Simplified differential equations approach for the calculation of multi-loop integrals

Chris Wever (N.C.S.R. Demokritos)

C. Papadopoulos, D. Tommasini, C. Wever [to appear]

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Motivation

- Mismatch between theory and experimental result
- Theory prediction up to NLO, full NNLO calculation might resolve the discrepancy

[CMS 2013]
Outline

- Introduction
- Differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook
EW importance

- First run LHC 2008-2013 at 7-8 TeV
- Second run to begin in 2015 at 13 TeV
- Experimental accuracy at LHC improving

Uncertainty for total Higgs production: Theory ~ 10%, Exp. ~10-20%

In particular EW processes important because of bigger signal to background ratio

HDECAY
[Djouadi, Kalinowski, Spira '97]

BR (%):

- 60
- 6
- 0.2
- 20
- 2.5
Calculations based on perturbative expansion:

\[ \sigma \sim \sigma_0 + \sigma_1 \alpha + \sigma_2 \alpha^2 + \cdots \]

- Improved results by including resummation
- Experimental accuracy better than theoretical prediction

By now factor ~1000 more luminosity!
Example at NLO

Imagine having to calculate a six-point gluon amplitude:

\[ A^{1\text{-loop}} = \sum_{\text{diagrams}} \int d^d k \left\{ \gamma_\mu, k, f^{abc}, \cdots \right\} \]

Amount of diagrams grows rapidly with the amount and diversity of external particles

Require higher-loop corrections for many processes

Do we need to calculate each diagram separately?

automation

use unitarity
NLO revolution: efficient reduction

- Automation possible because of existence of basis of **Master Integrals** (MI)
- Valid for any **renormalizable QFT** (up to $O(\epsilon)$):

\[
A^{1-\text{loop}} = \sum + \sum + \sum + \sum + \mathcal{R}
\]

- Traditional reduction: **PV-tensor reduction** [Pasarino & Veltman ’79]
- MI are known [’t Hooft & Veltman ’79], numerically in FF/LoopTools [Oldenborgh ’90, Hahn & Victoria ’98], QCDloop [Ellis & Zanderighi ’07] and OneLOop [A. van Hameren ’10]
NLO revolution: efficient reduction

- Automation possible because of existence of basis of **Master Integrals** (MI)
- Valid for any **renormalizable QFT** (up to $O(\epsilon)$):
  \[
  A^{1\text{-loop}} = \sum \text{square} + \sum \text{triangle} + \sum \text{circle} + \sum \text{one-loop} + \mathcal{R}
  \]

- Traditional reduction: **PV-tensor reduction** [Pasarino & Veltman ’79]
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Efficient reduction of scattering amplitudes by **unitarity cuts**: \[
\frac{1}{p^2 - m^2} \rightarrow \delta(p^2 - m^2)
\]

- (Generalized) unitarity methods [Bern, Dixon, Dunbar & Kosower ’94, Britto, Cachazo & Feng ’04,…]
- OPP integrand reduction OPP [Ossola, Papadopoulos & Pittau ’07,…]

Many numerical NLO tools: Formcalc [Hahn ’99], Golem (PV) [Binoth, Cullen et al ’08], Rocket [Ellis, Giele et al ’09], NJet [Badger, Biederman, Uwer & Yundin ’12], Blackhat [Berger, Bern, Dixon et al ’12], Helac-NLO [Bevilacqua, Czakon et al ’12], MadGolem, GoSam, OpenLoops, …
NLO revolution: 2 to $n$ particle processes

NLO timeline


2 $\rightarrow$ 1
2 $\rightarrow$ 2
2 $\rightarrow$ 3
2 $\rightarrow$ 4
2 $\rightarrow$ 5
2 $\rightarrow$ 6
6 $\rightarrow$ 7 (LHC)
6 $\rightarrow$ W+$+$j
2 $\rightarrow$ $W+$Z+$+j$
2 $\rightarrow$ $t+\bar{t}$
2 $\rightarrow$ $W+$4j, $Z+$4j
2 $\rightarrow$ $e+$e$-$

2010: NLO $W+$4j [BlackHat+Sherpa: Berger et al]
2011: NLO $W$+$W$jj [Rocket: Melia et al]
2011: NLO $Z+$4j [BlackHat+Sherpa: Ita et al]
2011: NLO 4j [BlackHat+Sherpa: Bern et al] [Njet+Sherpa: Badger et al]
2011: first automation [MadNLO: Hirschi et al]
2011: first automation [Helac NLO: Bevilacqua et al]
2011: first automation [GoSam: Cullen et al]
2011: $e+$e$-$ $\rightarrow$ 7j [Becker et al, leading colour]
2012: NLO $W+$5j [BlackHat, preliminary] [published in 2013]
2013: NLO 5j [Njet+Sherpa: Badger et al]
NNLO revolution?

Next step in automation: NNLO

Three types of corrections:

- Tree-level Feynman integrals
- One-loop Feynman integrals
- Two-loop Feynman integrals (bottleneck)

Real-real

Real-virtual

Virtual-virtual (may also have non-planar graphs!)

Entering NNLO

Tree-level Feynman integrals

One-loop Feynman integrals

Two-loop Feynman integrals (bottleneck)

Virtual-virtual corrections

Basically the same as NLO
Two-loop overview

- A finite basis of Master Integrals exists as well at two-loops:

\[ \mathcal{A}_{2\text{loop}} = \sum_{8\text{-prop}} + \cdots + \sum_{2\text{-prop}} + \mathcal{R} \]

- Master integrals may contain loop-dependent numerators as well (*tensor integrals*)

Coherent framework for reductions for two- and higher-loop amplitudes:

- In N=4 SYM [Bern, Carrasco, Johansson et al. ’09-’12]
- Maximal unitarity cuts in general QFT’s [Johansson, Kosower, Larsen et al. ’12-’13]
- Integrand reduction with polynomial division in general QFT’s [Ossola & Mastrolia ’11, Zhang ’12, Badger, Frellesvig & Zhang ’12-’13, Mastrolia, Mirabella, Ossola & Peraro ’12-’13, Kleis, Malamos, Papadopoulos & Verheynen ’12]
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Coherent framework for reductions for two- and higher-loop amplitudes:

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- By now reduction substantially understood for two- and multi-loop integrals [no public code as of yet available]

**Missing ingredient:** library of Master integrals (MI)

- Reduction to MI used for specific processes: Integration by parts (IBP) [Tkachov ’81, Chetyrkin & Tkachov ’81]
Reduction by IBP

Fundamental theorem of calculus: given integral, by IBP get linear system of equations

\[ G = \int \left( \prod_i d^d k_i \right) I \quad \text{IBP identities:} \quad \int \left( \prod_i d^d k_i \right) \frac{\partial}{\partial k_j^\mu} \left( v^\mu I \right) = \text{Boundary term} \overset{DR}{=} 0 \]

\[ I = \frac{\text{Num}(k, p)}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \quad D_i = c_{ij} k_j.k_l + c_{ij} k_j.p_j + m_i^2, \quad v \in \{k_1, \cdots, k_n, \text{external momenta}\} \]
**Reduction by IBP**

- **Fundamental theorem of calculus:** given integral, by IBP get linear system of equations

\[
G = \int \left( \prod_i d^d k_i \right) I \quad \Rightarrow \quad \int \left( \prod_i d^d k_i \right) \frac{\partial}{\partial k_j^\mu} (\nu^\mu I) = \text{Boundary term} \quad DR = 0
\]

\[
I = \frac{\text{Num}(k, p)}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \quad D_i = c_{ij} \cdot k_i \cdot k_l + c_{ij} \cdot k_j \cdot p_j + m_i^2, \quad \nu \in \{k_1, \cdots, k_n, \text{external momenta}\}
\]

