Twist-2 operators in $\mathcal{N} = 4$ SYM theory

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Outline

- Anomalous dimension matrix for twist-2 operators
  - Leading order in QCD and $\mathcal{N}=4$ SYM theory (one loop)
  - Next-to-leading order in $\mathcal{N}=4$ SYM theory (two loops)
    - Maximal transcendentality principle
  - NNLO in $\mathcal{N}=4$ SYM theory (three loops)
- AdS/CFT-correspondence
  - Integrability in $\mathcal{N}=4$ SYM theory
  - Asymptotic Bete-ansatz
- Calculations of the anomalous dimension in $\mathcal{N}=4$ SYM theory
  - Technique of calculations
  - Calculation of non-planar contribution at four loops
- Reconstruction of a general form of AD from a fixed values
  - Gribov-Lipatov reciprocity
  - Binomial harmonic sums
  - LLL-lagorithm
- Open problems

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Deep Inelastic Scattering

[Gross, Wilczek '73]

$$\mathcal{O}^{q}_{\mu_1,\ldots,\mu_j} = \bar{q}\,\gamma_{\mu_1}\mathcal{D}_{\mu_2}\ldots\mathcal{D}_{\mu_j}\,q + \text{symmetrisation} - \text{traces}$$

Twist = Canonical dimension - Lorentz spin j [Gross, Wilczek '73]

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$$\mathcal{O}^{g}_{\mu_1,\ldots,\mu_j} = G_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\ldots\mathcal{D}_{\mu_{j-1}}G_{\rho\mu_j} + \text{symmetrisation} - \text{traces}$$

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Operators mix under renormalization

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Operators mix under renormalization

$$\langle\bar{q}|\mathcal{O}^{q}_{\mu_1,\ldots,\mu_j}|q\rangle \to \gamma^{j}_{qq}$$

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$$\langle g|\mathcal{O}^{q}_{\mu_1,\ldots,\mu_j}|g\rangle \to \gamma^{j}_{qg}$$

$$\langle g|\mathcal{O}^{g}_{\mu_1,\ldots,\mu_j}|g\rangle \to \gamma^{j}_{gg}$$

$$\langle\bar{q}|\mathcal{O}^{g}_{\mu_1,\ldots,\mu_j}|q\rangle \to \gamma^{j}_{gq}$$

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Operators mix under renormalization → Matrix of anomalous dimensions

$$\langle\bar{q}|\mathcal{O}^{q}_{\mu_1,\ldots,\mu_j}|q\rangle \to \gamma^{j}_{qq}$$

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$$\Gamma = \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}$$

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$$\Gamma = \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}$$

$$\gamma_{qq} = 2C_F\left[4S_1(j) - 3 - \frac{2}{j(j+1)}\right] \qquad \gamma_{qg} = -8T_R\frac{j^2+j+2}{j(j+1)(j+2)}$$

$$\gamma_{gq} = -4C_F\frac{j^2+j+2}{(j-1)j(j+1)} \quad \gamma_{gg} = \left[8C_A\left(S_1(j) - \frac{1}{j(j-1)} - \frac{1}{(j+1)(j+2)} - \frac{11}{12}\right) + \frac{8}{3}T_R\right]$$

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$$S_1(j) = \sum_{k=1}^{j}\frac{1}{k} = \Psi(1) - \Psi(j+1)$$

$$\gamma_{qq} = 2C_F\left[4S_1(j) - 3 - \frac{2}{j(j+1)}\right] \qquad \gamma_{qg} = -8T_R\,\frac{j^2+j+2}{j(j+1)(j+2)}$$

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$$\gamma_{qq} = C_F \left[8S_1(j) - \frac{4}{j} + \frac{4}{j+1} - 6 \right]$$

$$\gamma_{qg} = T_R \left[-\frac{8}{j} + \frac{16}{j+1} - \frac{16}{j+2} \right] \qquad \gamma_{gq} = C_F \left[-\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1} \right]$$

$$\gamma_{gg} = C_A \left[8S_1(j) - \frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} - \frac{11}{12} \right] + \frac{8}{3} T_R$$

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$$\gamma_{gg} = C_A \left[\boxed{8S_1(j)} - \frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} \boxed{+\frac{8}{j+2}} - \frac{11}{12} \right] + \frac{8}{3} T_R$$

Quantum Chromodynamics: Wilson twist-2 operators

$$\gamma_{qq} = C_F \left[8S_1(j) - \frac{4}{j} + \frac{4}{j+1} - 6 \right]$$

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$$\gamma_{qg} = T_R \left[-\frac{8}{j} + \frac{16}{j+1} - \frac{16}{j+2} \right] \qquad \gamma_{gq} = C_F \left[-\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1} \right]$$

$$\gamma_{gg} = C_A \left[8S_1(j) - \frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} - \frac{11}{12} \right] + \frac{8}{3} T_R$$

$$C_F = C_A \qquad T_R = \frac{1}{2} C_A$$

$$\gamma_{qq} = C_F \left[8S_1(j) \; - \frac{4}{j} + \frac{4}{j+1} - 6 \right]$$

$$\gamma_{qg} = T_R \left[-\frac{8}{j} + \frac{16}{j+1} - \frac{16}{j+2} \right] \qquad \gamma_{gq} = C_F \left[-\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1} \right]$$

$$\gamma_{gg} = C_A \left[8S_1(j) \; - \frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} - \frac{11}{12} \right] + \frac{8}{3} T_R$$

$$C_F = C_A \qquad T_R = \frac{1}{2} C_A$$

$$\gamma_{qq} + \gamma_{gq} = \gamma_{gg} + \gamma_{qg} = C_A \left[8S_1(j) - \frac{8}{j-1} + \frac{4}{j} - 6 \right]$$

Quantum Chromodynamics: Wilson twist-2 operators

$$\gamma_{qq} = C_F \left[8S_1(j) - \frac{4}{j} + \frac{4}{j+1} - 6 \right]$$

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$$\gamma_{gg} = C_A \left[8S_1(j) - \frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} - \frac{11}{12} \right] + \frac{8}{3} T_R$$

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Dokshitzer relation

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$$\gamma_{qg} = T_R \left[-\frac{8}{j} + \frac{16}{j+1} - \frac{16}{j+2} \right] \qquad \gamma_{gq} = C_F \left[-\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1} \right]$$

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Origin:

Quantum Chromodynamics: Wilson twist-2 operators

$$\gamma_{qq} = C_F \left[8S_1(j) - \frac{4}{j} + \frac{4}{j+1} - 6 \right]$$

$$\gamma_{qg} = T_R \left[-\frac{8}{j} + \frac{16}{j+1} - \frac{16}{j+2} \right] \qquad \gamma_{gq} = C_F \left[-\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1} \right]$$

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Origin: $\mathcal{N} = 1$ Supersymmetric Yang-Mills theory

Wilson twist-2 operators:

[Gross, Wilczek '73]

$$\mathcal{O}^{g}_{\mu_1,\ldots,\mu_j} = G^a_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\ldots\mathcal{D}_{\mu_{j-1}}G^a_{\rho\mu_j}$$

$$\mathcal{O}^{\lambda}_{\mu_1,\ldots,\mu_j} = \bar{\lambda}^a_i\gamma_{\mu_1}\mathcal{D}_{\mu_2}\ldots\mathcal{D}_{\mu_j}\lambda^{a,\,i}$$

Wilson twist-2 operators:

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Wilson twist-2 operators:

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$$\mathcal{O}^{g}_{\mu_1,\ldots,\mu_j} = G^a_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\ldots\mathcal{D}_{\mu_{j-1}}G^a_{\rho\mu_j} \qquad \tilde{\mathcal{O}}^{g}_{\mu_1,\ldots,\mu_j} = G^a_{\rho\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\ldots\mathcal{D}_{\mu_{j-1}}\tilde{G}^a_{\rho\mu_j}$$

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Anomalous dimension matrix in leading order:

[Lipatov '00]

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Anomalous dimension matrix in leading order:

[Lipatov '00]

$$\Gamma = \begin{pmatrix} \gamma_{gg} & \gamma_{g\lambda} & \gamma_{g\phi} \\ \gamma_{\lambda g} & \gamma_{\lambda\lambda} & \gamma_{\lambda\phi} \\ \gamma_{\phi g} & \gamma_{\phi\lambda} & \gamma_{\phi\phi} \end{pmatrix}$$

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[Gross, Wilczek '73]

