coefficients of soft expansion of the N3LO are in close reach. The full result could be feasible.
- Have presented numerics for the soft virtual approximation at N3LO. The SV approximation gives an indication of the full N3LO, but has large uncertainties.

Backup

Shifting Logs and Collinear improved Soft-Virtual Approximation

$$\sigma_{gg\to H}^{N^3LO,SV} = \int \mathrm{d}x_1 \mathrm{d}x_2 f_i(x_1) f_j(x_2) \int \mathrm{d}z \delta(z - \frac{\tau}{x_1 x_2}) \cdot z \cdot g(z) \cdot \left[\lim_{z\to 1} \frac{\sigma_{ij}^{N^3LO}(z)}{z g(z)} \right]$$

The analytic structure of the partonic cross section can be written as

$$\frac{\sigma_{gg\to H}^{N^3LO}(z)}{z} = A\delta(1-z) + \sum_{n=1}^{5} B_n \left[\frac{\log^n(1-z)}{1-z}\right]_+ + C(z)$$

Can shift logs from plus to regular terms:

$$z\left[\frac{\log^n(1-z)}{1-z}\right]_+ = \left[\frac{\log^n(1-z)}{1-z}\right]_+ - \log^n(1-z)$$

Taking $g(z) = z$ reproduces the correct leading logarithm in the reguar part