In practice, **generate numerator with negative indices** such that w.l.o.g.:

\[
G_{a_1 \cdots a_n}(s) := \int \left( \prod_i \frac{d^d k_i}{i \pi^{d/2}} \right) \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}, \quad s = \{p_i \cdot p_j\}_{i,j}
\]

**IBP identities:**

\[
\sum_{a_1, \cdots a_n} \text{Rational}^{a_1 \cdots a_n}(s, d) G_{a_1 \cdots a_n}(s) = 0
\]

**Solve:**

\[
G_{a_1 \cdots a_n}(s) = \sum_{(b_1 \cdots b_n) \in \text{Master Integrals}} \text{Rational}^{b_1 \cdots b_n}(s, d) G_{b_1 \cdots b_n}(s)
\]

- **Systematic algorithm:** [Laporta '00]. Public implementations: AIR [Anastasiou & Lazopoulos '04 ], FIRE [A. Smirnov '08] Reduze [Studerus '09, A. von Manteuffel & Studerus '12-13], LiteRed [Lee '12], ...

- **Revealing independent IBP’s:** ICE [P. Kant '13]
Reduction by IBP: one-loop triangle

One-loop triangle example:

\[ G_{a_1a_2a_3} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^{2a_1} (k + p_1)^{2a_2} (k + p_1 + p_2)^{2a_3}}, \quad p_1^2 = m_1, \quad p_2^2 = m_2, \quad (p_1 + p_2)^2 = 0 \]

IBP identities:

\[ \int \frac{d^d k}{i\pi^{d/2}} \frac{\partial}{\partial k^\mu} \left( v^\mu \frac{1}{k^{2a_1} (k + p_1)^{2a_2} (k + p_1 + p_2)^{2a_3}} \right) = 0 \]

Choose \( v = k, p_1, p_2 \) respectively

\[ \begin{align*}
0 &= -a_3 G_{-1+a_1,a_2,1+a_3} - a_2 G_{-1+a_1,1+a_2,a_3} + (-2a_1 + d - a_2 - a_3) G_{a_1,a_2,a_3} + m_1 a_2 G_{a_1,1+a_2,a_3} \\
0 &= a_2 G_{-1+a_1,1+a_2,a_3} + (a_1 - a_2) G_{a_1,a_2,a_3} + a_3 (G_{-1+a_1,a_2,1+a_3} - G_{a_1,-1+a_2,1+a_3} + m_2 G_{a_1,a_2,1+a_3}) \\
0 &= -m_1 a_2 G_{a_1,1+a_2,a_3} - a_1 G_{1+a_1,-1+a_2,a_3} + a_1 m_1 G_{1+a_1,a_2,a_3} \\
0 &= a_3 G_{a_1,-1+a_2,1+a_3} + (a_2 - a_3) G_{a_1,a_2,a_3} - m_2 a_3 G_{a_1,a_2,1+a_3} - a_2 G_{a_1,1+a_2,-1+a_3} + m_2 a_2 G_{a_1,1+a_2,a_3} + a_1 (G_{1+a_1,-1+a_2,a_3} - G_{1+a_1,a_2,-1+a_3} - m_1 G_{1+a_1,a_2,a_3})
\end{align*} \]

Solve:

\[ \{ G_{110}, G_{011} \} \]

Triangle reduction by IBP:

\[ G_{111} = \frac{2(d - 3)}{(d - 4)(m_1 - m_2)} (G_{011} - G_{110}) \]
Methods for calculating MI

Rewriting of integrals in different representations:
- Parametric: Feynman/alpha parameters → Sector decomposition
- Mellin-Barnes [Bergere & Lam ’74, Ussyukina ’75, …, V. Smirnov ’99, Tausk ’99]

Using relations and/or cut identities:
- Dimensional shifting relations [Tarasov ’96, Lee ’10, Lee, V. Smirnov & A. Smirnov ’10]
- Schouten identities [Remiddi & Tancredi ’13]
- Integral reconstruction with cuts and using Goncharov polylogarithms (GP) and their properties [Abreu, Britto, Duhr & Gardi ’14]
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As solutions of differential equations:

- Differentiation w.r.t. invariants [Kotikov ’91, Remiddi ’97, Caffo, Cryz & Remiddi ’98, Gehrmann & Remiddi ’00]
- Differentiation w.r.t. externally introduced parameter [Papadopoulos ’14]

Many more: Dispersion relations, dualities, …
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As solutions of differential equations: Algorithmic solutions (next slides)
- Differentiation w.r.t. invariants [Kotikov ’91, Remiddi ’97, Caffo, Cryz & Remiddi ’98, Gehrmann & Remiddi ’00]
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Many more: Dispersion relations, dualities, …
For numerical purposes and phenomenology we are not interested in exact expressions for loop integrals, but efficient and manageable expressions.

We need $\epsilon$ expansion ($\epsilon = \frac{4-d}{2}$).

In practice it is easier to expand and then integrate instead of other way around:

- Make integrable by performing subtractions at integrand level
- Then expand in epsilon
- Finally integrate

\[
\int dx_1 \cdots dx_n G[\vec{x}, s, \epsilon] = \int dx_1 \cdots dx_n G_{\text{sing}}[\vec{x}, s, \epsilon] + \int dx_1 \cdots dx_n (G[\vec{x}, s, \epsilon] - G_{\text{sing}}[\vec{x}, s, \epsilon])
\]

\[
= G_{\text{sing}}[s, \epsilon] + \sum_k \epsilon^k \int dx_1 \cdots dx_n G^{(k)}_{\text{finite}}[\vec{x}, s]
\]

\[
= \sum_k \epsilon^k \left( G_{\text{sing}}^{(k)}[s] + \int dx_1 \cdots dx_n G^{(k)}_{\text{finite}}[\vec{x}, s] \right)
\]

Numerical extraction of coefficients in $\epsilon$ expansion are found in public codes: Secdec [Borowka, Carter & Heinrich], FIESTA [A. Smirnov, V. Smirnov & Tentyukov], sector_decomposition [Bogner & Weinzierl]
**Intermission: Functional basis for (class of) MI**

- The expansion in epsilon often leads to log's: \((\cdots)^{a\epsilon} = 1 + a\epsilon \log(\cdots) + \frac{a^2}{2} \epsilon^2 \log^2(\cdots) + \cdots\)

- Integrals if **parametrized correctly**: \(\sum \int (\text{Rational function}) \ast \log^n(\cdots)\)

- The above integrals (often) naturally lead to **Goncharov Polylogarithms** (GP) [Goncharov ’98, ’01, Remiddi & Vermaseren ’00]:

\[
GP(a_1, \cdots, a_n; x) := \int_0^x dx' \frac{GP(a_2, \cdots, a_n; x')}{x' - a_1}, \quad GP(; x) = 1, \quad GP(0, \cdots, 0; x) = \frac{1}{n!} \log^n(x)
\]

\[
GP(\bar{a}; x)GP(\bar{b}; x) = \sum_{\bar{c} = \text{shuffle}\{\bar{a}, \bar{b}\}} GP(\bar{c}; x), \quad \int_0^x dx' \text{Rational}(x') GP(a_1, \cdots, a_n; x') = \sum_{i=0}^{n+1} \sum_{b_0 \cdots b_i} \text{Rational}^{b_0 \cdots b_i}(x) GP(b_1, \cdots, b_i; x)
\]