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$$\gamma^{(0)}_{gg} = -4S_1(j) + \frac{4}{j-1} - \frac{4}{j} + \frac{4}{j+1} - \frac{4}{j+2} \qquad \gamma^{(0)}_{\lambda g} = \frac{8}{j} - \frac{16}{j+1} + \frac{16}{j+2}$$

$$\gamma^{(0)}_{\phi g} = \frac{12}{j+1} - \frac{12}{j+2} \qquad \gamma^{(0)}_{g\lambda} = \frac{4}{j-1} - \frac{4}{j} + \frac{2}{j+1} \qquad \gamma^{(0)}_{\lambda\phi} = \frac{8}{j} \qquad \gamma^{(0)}_{\phi\lambda} = \frac{6}{j+1}$$

$$\gamma^{(0)}_{\lambda\lambda} = -4S_1(j) + \frac{4}{j} - \frac{4}{j+1} \qquad \gamma^{(0)}_{\phi\phi} = -4S_1(j) \qquad \gamma^{(0)}_{g\phi} = \frac{4}{j-1} - \frac{4}{j}$$

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$$\mathcal{O}^{\lambda}_{\mu_1,\ldots,\mu_j} = \bar{\lambda}^a_i\gamma_{\mu_1}\mathcal{D}_{\mu_2}\ldots\mathcal{D}_{\mu_j}\lambda^{a,\,i} \qquad \tilde{\mathcal{O}}^{\lambda}_{\mu_1,\ldots,\mu_j} = \bar{\lambda}^a_i\gamma_5\gamma_{\mu_1}\mathcal{D}_{\mu_2}\ldots\mathcal{D}_{\mu_j}\lambda^{a,\,i}$$

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$$\widetilde{\Gamma} = \begin{pmatrix} \tilde{\gamma}_{gg} & \tilde{\gamma}_{g\lambda} \\ \tilde{\gamma}_{\lambda g} & \tilde{\gamma}_{\lambda\lambda} \end{pmatrix}$$

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$$\tilde{\gamma}^{(0)}_{g\lambda} = \frac{4}{j} - \frac{2}{j+1} \qquad \tilde{\gamma}^{(0)}_{\lambda\lambda} = -4S_1(j) + \frac{4}{j+1} - \frac{4}{j}$$

Anomalous dimension matrix in leading order: [Lipatov '00]

$$\Gamma^{(0)}=\left(\begin{array}{ccc}\gamma_{gg}^{(0)} & \gamma_{g\lambda}^{(0)} & \gamma_{g\phi}^{(0)}\\ \gamma_{\lambda g}^{(0)} & \gamma_{\lambda\lambda}^{(0)} & \gamma_{\lambda\phi}^{(0)}\\ \gamma_{\phi g}^{(0)} & \gamma_{\phi\lambda}^{(0)} & \gamma_{\phi\phi}^{(0)}\end{array}\right) \qquad \widetilde{\Gamma}^{(0)}=\left(\begin{array}{cc}\tilde{\gamma}_{gg}^{(0)} & \tilde{\gamma}_{g\lambda}^{(0)}\\ \tilde{\gamma}_{\lambda g}^{(0)} & \tilde{\gamma}_{\lambda\lambda}^{(0)}\end{array}\right)$$

Anomalous dimension matrix in leading order: [Lipatov '00]

$$\Gamma^{(0)} = \begin{pmatrix} \gamma_{gg}^{(0)} & \gamma_{g\lambda}^{(0)} & \gamma_{g\phi}^{(0)} \\ \gamma_{\lambda g}^{(0)} & \gamma_{\lambda\lambda}^{(0)} & \gamma_{\lambda\phi}^{(0)} \\ \gamma_{\phi g}^{(0)} & \gamma_{\phi\lambda}^{(0)} & \gamma_{\phi\phi}^{(0)} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{pmatrix}$$

$$\widetilde{\Gamma}^{(0)} = \begin{pmatrix} \tilde{\gamma}_{gg}^{(0)} & \tilde{\gamma}_{g\lambda}^{(0)} \\ \tilde{\gamma}_{\lambda g}^{(0)} & \tilde{\gamma}_{\lambda\lambda}^{(0)} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} S_1(j-1) & 0 \\ 0 & S_1(j+1) \end{pmatrix}$$

Anomalous dimension matrix in leading order: [Lipatov '00]

$$\Gamma^{(0)} = \begin{pmatrix} \gamma_{gg}^{(0)} & \gamma_{g\lambda}^{(0)} & \gamma_{g\phi}^{(0)} \\ \gamma_{\lambda g}^{(0)} & \gamma_{\lambda\lambda}^{(0)} & \gamma_{\lambda\phi}^{(0)} \\ \gamma_{\phi g}^{(0)} & \gamma_{\phi\lambda}^{(0)} & \gamma_{\phi\phi}^{(0)} \end{pmatrix} \qquad \widetilde{\Gamma}^{(0)} = \begin{pmatrix} \tilde{\gamma}_{gg}^{(0)} & \tilde{\gamma}_{g\lambda}^{(0)} \\ \tilde{\gamma}_{\lambda g}^{(0)} & \tilde{\gamma}_{\lambda\lambda}^{(0)} \end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{pmatrix} \qquad \begin{pmatrix} S_1(j-1) & 0 \\ 0 & S_1(j+1) \end{pmatrix}$$

Eigenvalues of anomalous dimension matrix are expressed trough the same function $\gamma_{uni}^{(0)}(j+2) = S_1(j)$ with shifted argument

Anomalous dimension matrix in leading order: [Lipatov '00]

$$\Gamma^{(0)} = \begin{pmatrix} \gamma_{gg}^{(0)} & \gamma_{g\lambda}^{(0)} & \gamma_{g\phi}^{(0)} \\ \gamma_{\lambda g}^{(0)} & \gamma_{\lambda\lambda}^{(0)} & \gamma_{\lambda\phi}^{(0)} \\ \gamma_{\phi g}^{(0)} & \gamma_{\phi\lambda}^{(0)} & \gamma_{\phi\phi}^{(0)} \end{pmatrix} \qquad \widetilde{\Gamma}^{(0)} = \begin{pmatrix} \tilde{\gamma}_{gg}^{(0)} & \tilde{\gamma}_{g\lambda}^{(0)} \\ \tilde{\gamma}_{\lambda g}^{(0)} & \tilde{\gamma}_{\lambda\lambda}^{(0)} \end{pmatrix}$$

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Eigenvalues of anomalous dimension matrix are expressed trough the same function $\gamma_{uni}^{(0)}(j+2) = S_1(j)$ with shifted argument

Origin:

Anomalous dimension matrix in leading order: [Lipatov '00]

$$\Gamma^{(0)} = \begin{pmatrix} \gamma_{gg}^{(0)} & \gamma_{g\lambda}^{(0)} & \gamma_{g\phi}^{(0)} \\ \gamma_{\lambda g}^{(0)} & \gamma_{\lambda\lambda}^{(0)} & \gamma_{\lambda\phi}^{(0)} \\ \gamma_{\phi g}^{(0)} & \gamma_{\phi\lambda}^{(0)} & \gamma_{\phi\phi}^{(0)} \end{pmatrix} \qquad \widetilde{\Gamma}^{(0)} = \begin{pmatrix} \tilde{\gamma}_{gg}^{(0)} & \tilde{\gamma}_{g\lambda}^{(0)} \\ \tilde{\gamma}_{\lambda g}^{(0)} & \tilde{\gamma}_{\lambda\lambda}^{(0)} \end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{pmatrix} \qquad \begin{pmatrix} S_1(j-1) & 0 \\ 0 & S_1(j+1) \end{pmatrix}$$

Eigenvalues of anomalous dimension matrix are expressed trough the same function $\gamma_{uni}^{(0)}(j+2) = S_1(j)$ with shifted argument

Origin: All multiplicatively renormalizable operators in $\mathcal{N}=4$ SYM theory belong to the same supermultiplet

Anomalous dimension matrix in leading order: [Lipatov '00]

$$\Gamma^{(0)} = \begin{pmatrix} \gamma_{gg}^{(0)} & \gamma_{g\lambda}^{(0)} & \gamma_{g\phi}^{(0)} \\ \gamma_{\lambda g}^{(0)} & \gamma_{\lambda\lambda}^{(0)} & \gamma_{\lambda\phi}^{(0)} \\ \gamma_{\phi g}^{(0)} & \gamma_{\phi\lambda}^{(0)} & \gamma_{\phi\phi}^{(0)} \end{pmatrix} \qquad \widetilde{\Gamma}^{(0)} = \begin{pmatrix} \tilde{\gamma}_{gg}^{(0)} & \tilde{\gamma}_{g\lambda}^{(0)} \\ \tilde{\gamma}_{\lambda g}^{(0)} & \tilde{\gamma}_{\lambda\lambda}^{(0)} \end{pmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{pmatrix} \qquad \begin{pmatrix} S_1(j-1) & 0 \\ 0 & S_1(j+1) \end{pmatrix}$$

Eigenvalues of anomalous dimension matrix are expressed trough the same function $\gamma_{uni}^{(0)}(j+2) = S_1(j)$ with shifted argument

Origin: All multiplicatively renormalizable operators in $\mathcal{N}=4$ SYM theory belong to the same supermultiplet

$$\mathcal{O}^{T}_{\mu_1,\ldots,\mu_j} = \mathcal{O}^{g}_{\mu_1,\ldots,\mu_j} + \mathcal{O}^{\lambda}_{\mu_1,\ldots,\mu_j} + \mathcal{O}^{\phi}_{\mu_1,\ldots,\mu_j}$$

$$\Gamma^{(1)} = \begin{pmatrix} \gamma^{(1)}_{gg} & \gamma^{(1)}_{g\lambda} & \gamma^{(1)}_{g\phi} \\ \gamma^{(1)}_{\lambda g} & \gamma^{(1)}_{\lambda\lambda} & \gamma^{(1)}_{\lambda\phi} \\ \gamma^{(1)}_{\phi g} & \gamma^{(1)}_{\phi\lambda} & \gamma^{(1)}_{\phi\phi} \end{pmatrix} \qquad \widetilde{\Gamma}^{(1)} = \begin{pmatrix} \tilde{\gamma}^{(1)}_{gg} & \tilde{\gamma}^{(1)}_{g\lambda} \\ \tilde{\gamma}^{(1)}_{\lambda g} & \tilde{\gamma}^{(1)}_{\lambda\lambda} \end{pmatrix}$$

$\mathcal{N}=4$ SYM theory: Two loops

$$\Gamma^{(1)}=\begin{pmatrix} \gamma_{gg}^{(1)} & \gamma_{g\lambda}^{(1)} & \gamma_{g\phi}^{(1)} \\ \gamma_{\lambda g}^{(1)} & \gamma_{\lambda\lambda}^{(1)} & \gamma_{\lambda\phi}^{(1)} \\ \gamma_{\phi g}^{(1)} & \gamma_{\phi\lambda}^{(1)} & \gamma_{\phi\phi}^{(1)} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \gamma_{uni}^{(1)}(j-2) & \Gamma_{21} & \Gamma_{31} \\ 0 & \gamma_{uni}^{(1)}(j) & \Gamma_{32} \\ 0 & 0 & \gamma_{uni}^{(1)}(j+2) \end{pmatrix}$$

$$\widetilde{\Gamma}^{(1)}=\begin{pmatrix} \tilde{\gamma}_{gg}^{(1)} & \tilde{\gamma}_{g\lambda}^{(1)} \\ \tilde{\gamma}_{\lambda g}^{(1)} & \tilde{\gamma}_{\lambda\lambda}^{(1)} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \gamma_{uni}^{(1)}(j-1) & \widetilde{\Gamma}_{21} \\ 0 & \gamma_{uni}^{(1)}(j+1) \end{pmatrix}$$

$\mathcal{N}=4$ SYM theory: Two loops

$$\Gamma^{(1)} = \begin{pmatrix} \gamma_{gg}^{(1)} & \gamma_{g\lambda}^{(1)} & \gamma_{g\phi}^{(1)} \\ \gamma_{\lambda g}^{(1)} & \gamma_{\lambda\lambda}^{(1)} & \gamma_{\lambda\phi}^{(1)} \\ \gamma_{\phi g}^{(1)} & \gamma_{\phi\lambda}^{(1)} & \gamma_{\phi\phi}^{(1)} \end{pmatrix} \qquad \widetilde{\Gamma}^{(1)} = \begin{pmatrix} \widetilde{\gamma}_{gg}^{(1)} & \widetilde{\gamma}_{g\lambda}^{(1)} \\ \widetilde{\gamma}_{\lambda g}^{(1)} & \widetilde{\gamma}_{\lambda\lambda}^{(1)} \end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix} \gamma_{uni}^{(1)}(j-2) & \Gamma_{21} & \Gamma_{31} \\ 0 & \gamma_{uni}^{(1)}(j) & \Gamma_{32} \\ 0 & 0 & \gamma_{uni}^{(1)}(j+2) \end{pmatrix} \qquad \begin{pmatrix} \gamma_{uni}^{(1)}(j-1) & \widetilde{\Gamma}_{21} \\ 0 & \gamma_{uni}^{(1)}(j+1) \end{pmatrix}$$

In the next-to-leading order (NLO) the matrix will triangle, but again the eigenvalues are expressed through the same function $\gamma_{uni}^{(1)}(j)$

$\mathcal{N}=4$ SYM theory: Two loops

$$\Gamma^{(1)} = \begin{pmatrix} \gamma_{gg}^{(1)} & \gamma_{g\lambda}^{(1)} & \gamma_{g\phi}^{(1)} \\ \gamma_{\lambda g}^{(1)} & \gamma_{\lambda\lambda}^{(1)} & \gamma_{\lambda\phi}^{(1)} \\ \gamma_{\phi g}^{(1)} & \gamma_{\phi\lambda}^{(1)} & \gamma_{\phi\phi}^{(1)} \end{pmatrix} \qquad \widetilde{\Gamma}^{(1)} = \begin{pmatrix} \tilde{\gamma}_{gg}^{(1)} & \tilde{\gamma}_{g\lambda}^{(1)} \\ \tilde{\gamma}_{\lambda g}^{(1)} & \tilde{\gamma}_{\lambda\lambda}^{(1)} \end{pmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

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In the next-to-leading order (NLO) the matrix will triangle, but again the eigenvalues are expressed through the same function $\gamma_{uni}^{(1)}(j)$

$$\gamma(j) \equiv \gamma_{uni}(j) = \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + \ldots,$$

[KLV '03]

$$\frac{1}{8}\,\gamma_{uni}^{(1)}(j+2) = (S_3(j) + S_{-3}(j)) - 2\,S_{-2,1}(j) + 2\,S_1(j)(S_2(j) + S_{-2}(j))$$

$\mathcal{N}=4$ SYM theory: Two loops

$$\Gamma^{(1)} = \begin{pmatrix} \gamma^{(1)}_{gg} & \gamma^{(1)}_{g\lambda} & \gamma^{(1)}_{g\phi} \\ \gamma^{(1)}_{\lambda g} & \gamma^{(1)}_{\lambda\lambda} & \gamma^{(1)}_{\lambda\phi} \\ \gamma^{(1)}_{\phi g} & \gamma^{(1)}_{\phi\lambda} & \gamma^{(1)}_{\phi\phi} \end{pmatrix} \qquad \widetilde{\Gamma}^{(1)} = \begin{pmatrix} \tilde{\gamma}^{(1)}_{gg} & \tilde{\gamma}^{(1)}_{g\lambda} \\ \tilde{\gamma}^{(1)}_{\lambda g} & \tilde{\gamma}^{(1)}_{\lambda\lambda} \end{pmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix} \gamma^{(1)}_{uni}(j-2) & \Gamma_{21} & \Gamma_{31} \\ 0 & \gamma^{(1)}_{uni}(j) & \Gamma_{32} \\ 0 & 0 & \gamma^{(1)}_{uni}(j+2) \end{pmatrix} \qquad \begin{pmatrix} \gamma^{(1)}_{uni}(j-1) & \widetilde{\Gamma}_{21} \\ 0 & \gamma^{(1)}_{uni}(j+1) \end{pmatrix}$$

In the next-to-leading order (NLO) the matrix will triangle, but again the eigenvalues are expressed through the same function $\gamma^{(1)}_{uni}(j)$

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Nested harmonic sums (level $\ell = |a| + |b| + |c| + \ldots$):

$$S_a(j) = \sum_{k=1}^{j} \frac{(\mathrm{sign}(a))^k}{k^a}, \qquad S_{a,b,c,\ldots}(j) = \sum_{k=1}^{j} \frac{(\mathrm{sign}(a))^k}{k^a}\, S_{b,c,\ldots}(k)$$

Two-loop result in $\mathcal{N}=4$ SYM theory:

[KLV '03]

$$\frac{1}{8}\,\gamma^{(1)}_{uni}(j+2) = -S_3(j) - S_{-3}(j) + 2\,S_{1,-2}(j) + 2\,S_{2,1}(j) + 2\,S_{1,2}(j)$$

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Two-loop result in QCD:

$$\mathbf{N}_{\pm}\, S_{\vec{m}} \;=\; S_{\vec{m}}(N\pm 1)$$

$$\begin{aligned}\gamma_{\rm ns}^{(1)+}(N) \;=\; & 4C_AC_F\left(2\mathbf{N}_+S_3-\frac{17}{24}-2S_{-3}-\frac{28}{3}S_1+(\mathbf{N}_-+\mathbf{N}_+)\left[\frac{151}{18}S_1+2S_{1,-2}-\frac{11}{6}S_2\right]\right)\\ &+4C_Fn_f\left(\frac{1}{12}+\frac{4}{3}S_1-(\mathbf{N}_-+\mathbf{N}_+)\left[\frac{11}{9}S_1-\frac{1}{3}S_2\right]\right)+4{C_F}^2\left(4S_{-3}+2S_1+2S_2-\frac{3}{8}\right.\\ &\left.+\mathbf{N}_-\left[S_2+2S_3\right]-(\mathbf{N}_-+\mathbf{N}_+)\left[S_1+4S_{1,-2}+2S_{1,2}+2S_{2,1}+S_3\right]\right)\end{aligned}$$

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[A.Kotikov, L.Lipatov arXiv:hep-ph/0112346]

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Main result of two-loop calculations:

confirmation of maximal transcendentality principle [KL '02]

$$S_{a,b,c,\cdots}(j) \;=\; \sum_{k=1}^{j} \frac{(\mathrm{sign}(a))^k}{k^a}\, S_{b,c,\cdots}(k)$$

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Using the maximal transcendentality principle we can obtain the universal anomalous dimension in $\mathcal{N}=4$ SYM theory without any calculations from the results obtained in QCD

Three-loop anomalous dimension in QCD: 10 years [Moch,Vermaseren,Vogt '04]

$\mathcal{N}=4$ SYM theory: Three loops

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\begin{aligned}
\gamma_{\rm ns}^{(2)+}(N) \;=\; & 16C_AC_Fn_f\Big(\frac{3}{2}\zeta_3-\frac{5}{4}+\frac{10}{9}S_{-3}-\frac{10}{9}S_3+\frac{4}{3}S_{1,-2}-\frac{2}{3}S_{-4}+2S_{1,1}-\frac{25}{9}S_2+\frac{257}{27}S_1-\frac{2}{3}S_{-3,1}-\mathbf{N}_+\Big[S_{2,1}-\frac{2}{3}S_{3,1}-\frac{2}{3}S_4\Big]-(\mathbf{N}_+-1)\Big[\frac{23}{18}S_3-S_2\Big]-(\mathbf{N}_-+\mathbf{N}_+)\Big[S_{1,1}\\
&+\frac{1237}{216}S_1+\frac{11}{18}S_3-\frac{317}{108}S_2+\frac{16}{9}S_{1,-2}-\frac{2}{3}S_{1,-2,1}-\frac{1}{3}S_{1,-3}-\frac{1}{2}S_{1,3}-\frac{1}{2}S_{2,1}-\frac{1}{3}S_{2,-2}+S_1\zeta_3+\frac{1}{2}S_{3,1}\Big]\Big)+16C_FC_A{}^2\Big(\frac{1657}{576}-\frac{15}{4}\zeta_3+2S_{-5}+\frac{31}{6}S_{-4}-4S_{-4,1}-\frac{67}{9}S_{-3}+2S_{-3,-2}\\
&+\frac{11}{3}S_{-3,1}+\frac{3}{2}S_{-2}-6S_{-2}\zeta_3-2S_{-2,-3}+3S_{-2,-2}-4S_{-2,-2,1}+8S_{-2,1,-2}-\frac{1883}{54}S_1-10S_{1,-3}-\frac{16}{3}S_{1,-2}+12S_{1,-2,1}+4S_{1,3}-4S_{2,-2}-\frac{5}{2}S_4+\frac{1}{2}S_5+\frac{176}{9}S_2+\frac{13}{3}S_3\\
&+(\mathbf{N}_-+\mathbf{N}_+-2)\Big[3S_1\zeta_3+11S_{1,1}-4S_{1,1,-2}\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{9737}{432}S_1-3S_{1,-4}+\frac{19}{6}S_{1,-3}+8S_{1,-3,1}+\frac{91}{9}S_{1,-2}-6S_{1,-2,-2}-\frac{29}{3}S_{1,-2,1}+8S_{1,1,-3}-16S_{1,1,-2,1}-4S_{1,1,3}-\frac{19}{4}S_{1,3}\\
&+4S_{1,3,1}+3S_{1,4}+8S_{2,-2,1}+2S_{2,3}-S_{3,-2}+\frac{11}{12}S_{3,1}-S_{4,1}-4S_{2,-3}+\frac{1}{6}S_{2,-2}-\frac{1967}{216}S_2+\frac{121}{72}S_3\Big]-(\mathbf{N}_--\mathbf{N}_+)\Big[3S_2\zeta_3+7S_{2,1}-3S_{2,1,-2}+2S_{2,-2,1}-\frac{1}{4}S_{2,3}-\frac{3}{2}S_{3,-2}-\frac{29}{6}S_{3,1}\\
&+\frac{11}{4}S_{4,1}+\frac{1}{2}S_{2,-3}-S_{2,-2}\Big]+\mathbf{N}_+\Big[\frac{28}{9}S_3-\frac{2376}{216}S_2-\frac{8}{3}S_4-\frac{5}{2}S_5\Big]\Big)+16C_Fn_f^2\Big(\frac{17}{144}-\frac{13}{27}S_1+\frac{2}{9}S_2+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{2}{9}S_1-\frac{11}{54}S_2+\frac{1}{18}S_3\Big]\Big)+16C_F{}^2C_A\Big(\frac{45}{4}\zeta_3-\frac{151}{64}-10S_{-5}\\
&-\frac{89}{6}S_{-4}+20S_{-4,1}+\frac{134}{9}S_{-3}-2S_{-3,-2}-\frac{31}{3}S_{-3,1}+2S_{-3,2}-\frac{9}{2}S_{-2}+18S_{-2}\zeta_3+10S_{-2,-3}-6S_{-2,-2}+8S_{-2,-2,1}-28S_{-2,1,-2}+46S_{1,-3}+\frac{26}{3}S_{1,-2}-48S_{1,-2,1}+\frac{28}{3}S_{1,2}-\frac{185}{6}S_3\\
&-8S_{1,3}+2S_{3,-2}-4S_5-(\mathbf{N}_-+\mathbf{N}_+-2)\Big[9S_1\zeta_3-\frac{133}{36}S_1+\frac{209}{6}S_{1,1}-14S_{1,1,-2}-\frac{242}{18}S_2+9S_{2,-2}+\frac{33}{4}S_4-3S_{3,1}+\frac{14}{3}S_{2,1}\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[17S_{1,-4}-\frac{107}{6}S_{1,-3}-32S_{1,-3,1}\\
&-\frac{173}{9}S_{1,-2}+16S_{1,-2,-2}+\frac{103}{3}S_{1,-2,1}-2S_{1,-2,2}-36S_{1,1,-3}+56S_{1,1,-2,1}+8S_{1,1,3}-\frac{109}{9}S_{1,2}-4S_{1,2,-2}+\frac{43}{3}S_{1,3}-8S_{1,3,1}-11S_{1,4}+\frac{11}{3}S_{2,2}+21S_{2,-3}-30S_{2,-2,1}-4S_{2,1,-2}\\
&-5S_{2,3}-S_{4,1}+\frac{31}{6}S_{2,-2}-\frac{67}{9}S_{2,1}\Big]+(\mathbf{N}_--\mathbf{N}_+)\Big[9S_2\zeta_3+2S_{2,-3}+4S_{2,-2,1}-12S_{2,1,-2}-2S_{2,3}+13S_{4,1}+\frac{1}{2}S_{2,-2}+\frac{11}{2}S_4-\frac{33}{2}S_{3,1}+\frac{59}{9}S_3+\frac{127}{6}S_{2,1}-\frac{1153}{72}S_2\Big]+\mathbf{N}_+\Big[8S_{3,-2}\\
&+\frac{4}{3}S_{3,1}-2S_{3,2}+14S_5+\frac{23}{6}S_4+\frac{73}{3}S_3+\frac{151}{24}S_2\Big]\Big)+16C_F{}^2n_f\Big(\frac{23}{16}-\frac{3}{2}\zeta_3+\frac{4}{3}S_{-3,1}-\frac{59}{36}S_2+\frac{4}{3}S_{-4}-\frac{20}{9}S_{-3}+\frac{20}{9}S_1-\frac{8}{3}S_{1,-2}-\frac{8}{3}S_{1,1}-\frac{4}{3}S_{1,2}+\mathbf{N}_+\Big[\frac{25}{9}S_3-\frac{4}{3}S_{3,1}-\frac{1}{3}S_4\Big]\\
&-(\mathbf{N}_+-1)\Big[\frac{67}{36}S_2-\frac{4}{3}S_{2,1}+\frac{4}{3}S_3\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[S_1\zeta_3-\frac{325}{144}S_1-\frac{2}{3}S_{1,-3}+\frac{32}{9}S_{1,-2}-\frac{4}{3}S_{1,-2,1}+\frac{4}{3}S_{1,1}+\frac{16}{9}S_{1,2}-\frac{4}{3}S_{1,3}+\frac{11}{18}S_2-\frac{2}{3}S_{2,-2}+\frac{10}{9}S_{2,1}+\frac{1}{2}S_4-\frac{2}{3}S_{2,2}-\frac{8}{9}S_3\Big]\Big)\\
&+16C_F{}^3\Big(12S_{-5}-\frac{29}{32}-\frac{15}{2}\zeta_3+9S_{-4}-24S_{-4,1}-4S_{-3,-2}+6S_{-3,1}-4S_{-3,2}+3S_{-2}+25S_3-12S_{-2}\zeta_3-12S_{-2,-3}+24S_{-2,1,-2}-52S_{1,-3}+4S_{1,-2}+48S_{1,-2,1}-4S_{3,-2}+\frac{67}{2}S_2-17S_4\\
&+(\mathbf{N}_-+\mathbf{N}_+-2)\Big[6S_1\zeta_3-\frac{31}{8}S_1+35S_{1,1}-12S_{1,1,-2}+S_{1,2}+10S_{2,-2}+S_{2,1}+2S_{2,2}-2S_{3,1}-3S_5\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[23S_{1,-3}-22S_{1,-4}+32S_{1,-3,1}-2S_{1,-2}-8S_{1,-2,-2}-30S_{1,-2,1}-6S_{1,3}\\
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&+(\mathbf{N}_--\mathbf{N}_+)\Big[12S_{2,1,-2}-6S_2\zeta_3-2S_{2,-3}+3S_{2,3}+2S_{3,-2}-\frac{81}{4}S_{2,1}+14S_{3,1}-5S_{2,-2}-\frac{1}{2}S_{2,2}+\frac{15}{8}S_2+\frac{1}{2}S_3-13S_{4,1}+4S_5\Big]+\mathbf{N}_+\Big[14S_4-\frac{265}{8}S_2-\frac{87}{4}S_3-4S_{4,1}-4S_5\Big]\Big)
\end{aligned}
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Three-loop anomalous dimension in QCD: 10 years [Moch, Vermaseren, Vogt '04]