*Assuming convergence of integral, i.e. after subtracting singularities*
Intermission: Functional basis for (class of) MI

- The expansion in epsilon often leads to log’s: \( (\cdots)^{a_\epsilon} = 1 + a_\epsilon \log (\cdots) + \frac{a_\epsilon^2}{2} \epsilon^2 \log^2 (\cdots) + \cdots \)
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\]

One-loop triangle example in Feynman parameters:

\[
G_{111} = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2} = (-m_1)^{-\epsilon} - (-m_2)^{-\epsilon} \Gamma[1+\epsilon] \frac{m_1 - m_2}{\epsilon} \int_0^1 x^{-1-\epsilon} (1-x)^{-\epsilon} dx
\]

\[
\int_0^1 x^{-1-\epsilon} (1-x)^{-\epsilon} dx = \int_0^1 x^{-1-\epsilon} dx + \int_0^1 (x^{-1-\epsilon} (1-x)^{-\epsilon} - x^{-1-\epsilon}) dx
\]

\[
= \frac{1}{-\epsilon} + \left( -\epsilon \int_0^1 dx \frac{GP(1; x)}{x} + \epsilon^2 \int_0^1 dx \frac{GP(0, 1; x) + GP(1, 0; x) + GP(1, 1; x)}{x} \right)
\]

\[
= \frac{1}{-\epsilon} + \left( -\epsilon GP(0, 1; 1) + \epsilon^2 (GP(0, 0, 1; 1) + GP(0, 1, 0; 1) + GP(0, 1, 1; 1)) \right)
\]

*Assuming convergence of integral, i.e. after subtracting singularities
The expansion in epsilon often leads to log's

\[(\cdots)^{a_\epsilon} = 1 + a_\epsilon \log (\cdots) + \frac{a_\epsilon^2}{2} \log^2 (\cdots) + \cdots\]

Integrals if parametrized correctly:

\[\sum \int (\text{Rational function}) \ast \log^n (\cdots)\]

The above integrals (often) naturally lead to Goncharov Polylogarithms (GP) [Goncharov '98, '01, Remiddi & Vermaseren '00]:

\[GP(a_1, \cdots, a_n; x) := \int_0^x dx' \frac{GP(a_2, \cdots, a_n; x')}{x' - a_1}, \quad GP(\epsilon; x) = 1, \quad GP(0, \cdots, 0; x) = \frac{1}{n!} \log^n (x)\]

\[GP(\bar{a}; x)GP(\bar{b}; x) = \sum_{\bar{c}=\text{shuffle}\{\bar{a}, \bar{b}\}} GP(\bar{c}; x), \quad \int_0^x dx' \text{Rational}(x')GP(a_1, \cdots, a_n; x') = \sum_{i=0}^{n+1} \sum \text{Rational}^{b_0 \cdots b_i} (x)GP(b_1, \cdots, b_i; x)\]

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\[G_{111} = \int \frac{d^d k}{i \pi^{d/2} k^2 (k + p_1)^2 (k + p_1 + p_2)^2} = \frac{(-m_1)^{-\epsilon} - (-m_2)^{-\epsilon}}{m_1 - m_2} \Gamma[1 + \epsilon] \frac{1}{\epsilon} \int_0^1 x^{-1-\epsilon} (1 - x)^{-\epsilon} dx\]

\[\int_0^1 x^{-1-\epsilon} (1 - x)^{-\epsilon} dx = \int_0^1 x^{-1-\epsilon} dx + \int_0^1 (x^{-1-\epsilon} (1 - x)^{-\epsilon} - x^{-1-\epsilon}) dx\]

\[= \frac{1}{-\epsilon} + \left( -\epsilon \int_0^1 dx \frac{GP(1; x)}{x} + \epsilon^2 \int_0^1 dx \frac{GP(0, 1; x) + GP(1, 0; x) + GP(1, 1; x)}{x} \right)\]

\[= \frac{1}{-\epsilon} + \left( -\epsilon GP(0, 1; 1) + \epsilon^2 (GP(0, 0, 1; 1) + GP(0, 1, 0; 1) + GP(0, 1, 1; 1)) \right)\]

GP's are fundamental building blocks for many MI

DE method takes advantage of this fact

*Assuming convergence of integral, i.e. after subtracting singularities
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- Application
- Summary and outlook
Assume one is interested in a multi-loop Feynman integral:

\[ G_{a_1 \ldots a_n}(\bar{s}) := \int \left( \prod_i \frac{d^d k_i}{i\pi^{d/2}} \right) \frac{1}{D_1^{a_1} D_2^{a_2} \ldots D_n^{a_n}}, \quad D_i = c_{ij} k_j \cdot k_i + c_{ij} k_j \cdot p_j + m_i^2, \quad \bar{s} = \{\bar{s}_1, \bar{s}_2 \ldots\} \]

- Define denominators such as to be able to generate all (irreducible) scalar products
- Numerators may be included as negative indices \(a_i < 0\)
- Invariants \(\bar{s}\) may be functions of scalar products: \(\bar{s} = \{f_1(p_i \cdot p_j), f_2(p_i \cdot p_j)\ldots\}\)
Assume one is interested in a multi-loop Feynman integral:

\[ G_{a_1 \ldots a_n}(\tilde{s}) := \int \left( \prod_i \frac{d^d k_i}{i \pi^{d/2}} \right) \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}, \quad D_i = c_{ij} k_i \cdot k_j + c_{ij} k_i \cdot p_j + m_i^2, \quad \tilde{s} = \{\tilde{s}_1, \tilde{s}_2 \ldots \} \]

Define denominators such as to be able to generate all (irreducible) scalar products

Numerators may be included as negative indices \( a_i < 0 \)

Invariants \( \tilde{s} \) may be functions of scalar products: \( \tilde{s} = \{f_1(p_i \cdot p_j), f_2(p_i \cdot p_j)\ldots\} \)

Differentiate w.r.t. external momenta:

\[ p_i \cdot \frac{\partial}{\partial p_j} G_{a_1 \ldots a_n}(\tilde{s}) = \int \left( \prod_i \frac{d^d k_i}{i \pi^{d/2}} \right) p_i \cdot \frac{\partial}{\partial p_j} \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} = \sum_{b_1 \ldots b_n} \tilde{c}_{a_1 \ldots a_n}(\tilde{s}) G_{b_1 \ldots b_n}(\tilde{s}) \]

Use IBP identities to reduce r.h.s. to MI:

If all MI on r.h.s. known, solve the above differential equation (DE) to get required \( G_{a_1 \ldots a_n} \)
DE method can be most efficiently used to calculate MI themselves

In that case the r.h.s. of previous DE may contain MI $G_{a_1...a_n}$ itself

Perform differentiation on every MI to get a matrix equation:

$$\mathcal{G}_{MI} = \{G_{b_1,...,b_n} | (b_1, \cdots b_n) \in \text{Master Integrals}\} =: \{G_1, \cdots, G_m\}$$

$$p_i \cdot \frac{\partial}{\partial p_j} \mathcal{G}_{MI}(\tilde{s}, \epsilon) = \overline{M}(\tilde{s}, \epsilon, i, j) \cdot \mathcal{G}_{MI}(\tilde{s}, \epsilon)$$
DE method can be most efficiently used to calculate MI themselves

In that case the r.h.s. of previous DE may contain MI $G_{a_1...a_n}$ itself

Perform differentiation on every MI to get a matrix equation:

$$G^{MI} = \{G_{b_1,...b_n} | (b_1, \cdots b_n) \in \text{Master Integrals} \} =: \{G_1, \cdots, G_m\}$$

$$p_i \frac{\partial}{\partial p_j} G^{MI}(\tilde{s}, \epsilon) = \overline{M}(\tilde{s}, \epsilon, i, j) \cdot G^{MI}(\tilde{s}, \epsilon)$$

Solve above linear system of DE to get all MI in one go

In practice first need to change to a DE in differentiation w.r.t. invariants $\tilde{s}$:

$$p_i \frac{\partial}{\partial p_j} F(\tilde{s}) = \sum_k p_i \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} F(\tilde{s})$$

$$\frac{\partial}{\partial \tilde{s}_k} G^{MI}(\tilde{s}, \epsilon) = \overline{M}_k(\tilde{s}, \epsilon) \cdot G^{MI}(\tilde{s}, \epsilon)$$

Boundary condition $G^{MI}(\tilde{s}_k = \tilde{s}_{k,0})$ may be found (among other ways) by plugging in special kinematical values for invariants (if regular), via symmetries, via analytical/regularity constraints, solving DE's in the other invariants or asymptotic expansion
**Uniform weight solution**

In general matrix in DE is dependent on $\epsilon$:

$$\frac{\partial}{\partial \tilde{s}_k} \tilde{G}^{MI}(\tilde{s}, \epsilon) = \overline{M}_k(\tilde{s}, \epsilon) \cdot \tilde{G}^{MI}(\tilde{s}, \epsilon)$$

**Conjecture**: possible to make a rotation $\tilde{G}^{MI} \rightarrow \overline{A} \cdot \tilde{G}^{MI}$ such that:

$$\frac{\partial}{\partial \tilde{s}_k} \tilde{G}^{MI}(\tilde{s}, \epsilon) = \epsilon \overline{M}_k(\tilde{s}) \cdot \tilde{G}^{MI}(\tilde{s}, \epsilon)$$  \[Henn '13\]

Explicitly shown to be true for many examples [Henn '13, Henn, Smirnov et al '13-'14]

If set of invariants $\tilde{s} = \{f(p_i, p_j)\}$ chosen correctly: $\overline{M}_k(\tilde{s}) = \sum_{\text{poles } \tilde{s}^{(0)}_k} \frac{\overline{s}^{(0)}_k}{\overline{s}_k - \tilde{s}^{(0)}_k}$

Solution is uniform in weight of GP's:

$$\tilde{G}^{MI}(\tilde{s}, \epsilon) = Pe^\epsilon \int_{C[0, \tilde{s}]} \overline{M}_k(\tilde{s}'_k) \tilde{G}^{MI}(0, \epsilon) = (1 + \epsilon \int_0^{\tilde{s}_k} \overline{M}_k(\tilde{s}'_k) + \cdots) \tilde{G}^{MI}(0, \epsilon)$$

$$= \frac{G^{MI}_{0} + \epsilon (\frac{G^{MI}_{1}}{G^{MI}_{0}}) + \sum_{\text{poles } \tilde{s}^{(0)}_k} \left( \int_0^{\tilde{s}_k} \frac{d\tilde{s}'_k}{(\tilde{s}'_k - \tilde{s}^{(0)}_k)} \right) \frac{\overline{s}^{(0)}_k}{\overline{s}_k - \tilde{s}^{(0)}_k} \cdot \frac{G^{MI}_{0}}{G^{MI}_{0}} + \cdots}{G^{MI}_{0} + \epsilon G^{MI}_{1} + \cdots}$$
Outline

- Introduction
- Differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook
**x-Parametrization**

- Method up till now only developed for *massless internal lines* (assume in following)
- Introduce extra parameter $x$ in the denominators of loop integral
- $x$-parameter describes off-shellness of external legs

\[ G_{a_1 \cdots a_n}(s, \epsilon) = \int \left( \prod_i d^d k_i \right) \frac{1}{D_1^{2a_1}(k, p) \cdots D_n^{2a_n}(k, p)} \]

\[ G_{a_1 \cdots a_n}(x, s, \epsilon) = \int \left( \prod_i d^d k_i \right) \frac{1}{D_1^{2a_1}(k, p(x)) \cdots D_n^{2a_n}(k, p(x))} \]

\[ p_i(x) = p_i + (1-x)q_i, \quad \sum_i q_i = 0, \quad D_i(k, p) = c_{ij}k_j + d_{ij}p_j, \quad s = \{p_i \cdot p_j\}_{i,j} \]

One-loop triangle example:
x-Parametrization

- Method up till now only developed for massless internal lines (assume in following)
- Introduce extra parameter $x$ in the denominators of loop integral
- $x$-parameter describes off-shellness of external legs

\[
G_{a_1 \ldots a_n}(s, \epsilon) = \int \left( \prod_i \frac{d^d k_i}{D_i^{2a_1}(k, p) \cdots D_i^{2a_n}(k, p)} \right) \frac{1}{D_i^{2a_1}(k, p) \cdots D_i^{2a_n}(k, p)}
\]

\[
p_i(x) = p_i + (1 - x)q_i, \quad \sum_i q_i = 0, \quad D_i(k, p) = c_{ij} k_j + d_{ij} p_j, \quad s = \{p_i \cdot p_j\}\bigg|_{i,j}
\]

One-loop triangle example:

- Take derivative of integral $G$ w.r.t. $x$-parameter instead of w.r.t. invariants and reduce r.h.s. by IBP identities

\[
\frac{\partial}{\partial x} \tilde{G}^{MI}(x, s, \epsilon) \overset{\text{IBP}}{=} \tilde{M}(x, s, \epsilon) \cdot \tilde{G}^{MI}(x, s, \epsilon), \quad s = \{p_i \cdot p_j\}\bigg|_{i,j}
\]
Bottom-up approach

Notation: upper index “(m)” in integrals $G_{\{a_1...a_n\}}^{(m)}$ denotes amount of positive indices, i.e. amount of denominators/propagators

$$G_{a_1...a_n}^{(m)} = \int \left( \prod_i d^d k_i \right) \frac{1}{D_1^{2a_1}(k, p) \cdots D_n^{2a_n}(k, p)}$$

$m$ propagators, (positive indices) $a_i$

In practice individual DE’s of MI are of the form:

$$\frac{\partial}{\partial x} G_{a_1...a_n}^{(m)}(x, s, \epsilon) = \sum_{m'=m_0}^m \sum_{b_1,...,b_n} \text{Rational}_{a_1...a_n}^{b_1,...,b_n}(x, s, \epsilon) G_{b_1...b_n}^{(m')} (x, s, \epsilon)$$

Bottom-up:

- Solve first for all MI with least amount of denominators $m_0$ (these are often already known to all orders in $\epsilon$ or often calculable with other methods)
- After solving all MI with $m$ denominators ($m \geq m_0$), solve all MI with $m + 1$ denominators

In practice:

$$G_{a_1...a_n}^{(m_0)} (x, s, \epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x) GP(\cdots ; x) \right)$$
Choice of $x$-parametrization:

- **Main criteria for choice of $x$-parametrization:** constant term ($\epsilon = 0$) of residues of homogeneous term for every DE needs to be an integer.