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&+\frac{4}{3}S_{3,1}-2S_{3,2}+14S_5+\frac{23}{6}S_4+\frac{73}{3}S_3+\frac{151}{24}S_2\bigg]\bigg)+16C_F{}^2n_f\bigg(\frac{23}{16}-\frac{3}{2}\zeta_3+\frac{4}{3}S_{-3,1}-\frac{59}{36}S_2+\frac{4}{3}S_{-4}-\frac{20}{9}S_{-3}+\frac{20}{9}S_1-\frac{8}{3}S_{1,-2}-\frac{8}{3}S_{1,1}-\frac{4}{3}S_{1,2}+\mathbf{N}_+\bigg[\frac{25}{9}S_3-\frac{4}{3}S_{3,1}-\frac{1}{3}S_4\bigg]\\
&-(\mathbf{N}_+-1)\bigg[\frac{67}{36}S_2-\frac{4}{3}S_{2,1}+\frac{4}{3}S_3\bigg]+(\mathbf{N}_-+\mathbf{N}_+)\bigg[S_1\zeta_3-\frac{325}{144}S_1-\frac{2}{3}S_{1,-3}+\frac{32}{9}S_{1,-2}-\frac{4}{3}S_{1,-2,1}+\frac{4}{3}S_{1,1}+\frac{16}{9}S_{1,2}-\frac{4}{3}S_{1,3}+\frac{11}{18}S_2-\frac{2}{3}S_{2,-2}+\frac{10}{9}S_{2,1}+\frac{1}{2}S_4-\frac{2}{3}S_{2,2}-\frac{8}{9}S_3\bigg]\bigg)\\
&+16C_F{}^3\bigg(12S_{-5}-\frac{29}{32}-\frac{15}{2}\zeta_3+9S_{-4}-24S_{-4,1}-4S_{-3,-2}+6S_{-3,1}-4S_{-3,2}+3S_{-2}+25S_3-12S_{-2}\zeta_3-12S_{-2,-3}+24S_{-2,1,-2}-52S_{1,-3}+4S_{1,-2}+48S_{1,-2,1}-4S_{3,-2}+\frac{67}{2}S_2-17S_4\\
&+(\mathbf{N}_-+\mathbf{N}_+-2)\bigg[6S_1\zeta_3-\frac{31}{8}S_1+35S_{1,1}-12S_{1,1,-2}+S_{1,2}+10S_{2,-2}+S_{2,1}+2S_{2,2}-2S_{3,1}-3S_5\bigg]+(\mathbf{N}_-+\mathbf{N}_+)\bigg[23S_{1,-3}-22S_{1,-4}+32S_{1,-3,1}-2S_{1,-2}-8S_{1,-2,-2}-30S_{1,-2,1}-6S_{1,3}\\
&+4S_{1,-2,2}+40S_{1,1,-3}-48S_{1,1,-2,1}+8S_{1,2,-2}+4S_{1,2,2}+8S_{1,3,1}+4S_{1,4}+28S_{2,-2,1}+4S_{2,1,2}+4S_{2,2,1}+4S_{3,1,1}-4S_{3,2}+8S_{2,1,-2}-26S_{2,-3}-2S_{2,3}-4S_{3,-2}-3S_{2,-2}-3S_{2,2}+\frac{3}{2}S_4\bigg]\\
&+(\mathbf{N}_--\mathbf{N}_+)\bigg[12S_{2,1,-2}-6S_2\zeta_3-2S_{2,-3}+3S_{2,3}+2S_{3,-2}-\frac{81}{4}S_{2,1}+14S_{3,1}-5S_{2,-2}-\frac{1}{2}S_{2,2}+\frac{15}{8}S_2+\frac{1}{2}S_3-13S_{4,1}+4S_5\bigg]+\mathbf{N}_+\bigg[14S_4-\frac{265}{8}S_2-\frac{87}{4}S_3-4S_{4,1}-4S_5\bigg]\bigg)
\end{aligned}
$$

$\mathcal{N} = 4$ SYM theory: Three loops

Three-loop anomalous dimension in QCD: 10 years [Moch,Vermaseren,Vogt '04]

Applied maximal transcendentality principle: immediately [KLOV '04]

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$$\begin{aligned}\frac{1}{32}\,\gamma^{(2)}_{uni}(j) \quad = \quad & 2\,S_{-3}\,S_2 - S_5 - 2\,S_{-2}\,S_3 - 3\,S_{-5} + 24\,S_{-2,1,1,1} \\ & +6(S_{-4,1} + S_{-3,2} + S_{-2,3}) - 12(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) \\ & -(S_2 + 2\,S_1^2)(3\,S_{-3} + S_3 - 2\,S_{-2,1}) - S_1(8\,S_{-4} + S^2_{-2} \\ & +4\,S_2\,S_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,S_{-3,1} - 10\,S_{-2,2} + 16\,S_{-2,1,1})\end{aligned}$$

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In particular case $j=2$

$$\gamma_{uni}(2) = 12\,g^2 - 48\,g^4 + 336\,g^6 + \cdots\,, \qquad g^2 = \frac{g^2_{YM}\,N}{(4\pi^2)^2}$$

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Confirmation of the result for the anomalous dimension of Konishi in $\mathcal{N}=4$ SYM theory from integrability (ABA)