\[
\frac{\partial}{\partial x} G^{(m)}_{a_1 \ldots a_n}(x, \epsilon) = \frac{\partial}{\partial x} \left[ H(x, \epsilon) G^{(m)}_{a_1 \ldots a_n}(x, \epsilon) + \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x) G_P(\cdots ; x) \right) \right]
\]

\[
= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1 \ldots a_n}(x, \epsilon) + \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x) G_P(\cdots ; x) \right) \rightarrow
\]

\[
\frac{\partial}{\partial x} (M(x, \epsilon) G^{(m)}_{a_1 \ldots a_n}(x, \epsilon)) = M(x, \epsilon) \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x) G_P(\cdots ; x) \right), \quad M(x, \epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)}) \frac{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}
\]

For all MI that we have calculated, the criteria could be easily met. Often it was enough to choose the external legs such that the corresponding massive MI triangles (found by pinching external legs) are as follows:
Choice of x-parametrisation

**Main criteria for choice of x-parametrisation:** constant term \((\epsilon = 0)\) of residues of homogeneous term for every DE needs to be an integer:

\[
\frac{\partial}{\partial x} G^{(m)}_{a_1 \ldots a_n}(x, \epsilon) = H(x, \epsilon) G^{(m)}_{a_1 \ldots a_n}(x, \epsilon) + \sum_{n,l} x^{-n+l} \left( \sum \text{Rational}(x) \text{GP}(\cdots; x) \right)
\]

For all MI that we have calculated, the criteria could be easily met. Often it was enough to choose the external legs such that the corresponding massive MI triangles (found by pinching external legs) are as follows:

**Boundary condition:**

- Boundary condition almost always zero in bottom-up approach
- Except in three cases, all loop integrals we have come across: \((M \ast G^{(m)}_{a_1 \ldots a_n})_{x \to 0} = 0\) Not well understood yet why this is so!
- If not zero, boundary condition \((M \ast G^{(m)}_{a_1 \ldots a_n})_{x \to 0}\) may be found (in principle) by plugging in special values for \(x\), via analytical/regularity constraints, asymptotic expansion in \(x \to 0\) or some modular transformation like \(x \to 1/x\)
Example: one-loop triangle

\[ G_{111}(m_1, m_2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 (k + p_1)^2 (k + p_1 + p_2)^2} \]

\[ p_1^2 = m_1, \quad p_2^2 = m_2, \quad (p_1 + p_2)^2 = 0 \]

Parametrize, the off-shellness with \( x \)

\[ G_{111}(m_1, x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 (k + x p_1)^2 (k + p_1 + p_2)^2} \]

\[ p_1^2 = m_1, \quad p_2^2 = 0, \quad (p_1 + p_2 - x p_1)^2 \neq 0, \quad (p_1 + p_2)^2 = 0 \]
Example: one-loop triangle

\[ G_{111}(m_1, m_2) = \int \frac{d^d k}{i \pi^{d/2}} \frac{1}{k^2 (k + p_1)^2 (k + p_1 + p_2)^2} \]

\[ p_1^2 = m_1, p_2^2 = m_2, (p_1 + p_2)^2 = 0 \]

\[ G_{111}(m_1, x) = \int \frac{d^d k}{i \pi^{d/2}} \frac{1}{k^2 (k + x p_1)^2 (k + p_1 + p_2)^2} \]

\[ p_1^2 = m_1, p_2^2 = 0, (p_1 + p_2 - x p_1)^2 \neq 0, (p_1 + p_2)^2 = 0 \]

Differentiate to \( x \) and use IBP to reduce:

\[ \frac{\partial}{\partial x} G_{111}(x) = \frac{-x^{-2-\epsilon}}{\epsilon^2 m_1} \left( (m_1 - i.0)^{-\epsilon} (1 + 2\epsilon) x^{-\epsilon} - (m_1 - i.0)^{-\epsilon} (1 - x)^{-1-\epsilon} (1 + \epsilon - x(1 + 2\epsilon)) \right) \]

Subtracting the singularities and expanding the finite part leads to:

\[ G_{111}(x) = G_{111}(0) + \int_0^x dx' \frac{-x'^{-2-\epsilon}}{\epsilon^2 m_1} \left( (m_1 - i.0)^{-\epsilon} (1 + 2\epsilon) x'^{-\epsilon} - (m_1 - i.0)^{-\epsilon} (1 - x')^{-1-\epsilon} (1 + \epsilon - x'(1 + 2\epsilon)) \right) \]

\[ = G_{111}(0) + \frac{-(m_1 - i.0)^{-\epsilon} x^{-\epsilon} + (m_1 - i.0)^{-\epsilon} x^{-2\epsilon}}{m_1 x \epsilon^2} + \frac{(m_1 - i.0)^{-\epsilon} (-x^{-\epsilon} + (x + GP(1;x)))}{m_1 x \epsilon} + \mathcal{O}(\epsilon^0) \]

Agrees with expansion of exact solution:

\[ G_{111}(m_1 x^2, m_2 = (m_1) x(1-x)) = \frac{\epsilon(\epsilon) \left( -m_1 x^2 \right)^{-\epsilon} - \left( -(m_1) x(1-x) \right)^{-\epsilon}}{\epsilon^2 \left( m_1 x^2 - (m_1) x(1-x) \right)} \]
Comparison of DE methods

Traditional DE method:

- Choose \( \tilde{s} = \{ f(p_i, p_j) \} \) and use chain rule to relate differentials of (independent) momenta and invariants:
  \[
p_i \frac{\partial}{\partial p_j} F(\tilde{s}) = \sum_k p_i \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} F(\tilde{s})
\]

- Solve above linear equations:
  \[
  \frac{\partial}{\partial \tilde{s}_k} = g_k(\{ p_i \cdot \frac{\partial}{\partial p_j} \})
  \]

- Differentiate w.r.t. invariant(s) \( \tilde{s}_k \):
  \[
  \frac{\partial}{\partial \tilde{s}_k} \tilde{G}^{MI}(\tilde{s}, \epsilon) = g_k(\{ p_i \cdot \frac{\partial}{\partial p_j} \}) \tilde{G}^{MI}(\tilde{s}, \epsilon)
  \]

- Make rotation \( \tilde{G}^{MI} \rightarrow \overline{A} \tilde{G}^{MI} \) such that:
  \[
  \frac{\partial}{\partial \tilde{s}_k} \tilde{G}^{MI}(\tilde{s}, \epsilon) = \epsilon \overline{M}_k(\tilde{s}) \tilde{G}^{MI}(\tilde{s}, \epsilon)
  \] [Henn '13]

- Solve perturbatively in \( \epsilon \) to get GP's if \( \tilde{s} = \{ f(p_i, p_j) \} \) chosen properly

- Solve DE of different \( \tilde{s}_k \), to capture boundary condition

Simplified DE method:

- Introduce external parameter \( x \) to capture off-shellness of external momenta:

- Parametrization: pinched massive triangles should have legs (not fully constraining):
  \[
  q_1(x) = xp', q_2(x) = p'' - xp', p'^2 = m_1, p''^2 = m_3
  \]

- Differentiate w.r.t. parameter \( x \):

- Check if constant term (\( \epsilon = 0 \)) of residues of homogeneous term for every DE is an integer:
  1) if yes, solve DE by “bottom-up” approach to express in GP's; 2) if no, change parametrization and check DE again

- Boundary term almost always captured, if not: try \( x \rightarrow 1/x \) or asymptotic expansion
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Two-loop planar double-box