AdS/CFT-correspondence

[Maldacena '97] [Witten '98] [Gubser, Klebanov, Polyakov '98]

Maximally extended supersymmetric gauge theory	$\Leftrightarrow$	IIB string on $AdS_5 \times S^5$

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spin	Maximally extended supersymmetric gauge theory	$\Leftrightarrow$	IIB string on $AdS_5 \times S^5$
1	Gauge field $\mathcal{A}_\mu$		
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0	3 complex scalars Φ^r		
	Conformal Field Theory		

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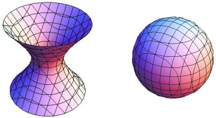
	Conformal Field Theory		AdS - Anti-de Sitter space with negative curvature



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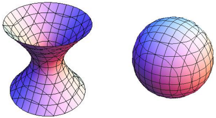

The same symmetry

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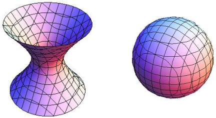

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Operators – $\mathcal{O}_A(x) = \text{Tr}\,\mathcal{A}\ldots\Psi\ldots\Phi$ ⇔ $|\mathcal{O}_A\rangle$ – String states

Dimension – $\Delta = 2 + \sqrt{4 + m^2 R^2}$ ⇔ m – Mass

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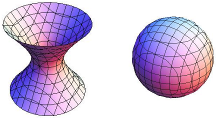

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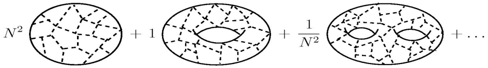

$1/N \;\Leftrightarrow\; g_{st}$

[Maldacena '97] [Witten '98] [Gubser, Klebanov, Polyakov '98]

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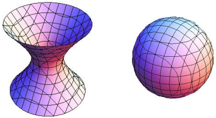

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$$1/N \quad \Leftrightarrow \quad g_{st}$$

$$\lambda = g_{YM}^2 N \quad \Leftrightarrow \quad \lambda = R^4/\alpha'^2$$

$$\langle \mathcal{O}_A(x)\mathcal{O}_B(y)\rangle \sim \frac{\delta_{A,B}}{(x-y)^{2\Delta\left(\lambda,\frac{1}{N}\right)}} \quad \Leftrightarrow \quad \mathcal{H}_{String}|\mathcal{O}_A\rangle = E_A\Big(\tfrac{1}{\sqrt{\lambda}}, g_S\Big)|\mathcal{O}_A\rangle$$

$$\Delta\left(\lambda, \tfrac{1}{N}\right) \quad \lambda \ll 1 \quad = \quad E\left(\tfrac{1}{\sqrt{\lambda}}, g_S\right) \quad \lambda \gg 1$$

BMN-operators:

[Berenstein, Maldacena and Nastase '02]

$$\text{Tr}\, Z^J = \text{Tr}\, ZZZZZZZZZZZZ\ldots$$

$$\text{Tr}\, XZ^J = \text{Tr}\, XZZ\ldots = \text{Tr}\, ZXZ\ldots$$

$$\sum_k \text{Tr}\ XZ^k XZ^{J-k}$$

$X = \Phi^1, Y = \Phi^2$ and $Z = \Phi^3$ – scalar fields

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Two dimensional quantum field theory in flat space

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Computations of anomalous dimension of BMN-operators: $\text{Tr}\; XZXZ$

Integrability in $\mathcal{N}=4$ SYM

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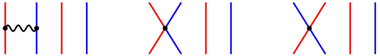

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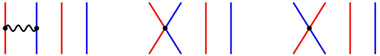

$$\mathrm{Tr}\, XZXZ_{\mathrm{Ren}} = A_1(\lambda)\,\mathrm{Tr}\, XZXZ + A_2(\lambda)\,\mathrm{Tr}\, ZXXZ$$

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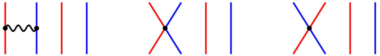

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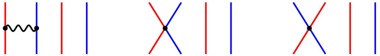

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Relation with Heisenberg spin chain [Minahan and Zarembo '02]

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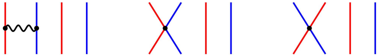

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Two dimensional quantum field theory in flat space

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$\mathcal{H}_1 = \frac{1}{2}\left(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1}\right)$ – Hamiltonian of $\text{XXX}_{1/2}$ spin chain

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Computations of anomalous dimension of BMN-operators: Tr $XZXZ$

$$\mathcal{D}(\lambda)\begin{pmatrix}\text{Tr}\, XZXZ\\ \text{Tr}\, ZXXZ\end{pmatrix} = \mathbb{A}(\lambda)\begin{pmatrix}\text{Tr}\, XZXZ\\ \text{Tr}\, ZXXZ\end{pmatrix} \quad \Leftrightarrow \quad \mathcal{H}(\lambda)\begin{pmatrix}|\downarrow\uparrow\downarrow\uparrow\rangle\\ |\uparrow\downarrow\downarrow\uparrow\rangle\end{pmatrix} = \mathbb{E}(\lambda)\begin{pmatrix}|\downarrow\uparrow\downarrow\uparrow\rangle\\ |\uparrow\downarrow\downarrow\uparrow\rangle\end{pmatrix}$$

$\mathcal{H}_1 = \frac{1}{2}\left(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1}\right)$ – Hamiltonian of $\text{XXX}_{1/2}$ spin chain

Exact solution: INTEGRABILITY (Bethe-Anzatz) [Bethe '31, Faddeev... '80]

BMN-operators: [Berenstein, Maldacena and Nastase '02]

Relation with Heisenberg spin chain [Minahan and Zarembo '02]

$$\text{Tr}\, Z^J = \text{Tr}\, ZZZZZZZZZZZZ\ldots \quad\Leftrightarrow\quad |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle$$

$$\text{Tr}\, XZ^J = \text{Tr}\, XZZ\ldots = \text{Tr}\, ZXZ\ldots \quad\Leftrightarrow\quad |\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle$$

$$\sum_k \text{Tr}\, XZ^kXZ^{J-k} \quad\Leftrightarrow\quad |\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle$$

$X=\Phi^1, Y=\Phi^2$ and $Z=\Phi^3$ – scalar fields $\Leftrightarrow$ spin up and spin down

BMN-limit: $N, J\to\infty,\ \lambda' = \dfrac{g_{YM}^2 N}{J^2}$

Two dimensional quantum field theory in flat space

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Exact solution: INTEGRABILITY (Bethe-Anzatz) [Bethe '31, Faddeev... '80]

$\mathcal{H}_2$ and $\mathcal{H}_3$ were computed: TEST [Beisert, Staudacher '03]

Bethe-anzatz: $\mathcal{H}\,|\Psi\rangle = \mathbb{E}\,|\Psi\rangle$ [Bethe '31]

$$\psi(x_1,x_2) = e^{ip_1x_1+ip_2x_2} + S(p_1,p_2)\,e^{ip_2x_1+ip_1x_2}$$

p_j are fixed by periodic boundary conditions $\psi(x_1,x_2) = \psi(x_2,x_1+L)$

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Leading order: $|\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle$ [Minahan and Zarembo '02]

$$H_{SU(2)} \quad = \quad \frac{\lambda}{16\pi^2}\sum_{x=1}^{L}(I_x\cdot I_{x+1}-\vec{\sigma}_x\cdot\vec{\sigma}_{x+1})$$

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$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M}\frac{u_k-u_j+i}{u_k-u_j-i} \qquad E_0(M)=\sum_{k=1}^{M}\frac{1}{u_k^2+\frac{1}{4}} \qquad u_k=\frac{1}{2}\cot\left(\frac{p_k}{2}\right)$$

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$$x(u)=u$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M}\frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \qquad E_0(M)=2\sum_{k=1}^{M}\left(\frac{i}{x_k^+}-\frac{i}{x_k^-}\right) \qquad x_k^\pm = x(u_k\pm\tfrac{i}{2})$$

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All-loop Bethe anzatz: $\lambda = g^2 N$ [Beisert and Staudacher, Kazakov '04-'06]

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The same Bethe-anzatz from the field theory and from the string

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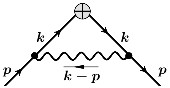
Solution: In leading order – system of non-linear equations

Explicit solution: $Q_M(u) = C_M\prod_{k=1}^{M}(u-u_k) = {}_3F_2[-M, M+1, \frac{1}{2} - iu; 1, 1; 1]$

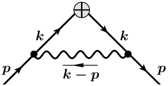
At higher orders – system of linear equations