Example of planar diagrams:

$$pp' \rightarrow V_1V_2, \quad m_{V_1} \neq m_{V_2} \neq 0$$

Require 4-point two-loop MI with 2 off-shell legs and massless internal legs (at LHC light-flavor quarks are massless to good degree): diboson production
Two-loop planar double-box

Example of planar diagrams:

\[ pp' \rightarrow V_1 V_2, \quad m_{V_1} \neq m_{V_2} \neq 0 \]

Require 4-point two-loop MI with 2 off-shell legs and massless internal legs (at LHC light-flavor quarks are massless to good degree): diboson production

- **On-shell legs**: \( q_1^2 = \cdots = q_4^2 = 0 \) [planar: V. Smirnov '99, V. Smirnov & Veretin '99, non-planar: Tausk '99, Anastasiou, Gehrmann, Oleari, Remiddi & Tausk '00]
- **One off-shell leg (pl.+non-pl.)**: \( q_1^2 = q_2^2 = q_3^2 = q_4^2 = 0 \) [Gehrmann & Remiddi '00-'01]
- **Two off-shell legs with same masses**: \( q_1^2 = q_2^2 = q_3^2 = q_4^2 = 0 \) [planar: Gehrmann, Tancredi & Weihs '13, non-planar: Gehrman, Manteuffel, Tancredi & Weihs '14]
- **Two off-shell legs with different masses**: \( q_1^2 \neq 0, q_2^2 \neq 0, q_3^2 = q_4^2 = 0 \) [planar: Henn, Melnikov & Smirnov '14, non-planar: Caola, Henn, Melnikov & Smirnov '14]
Double planar box: IBP families

- MI form 3 families of coupled DE's

1$^{\text{st}}$ Adjacent mass topology $\rightarrow$ 31 MI

2$^{\text{nd}}$ Adjacent mass topology $\rightarrow$ 29 MI

Opposite mass topology $\rightarrow$ 28 MI
Double planar box: Parametrization

1\textsuperscript{st} Adjacent mass topology $\rightarrow$ 31 MI

\[ G_{a_1...a_9}^{(1)}(x) := \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1 + x p_1)^{2a_2}(k_1 + x p_{12})^{2a_3}(k_1 + p_{123})^{2a_4} k_2^{2a_5}(k_2 - x p_1)^{2a_6}(k_2 - x p_{12})^{2a_7}(k_2 - p_{123})^{2a_8}(k_1 + k_2)^{2a_9}} \]

2\textsuperscript{nd} Adjacent mass topology $\rightarrow$ 29 MI

\[ G_{a_1...a_9}^{(2)}(x) := \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1 + x p_1)^{2a_2}(k_1 + x p_{12})^{2a_3}(k_1 + p_{123})^{2a_4} k_2^{2a_5}(k_2 - x p_1)^{2a_6}(k_2 - p_{123})^{2a_7}(k_2 - p_{123})^{2a_8}(k_1 + k_2)^{2a_9}} \]

Opposite mass topology $\rightarrow$ 28 MI

\[ G_{a_1...a_9}^{(3)}(x) := \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1 + x p_1)^{2a_2}(k_1 + p_{123} - x p_2)^{2a_3}(k_1 + p_{123})^{2a_4} k_2^{2a_5}(k_2 - p_1)^{2a_6}(k_2 + x p_2 - p_{123})^{2a_7}(k_2 - p_{123})^{2a_8}(k_1 + k_2)^{2a_9}} \]
Bottom-up approach

- The DE’s of 2nd family:

  7-denominators: \( \frac{\partial}{\partial x} \)

  \[
  \begin{align*}
  (p_{123} - x_{12}) p_{123} & \quad = \\
  & \quad \vdots \\
  & \quad \leq 7\text{-denominators}
  \end{align*}
  \]

  4-denominators: \( \frac{\partial}{\partial x} \)

  \[
  \begin{align*}
  x_{p_1} & \quad = \\
  & \quad \vdots \\
  & \quad \leq 4\text{-denominators}
  \end{align*}
  \]

  3-denominators:

  \[
  \begin{align*}
  p & \quad = \\
  & \quad \text{constant } \times (-p^2)^{1-2\epsilon}
  \end{align*}
  \]

- The 3-denominator MI are trivially known sunset diagrams

- We solve first the 4-denominator MI, then the 5-denominator MI etc.
Application

Solutions in GP

\[ G_{011111011}(x) = \]

\[ x_{p_2} \]

\[ \begin{align*}
G_{011111011}(x) &= A_3(x) \left( \frac{1}{e^2} - \text{GP} \left( \frac{m_4}{s_{12}}; x \right) + 2\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) + 2\text{GP}(0; x) - \text{GP}(1; x) + \log(-s_{12}) + \frac{9}{4} \right) \\
&+ \frac{1}{4e^2} \left( 18\text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 36\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) - 8\text{GP} \left( 0, \frac{m_4}{s_{12}}; x \right) + 16\text{GP} \left( 0, \frac{m_4}{m_4 - s_{23}}; x \right) + 8\text{GP} \left( \frac{s_{23}}{s_{12}} + 1, \frac{m_4}{m_4 - s_{23}}; x \right) \\
&+ 8\text{GP} \left( \frac{m_4}{s_{12}}, \frac{s_{23}}{s_{12}} + 1; x \right) - 8\text{GP} \left( \frac{m_4}{s_{12}}, \frac{m_4}{s_{12} - m_4}; x \right) + 8\text{GP} \left( \frac{m_4}{s_{12}}, 1; x \right) + 4 \left( -2\text{GP} \left( \frac{s_{23}}{s_{12}} + 1, \frac{m_4}{s_{12}}; x \right) \right) \text{GP} \left( \frac{m_4}{s_{12}}; x \right) \\
&+ 2\text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 2\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) + \text{GP}(1; x) \right) + \text{GP}(0; x) \left( 4\text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 8\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) \\
&+ 4\log(-s_{12}) - 9 \right) + 2\log(-s_{12}) \left( \text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 2\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) + \text{GP}(1; x) \right) - 4\text{GP}(0; x) \right)^2 \right. \\
&\left. - \log^2(-s_{12}) \right) \\
&\left[ -8\text{GP} \left( \frac{s_{23}}{s_{12}} + 1, 1; x \right) + 18\text{GP}(1; x) - \text{GP}(0, 1; x) - 18\log(-s_{12}) - 9 \right] + \frac{1}{e} ( \cdots ) \\
&\left[ (-3\text{GP} \left( 0, \frac{m_4}{s_{12}}; x \right) \right]^2 - 18\text{GP} \left( 0, \frac{m_4}{s_{12}}; x \right) ^2 - \text{GP} \left( \frac{s_{23}}{s_{12}} + 1, \frac{m_4}{s_{12} - m_4}; x \right) ^2 - \text{GP} \left( \frac{m_4}{s_{12}}, \frac{s_{23}}{s_{12}} + 1; x \right) ^2 + \text{GP} \left( \frac{m_4}{s_{12}}, \frac{m_4}{s_{12} - m_4}; x \right) ^2 \\
&\left. + GP \left( \frac{m_4}{s_{12}}, 1; x \right) ^2 - 2 (4\text{GP} \left( 0, 0, 0, \frac{m_4}{s_{12}}; x \right) - 8\text{GP} \left( 0, 0, 0, \frac{m_4}{m_4 - s_{23}}; x \right) + \text{GP} \left( 0, 0, \frac{m_4}{s_{12}}; x \right) + 7\text{GP} \left( 0, 0, 1, \frac{m_4}{s_{12}}; x \right) + 6}\text{GP} \left( 0, 0, 1, \frac{m_4 s_{12} - \sqrt{m_4 s_{12} s_{23} (s_{12} - s_{12} - s_{23})}}{s_{12} (s_{12} - s_{23})}; x \right) \right. \\
&\left. + \text{GP} \left( 0, 0, 1, \frac{m_4 s_{12} - \sqrt{m_4 s_{12} s_{23} (s_{12} - s_{12} - s_{23})}}{s_{12} (s_{12} - s_{23})}; x \right) \right) \\
&\left. - 10\text{GP} \left( 0, 0, 1, \frac{s_{23}}{s_{12}} + 1; x \right) + 4\text{GP}(0, 0, 0, 1; x) - \text{GP}(0, 0, 1, 1; x) \right) - \text{GP} \left( \frac{s_{23}}{s_{12}} + 1, 1; x \right) ^2 - 3\text{GP}(0, 1; x) ^2 + (\cdots ) \right] \\
\end{align*} \]
Solutions in GP