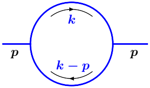
Calculations



Calculations

two-point

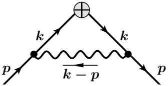


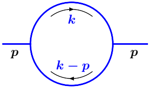


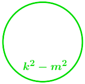
$$\int d^D k \, \frac{1}{k^2 \, (k-p)^2}$$

Calculations

IR-rearrangement: two-point $\rightarrow$ bubble [Vladimirov '78] [Misiak, Munz '94]







$$\int d^D k \, \frac{1}{k^2 \, (k-p)^2} \overset{\text{UV-pole}}{=} \int d^D k \, \frac{1}{(k^2-m^2)^2}$$

Calculations

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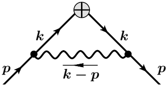
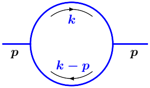
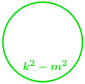






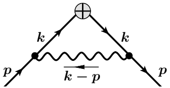
$$\int d^D k \, \frac{1}{k^2\,(k-p)^2} \overset{\text{UV-pole}}{=} \int d^D k \, \frac{1}{(k^2-m^2)^2}$$

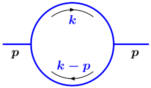
Expansion over p:
$$\frac{1}{(k-p)^2 - m^2} = \frac{1}{k^2 - m^2} + \frac{2\,k\,p - p^2}{k^2 - m^2}\,\frac{1}{(k-p)^2 - m^2}$$

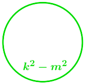
Calculations

IR-rearrangement: two-point → bubble

[Vladimirov '78]
[Misiak, Munz '94]







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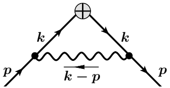
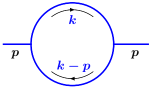
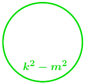
Calculations

IR-rearrangement: two-point $\rightarrow$ bubble

[Vladimirov '78]
[Misiak, Munz '94]







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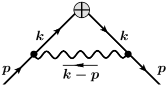

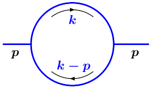

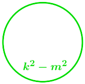

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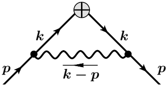

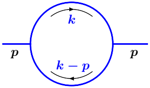

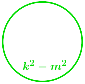

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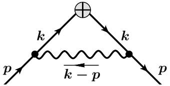

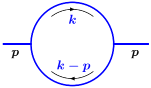

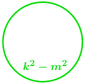

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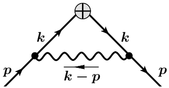

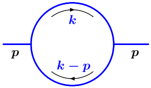

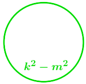

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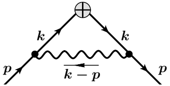

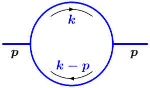

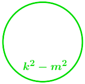

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Multiplicatively renormalizable operators:

$$\begin{aligned}
\mathcal{O}^{T_j}_{\mu_1,\dots,\mu_j} &= \mathcal{O}^{g}_{\mu_1,\dots,\mu_j} + \mathcal{O}^{\lambda}_{\mu_1,\dots,\mu_j} + \mathcal{O}^{\phi}_{\mu_1,\dots,\mu_j} \\
\mathcal{O}^{\Sigma_j}_{\mu_1,\dots,\mu_j} &= -2\,(j-1)\,\mathcal{O}^{g}_{\mu_1,\dots,\mu_j} + \mathcal{O}^{\lambda}_{\mu_1,\dots,\mu_j} + \frac{2}{3}\,(j+1)\,\mathcal{O}^{\phi}_{\mu_1,\dots,\mu_j} \\
\mathcal{O}^{\Xi_j}_{\mu_1,\dots,\mu_j} &= -\frac{j-1}{j+2}\,\mathcal{O}^{g}_{\mu_1,\dots,\mu_j} + \mathcal{O}^{\lambda}_{\mu_1,\dots,\mu_j} - \frac{j+1}{j}\,\mathcal{O}^{\phi}_{\mu_1,\dots,\mu_j}
\end{aligned}$$

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\end{aligned}
$$

Relations: $\gamma_{ab} = \gamma^{(0)}_{ab}(j)$

$$
\begin{aligned}
\gamma_{gg} + \gamma_{\lambda g} + \gamma_{\phi g} &= \gamma^{(0)}_{\mathrm{uni}}(j) \\
\gamma_{g\lambda} + \gamma_{\lambda\lambda} + \gamma_{\phi\lambda} &= \gamma^{(0)}_{\mathrm{uni}}(j) \\
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\end{aligned}$$

Relations: $\gamma_{ab} = \gamma^{(0)}_{ab}(j)$

$$\begin{aligned}
\gamma_{gg} - \frac{1}{2(j-1)}\,\gamma_{\lambda g} + \frac{1}{3}\,\frac{j+1}{j-1}\,\gamma_{\phi g} &= \gamma^{(0)}_{\text{uni}}(j+2) \\
-2\,(j-1)\,\gamma_{g\lambda} + \gamma_{\lambda\lambda} + \frac{2}{3}\,(j+1)\,\gamma_{\phi\lambda} &= \gamma^{(0)}_{\text{uni}}(j+2) \\
-3\,\frac{j-1}{j+1}\,\gamma_{g\phi} + \frac{3}{2(j+1)}\,\gamma_{\lambda\phi} + \gamma_{\phi\phi} &= \gamma^{(0)}_{\text{uni}}(j+2)
\end{aligned}$$

$\mathcal{N}=4$ SYM theory: One loop

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Relations: $\gamma_{ab} = \gamma^{(0)}_{ab}(j)$

$$\gamma_{gg} - \frac{j+2}{j-1}\,\gamma_{\lambda g} + \frac{j+1}{j}\,\frac{j+2}{j-1}\,\gamma_{\phi g} = \gamma^{(0)}_{\rm uni}(j+4)$$

$$-\frac{j-1}{j+2}\,\gamma_{g\lambda} + \gamma_{\lambda\lambda} - \frac{j+1}{j}\gamma_{\phi\lambda} = \gamma^{(0)}_{\rm uni}(j+4)$$

$$\frac{j-1}{j+2}\,\frac{j}{j+1}\,\gamma_{g\phi} - \frac{j}{j+1}\,\gamma_{\lambda\phi} + \gamma_{\phi\phi} = \gamma^{(0)}_{\rm uni}(j+4)$$

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We need only $\gamma_{ab}(j)$ to find $\gamma_{\text{uni}}(j+4)$

$\mathcal{N}=4$ SYM theory: Fixed j

$$
\begin{aligned}
\gamma_{g\lambda} + \gamma_{\lambda\lambda} + \gamma_{\phi\lambda} &= \gamma^{(0)}_{\rm uni}(j) \\
-2\,(j-1)\,\gamma_{g\lambda} + \gamma_{\lambda\lambda} + \frac{2}{3}\,(j+1)\,\gamma_{\phi\lambda} &= \gamma^{(0)}_{\rm uni}(j+2) \\
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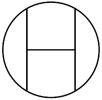
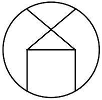
$$-3\,\frac{j-1}{j+1}\,\gamma_{g\phi} + \frac{3}{2(j+1)}\,\boxed{\gamma_{\lambda\phi}} + \gamma_{\phi\phi} = \boxed{\gamma_{\rm uni}^{(0)}(j+2)}$$

$$\frac{j-1}{j+2}\,\frac{j}{j+1}\,\gamma_{g\phi} - \frac{j}{j+1}\,\boxed{\gamma_{\lambda\phi}} + \gamma_{\phi\phi} = \gamma_{\rm uni}^{(0)}(j+4)$$

We need only $\gamma_{\lambda\phi}(j)$ to find $\gamma_{\rm uni}(j+4)$

Basic parent four-loop planar and non-planar tadpole topologies





Two loops: QCD vs. $\mathcal{N} = 4$ SYM theory

Basic parent four-loop planar and non-planar tadpole topologies

Non-planar contribution appear at four loops for the first time
No Renormalization

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$$\hat{\gamma}^{(3),\mathrm{np}}_{\lambda\lambda}(4) = \left(\frac{12901}{225} - \frac{68}{3}\,\mathsf{S}_2 + \frac{417091}{2430}\,\zeta_3 - \frac{9962}{15}\,\zeta_5\right)\frac{48\,g^8}{N_c^2}$$

$$\hat{\gamma}^{(3),\mathrm{np}}_{\lambda\phi}(4) = \left(\frac{809357}{1728} + \frac{91943}{450}\,\mathsf{S}_2 + \frac{742594303}{466560}\,\zeta_3 - \frac{23072}{9}\,\zeta_5\right)\frac{48\,g^8}{N_c^2}$$

Two loops: QCD vs. $\mathcal{N} = 4$ SYM theory

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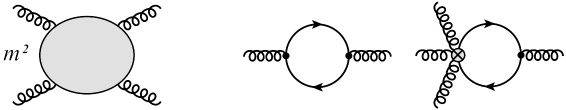
Gauge-non-invariant operator



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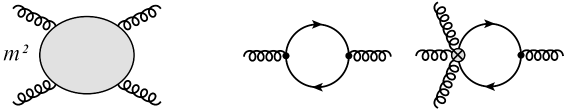
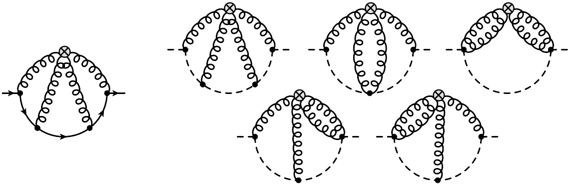


Diagrams with gauge-non-invariant operators



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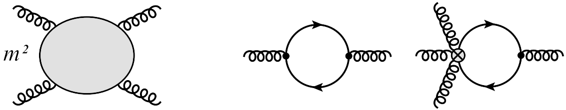

Renormalization

$$\gamma_{\lambda\lambda}^{(3),\mathrm{np,ren}}(4) = \left(-\frac{4}{3}+\frac{68}{3}\,\mathsf{S}_2+\frac{3781}{486}\,\zeta_3\right)\frac{48\,g^8}{N_c^2}$$

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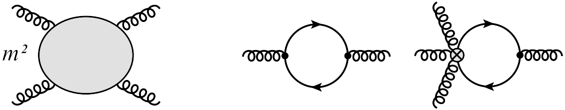

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Four-loop Konishi $\mathcal{O}_{\text{Konishi}} = \bar{\phi}_r \phi^r$ [0808.3832],[0902.4646]

$$\gamma_{\text{Konishi}}^{4-\text{loop}} = \left(-2496 + 576\,\zeta_3 - 1440\,\zeta_5 - \frac{17280}{N_c^2}\,\zeta_5\right) g^8, \quad g^2 = \frac{g_{YM}^2\,N_c}{(4\,\pi)^2}$$

$\mathcal{N}=4$ SYM theory: Non-planarity for fixed j

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Anomalous dimension from twist-2 operators with $j=2$ [1008.2752]

$$\gamma_{\text{uni,np}}(4) = -360\,\zeta_5\,\frac{48\,g^8}{N_c^2} + \mathcal{O}(g^{10})$$

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Anomalous dimension from twist-2 operators with $j=4$ [1404.7107]

$$\gamma_{\lambda\lambda}^{(3),\text{np}}(4) = \left(\frac{12601}{225} + \frac{8074}{45}\,\zeta_3 - \frac{9962}{15}\,\zeta_5\right)\frac{48\,g^8}{N_c^2}$$

$$\gamma_{\lambda\phi}^{(3),\text{np}}(4) = \left(\frac{21452}{45} + \frac{13648}{9}\,\zeta_3 - \frac{23072}{9}\,\zeta_5\right)\frac{48\,g^8}{N_c^2}$$

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Reconstruction of a general expression for arbitrary j ?

Twist-2 in $\mathcal{N} = 4$ SYM: Reconstruction

Anomalous dimension of twist-2 operators $\mathrm{Tr} Z \mathcal{D}_{\mu_1} \mathcal{D}_{\mu_2} \ldots \mathcal{D}_{\mu_M} Z$

1. Nested harmonic sums

$$S_{a,b,c,\cdots}(j) = \sum_{k=1}^{j} \frac{(\mathrm{sign}(a))^k}{k^a} S_{b,c,\cdots}(k)$$

2. Maximal transcendentality principle

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$$\gamma(M) = x_1 B_1(M) + x_2 B_2(M) + \ldots + x_n B_n(M)$$

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Transcend.	1	2	3	4	5	6	7	8	9	10	11
	1	3	7	17	41	99	239	577	1393	3363	8119

Spliting function:

$$P_{gg}^{(0)}(z) = 2C_A\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] + \left(\frac{11}{6}C_A - \frac{4}{3}T_F\right)\delta(z-1)$$

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"Reciprocity-respecting" function $\mathcal{P}(M)$:

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Symmetry: Reciprocity for anomalous dimension

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All is much simpler

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There are remarkable objects - binomial harmonic sums

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Binomial harmonic sums: Reconstruction

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Reconstruction of $\mathcal{P}(M)$ from results for fixed M:

$$\mathcal{P}(M) = x_1\mathbb{B}_1(M) + x_2\mathbb{B}_2(M) + \ldots + x_n\mathbb{B}_n(M)$$

Assumptions about basis $\mathbb{B}_i(M)$:

1. Binomial harmonic sums $\mathbb{S}_{a,b,c,\cdots}(j)$ and ζ_a
2. Maximal transcendentality
3. Absence of ζ_a with even a ($\zeta_{2i} \sim \pi^{2i}$)

Reconstruction: LLL-algorithm

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The Lenstra-Lenstra-Lovász (LLL) lattice basis reduction algorithm

Polynomial time lattice reduction algorithm invented by
Arjen Lenstra, Hendrik Lenstra and László Lovász in 1982
The LLL algorithm outputs an LLL-reduced (short, nearly orthogonal) lattice basis under the Euclidean norm

- start from the system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ \frac{15}{32}x_1 + \frac{27}{32}x_2 + \frac{33}{32}x_3 + \frac{39}{32}x_4 + \frac{21}{32}x_5 &= \frac{9}{16} \end{aligned}$$

- take matrix (divide to the greatest common divisor – `GCD` function):

$$\mathsf{SE} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 5 & 9 & 11 & 13 & 7 & -6 \end{pmatrix}$$

- multiply SE to some huge integer number, for example 8^8
- create unity matrix $\mathbb{I}$ with rank equal to the length of row in SE
- append transpose SE to the right side of the unity matrix $\mathbb{I}$:

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$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 8^8 & 5\times 8^8 \\ 0 & 1 & 0 & 0 & 0 & 0 & 8^8 & 9\times 8^8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 8^8 & 11\times 8^8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 8^8 & 13\times 8^8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 8^8 & 7\times 8^8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -6\times 8^8 \end{pmatrix}$$

- apply `LatticeReduce` to this matrix

- start from the system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ \frac{15}{32}x_1 + \frac{27}{32}x_2 + \frac{33}{32}x_3 + \frac{39}{32}x_4 + \frac{21}{32}x_5 &= \frac{9}{16} \end{aligned}$$

- take matrix (divide to the greatest common divisor – `GCD` function):

$$\mathsf{SE} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 5 & 9 & 11 & 13 & 7 & -6 \end{pmatrix}$$

- multiply SE to some huge integer number, for example 8^8
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- apply `LatticeReduce` to this matrix

As result we obtain the following matrix:

$$\text{RSE} = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -8^8 & -8^8 \\ -1 & 0 & 0 & 0 & 0 & -1 & -8^8 & 8^8 \end{pmatrix}$$

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Origin: Application of LLL-algorithm to solution of Diophantine equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -a_1 & -b_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -a_2 & -b_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -a_3 & -b_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -a_4 & -b_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & -a_5 & -b_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & -a_6 & -b_6 \end{pmatrix}$$

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LLL-algorithm: System of linear diophantine equations

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We need about ten times less values of $\mathcal{P}(M)$ for fixed M

Open problems

1. Reconstruction of the non-planar contribution to the four-loop universal anomalous dimension in $\mathcal{N} = 4$ SYM theory

2. Calculation of odd or higher even moments of the non-planar contribution to the four-loop universal anomalous dimension in $\mathcal{N} = 4$ SYM theory

3. Calculation of higher moments for four-loop anomalous dimension of non-singlet twist-2 operator in QCD

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Have not found any reasonable solution for binomial harmonic sums:

$$\mathbb{S}_{i_1,\ldots,i_k}(M) = \sum_{j=1}^{M}(-1)^j \binom{M}{j}\binom{M+j}{j} S_{i_1,\ldots,i_k}(j)$$

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Non-singlet twist-2 operators in QCD: $\bar{q}\,\gamma_{\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\ldots\mathcal{D}_{\mu_M}\,q$