Application

Numerical agreement in Euclidean region found with Secdec [Borowka, Carter & Heinrich]:

\[ G^{(2)}_{01111011}(x) = s_{12} = p_{12}^2, \ s_{23} = p_{23}^2, \ m_4 = p_{123}^2 \]

\[ G^{(2)}_{01111011}(x) = \frac{A_3(\epsilon)}{x^2 s_{12} (m_4 + x)(m_4 - x)} \left( \frac{2\epsilon^3}{\epsilon^4} - \text{GP} \left( \frac{m_4}{s_{12}}; x \right) + 2\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) + 2\text{GP}(0;x) - \text{GP}(1;x) + \log(-s_{12}) + \frac{9}{4} \right) \]

\[ + \frac{1}{4\epsilon^2} \left( 8\text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 36\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) - 8\text{GP} \left( \frac{m_4}{s_{12}}; x \right) + 16\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) + 8\text{GP} \left( \frac{s_{23}}{s_{12}} + 1, \frac{m_4}{m_4 - s_{23}}; x \right) \right) \]

\[ + 2\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) \left( 2\text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 2\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) + \text{GP}(1;x) \right) + \text{GP}(0;x) \left( 4\text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 8\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) \right) \]

\[ + 4\log(-s_{12}) - 9 \right) + 2\log(-s_{12}) \left( \text{GP} \left( \frac{m_4}{s_{12}}; x \right) - 2\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right) + \text{GP}(1;x) \right) - 4\text{GP}(0;x)^2 - \log^2(-s_{12}) \]

\[ - \text{GP} \left( \frac{s_{23}}{s_{12}} + 1, 1, x \right) + 18\text{GP}(1;x) - 8\text{GP}(0, 1, x) - 18\log(-s_{12}) - 9 \right) + \frac{1}{\epsilon} (\ldots) \]

\[ + \left( -3\text{GP} \left( \frac{m_4}{s_{12}}; x \right)^2 - 18\text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right)^2 - \text{GP} \left( \frac{s_{23}}{s_{12}} + 1, \frac{m_4}{m_4 - s_{23}}; x \right)^2 - \text{GP} \left( \frac{m_4}{s_{12}}; x \right) \right) \]

\[ + \text{GP} \left( \frac{m_4}{m_4 - s_{23}}; x \right)^2 - 2 \left( 4\text{GP} \left( 0, 0, \frac{m_4}{s_{12}}; x \right) - \text{GP} \left( 0, 0, \frac{m_4}{m_4 - s_{23}}; x \right) - \text{GP} (0, 0, 1, \frac{m_4}{s_{12}}; x) + 7\text{GP} \left( 0, 0, 1, \frac{m_4}{m_4 - s_{23}}; x \right) \right) \]

\[ + 6 \left[ \text{GP} \left( 0, 0, 1, \frac{m_4 s_{12}}{s_{12}(m_4 - s_{23})}; x \right) - \text{GP} \left( 0, 0, 1, \frac{m_4 s_{12}}{m_4 - s_{23}}; x \right) - \text{GP} \left( \frac{s_{23}}{s_{12}} + 1, 1, x \right)^2 - 3\text{GP}(0, 1, x)^2 \right] + \left( \ldots \right) \]

\[ e^0 \text{ terms} \]
Outline

- Introduction
- Differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook
Summary

- In LHC era multi-loop calculations are compulsory
- Two-loop automation is the next step: reduction substantially understood, library of MI mandatory but still missing
- Functional basis for large class of MI: Goncharov polylogarithms
- DE method is very fruitful for deriving MI in terms of GP
- Simplified DE method [Papadopoulos ’14] captures GP solution naturally, boundary constraints taken into account, very algorithmic

Outlook

- Application to non-planar graphs
- Application/extension to diagrams with massive propagators
Summary

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- Application to non-planar graphs
- Application/extension to diagrams with massive propagators
Backup slides
Assume for $m' < m$ denominators:

$$G^{(m')}_{a_1 \cdots a_n}(x, s, \epsilon) = \sum_{n, l} x^{-n+l} \left( \sum \text{Rational}(x) \text{GP} \cdots \right), \quad m' < m$$

For simplicity we assume here a non-coupled DE for a MI with $m$ denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1 \cdots a_n}(x, s, \epsilon) = H(x, s, \epsilon) G^{(m)}_{a_1 \cdots a_n}(x, s, \epsilon) + \sum_{m' = 1}^{m-1} \sum \text{Rational}^{(b_1, \cdots, b_n)}(x, s, \epsilon) G^{(m')}_{b_1 \cdots b_n}(x, s, \epsilon)$$
GP-structure of solution

Assume for $m' < m$ denominators:

$$G_{a_1 \ldots a_n}^{(m')}(x, s, \epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x)GP(\cdots;x) \right), \quad m' < m$$

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Dependence on invariants $s$ suppressed:

$$\frac{\partial}{\partial x} G_{a_1 \ldots a_n}^{(m)}(x, \epsilon) = H(x, \epsilon) G_{a_1 \ldots a_n}^{(m)}(x, \epsilon) + \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x)GP(\cdots;x) \right)$$

$$= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon \epsilon_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G_{a_1 \ldots a_n}^{(m)}(x, \epsilon) + \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x)GP(\cdots;x) \right)$$

$$\frac{\partial}{\partial x} (M(x, \epsilon) G_{a_1 \ldots a_n}^{(m)}(x, \epsilon)) = M(x, \epsilon) \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x)GP(\cdots;x) \right), \quad M(x, \epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon \epsilon_{x^{(0)}}(\epsilon)}$$
Assume for $m' < m$ denominators:

$$G_{a_1 \cdots a_n}^{(m')} (x, s, \epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x) GP(\cdots ; x) \right), \ m' < m$$

For simplicity we assume here a non-coupled DE for a MI with $m$ denominators:

$$\frac{\partial}{\partial x} G_{a_1 \cdots a_n}^{(m)} (x, s, \epsilon) = H(x, s, \epsilon) G_{a_1 \cdots a_n}^{(m)} (x, s, \epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1, \cdots, b_n} \text{Rational}^{(b_1, \cdots, b_n)} (x, s, \epsilon) G_{b_1, \cdots, b_n}^{(m')} (x, s, \epsilon)$$

Formal solution:

$$M(x, \epsilon) G_{a_1 \cdots a_n}^{(m)} (x, s, \epsilon) = (M * G^{(m)})_{x \to 0} + \sum_{n,l} \prod_{\text{poles } x^{(0)}} \int_0^x dx' \left( x'^{-n+l\epsilon} (x' - x^{(0)})^{-\epsilon x^{(0)}} \right) \left( \sum (x' - x^{(0)})^{-r_x^{(0)}} \text{Rational}(x') GP(\cdots ; x') \right)$$

$$= (M * G^{(m)})_{x \to 0} + \sum_{n,l} \int_0^\infty dx' x'^{-\tilde{n}+l\epsilon} I_{n,l}(\epsilon) + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \int_0^\infty dx' (x' - x^{(0)})^{-r_x^{(0)}} \text{Rational}_k (x') GP(\cdots ; x')$$

if $r_x^{(0)} \in \mathbb{Z}$.
**GP-structure of solution**

Assume for $m' < m$ denominators:

$$G_{a_1 \ldots a_n}^{(m')} (x, s, \epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x)GP(\cdots;x) \right), \quad m' < m$$

For simplicity we assume here a non-coupled DE for a MI with $m$ denominators:

$$\frac{\partial}{\partial x} G_{a_1 \ldots a_n}^{(m)} (x, s, \epsilon) = H(x, s, \epsilon)G_{a_1 \ldots a_n}^{(m)} (x, s, \epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1 \ldots b_n} \text{Rational}^{(b_1 \ldots b_n)} (x, s, \epsilon)G_{b_1 \ldots b_n}^{(m')} (x, s, \epsilon)$$

Formal solution:

$$M(x, \epsilon)G_{a_1 \ldots a_n}^{(m)} (x, s, \epsilon) = (M \ast G_{a_1 \ldots a_n}^{(m)})_{x \to 0} + \sum_{n,l} \prod_{\text{poles } x^{(0)}} \int_0^x dx' \left( x'^{-n+l\epsilon} (x' - x^{(0)})^{-\epsilon c_s(x^{(0)})} \right) \left( \sum (x' - x^{(0)})^{-r_x(0)} \text{Rational}^{(x')} GP(\cdots;x') \right)$$

**MI expressible in GP’s:**

$$G_{a_1 \ldots a_n}^{(m)} (x, s, \epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left( \sum \text{Rational}(x)GP(\cdots;x) \right)$$

Fine print for coupled DE’s: if the non-diagonal piece of $\epsilon = 0$ term of matrix $H$ is nilpotent (e.g. triangular) and if diagonal elements of matrices $r_x^{(0)}$ are integers, then above “GP-argument” is still valid.
Example of tradition DE method: one-loop triangle (1/2)

- Consider again one-loop triangles with 2 massive legs and massless propagators:

\[
G_{a_1 a_2 a_3} (\hat{s}) = \int \frac{d^d k}{i \pi^{d/2}} \frac{1}{k^{2a_1} (k + p_1)^{2a_2} (k + p_1 + p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1 + p_2)^2 = m_3 = 0
\]

- General function:

\[
p_i \cdot \frac{\partial}{\partial p_j} F(m_1, m_2, m_3) = \sum_{k=1}^{3} p_i \cdot \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} F(m_1, m_2, m_3), \quad i, j \in \{1, 2\}
\]

\[
\tilde{s}_1 = p_1^2 = m_1, \tilde{s}_2 = p_2^2 = m_2, \tilde{s}_3 = (p_1 + p_2)^2 = m_3
\]

- Four linear equations, of which three independent because of invariance under Lorentz transformation [Remiddi & Gehrmann '00], in three unknowns:

\[
\left\{ \frac{\partial}{\partial m_1}, \frac{\partial}{\partial m_2}, \frac{\partial}{\partial m_3} \right\}
\]

- Solve linear equations:

\[
\frac{\partial}{\partial m_k} = g_k (p_1 \cdot \frac{\partial}{\partial p_1}, p_2 \cdot \frac{\partial}{\partial p_2}, p_2 \cdot \frac{\partial}{\partial p_1}), \quad k = 1, 2, 3
\]

\[
\frac{\partial}{\partial m_1} G_{111} = \frac{1 - 2\epsilon}{\epsilon (m_1 - m_2)^2} (G_{011} - (1 + \epsilon (1 - \frac{m_2}{m_1})) G_{110}), \quad \frac{\partial}{\partial m_2} G_{111} = \frac{\partial}{\partial m_1} G_{111} (m_1 \leftrightarrow m_2, G_{011} \leftrightarrow G_{110})
\]
Example of tradition DE method: one-loop triangle (2/2)

\[ \frac{\partial}{\partial m_1} G_{111} = \frac{1}{\epsilon^2(m_1 - m_2)^2} \left( (-m_2)^{-\epsilon} + (-m_1)^{-\epsilon} (1 + \epsilon) - \epsilon m_2 (-m_1)^{-1-\epsilon} \right) =: F[m_1, m_2] \]
\[ \frac{\partial}{\partial m_2} G_{111} = F[m_2, m_1] \]

- Solve by usual subtraction procedure:
  \[ F_{\text{sing}}[m_1, m_2] = \frac{-1}{\epsilon m_2} (-m_1)^{-1-\epsilon} \]

\[ G_{111}(m_1, m_2) = G_{111}(0, m_2) + \int_0^{m_1} F_{\text{sing}}[m'_1, m_2] + \int_0^{m_1} \left( F[m'_1, m_2] - F_{\text{sing}}[m'_1, m_2] \right) \]
\[ = G_{111}(0, m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \int_0^{m_1} \left( \frac{(1 - (-m_2)^{-\epsilon}) G P(-m'_1)}{\epsilon^2 (m_2 - m'_1)^2} - \frac{(m_2 - m'_1) G P(-m'_1) + m_2 G P(0; -m'_1)}{\epsilon m_2 (m_2 - m'_1)^2} \right) + O(\epsilon^0) \]
\[ = G_{111}(0, m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \left( \frac{m_1 (1 - (-m_2)^{-\epsilon})}{\epsilon^2 m_2 (m_1 - m_2)} + \frac{m_1 G P(0; -m_1)}{\epsilon m_2 (m_2 - m_1)} \right) + O(\epsilon^0) \]

- Boundary condition follows by plugging in above solution in
  \[ \frac{\partial}{\partial m_2} G_{111} = F[m_2, m_1] \]

\[ \frac{\partial}{\partial m_2} G_{111}(0, m_2) = \frac{(1 + \epsilon)}{\epsilon^2} (-m_2)^{-2-\epsilon} \rightarrow G_{111}(0, m_2) = \frac{(-m_2)^{-1-\epsilon}}{\epsilon^2} \]
\[ + G_{111}(0, 0) = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2} \]
\[ \text{scaleless=0} \]

- Agrees with exact solution:
  \[ G_{111} = \frac{c_r(\epsilon)}{\epsilon^2} \frac{(-m_1)^{-\epsilon} - (-m_2)^{-\epsilon}}{m_1 - m_2} = \frac{c_r(\epsilon)}{m_1 - m_2} \left( -\frac{1}{\epsilon} \log(\frac{-m_1}{-m_2}) + O(\epsilon^0) \right) \]
Open questions

- Is there a way to pre-empt the choice of $x$-parametrization without having to calculate the DE?
- Why are the boundary conditions (almost always) naturally taken into account?
- How do the DE in the $x$-parametrization method relate exactly to those in the traditional DE method?
- How to easily extend parameter $x$ to whole real axis and extend the invariants to the physical region?