Simplified differential equations approach for the calculation of multi-loop integrals

Chris Wever (N.C.S.R. Demokritos)

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C. Papadopoulos [arXiv: 1401.6057 [hep-ph]]
C. Papadopoulos, D. Tommasini, C. Wever [to appear]

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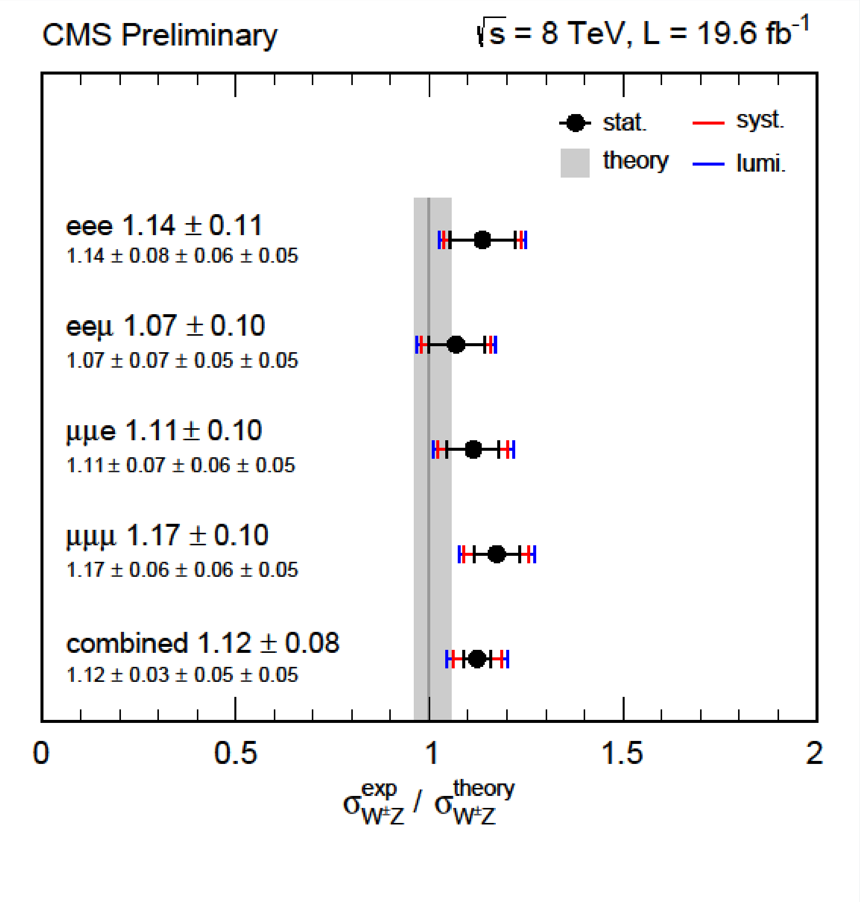
Motivation



[CMS 2013]

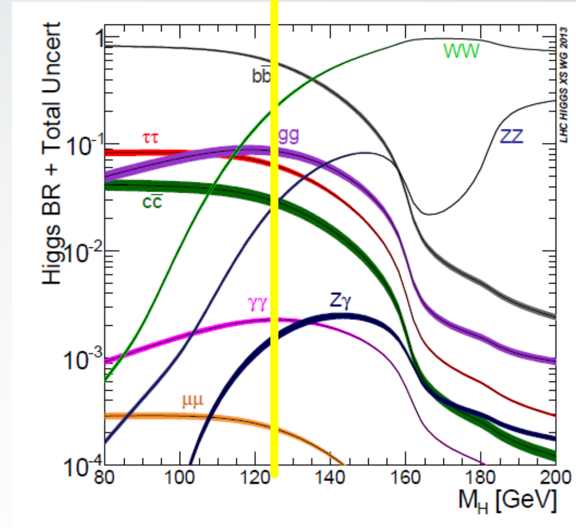
- Mismatch between theory and experimental result
- Theory prediction up to NLO, full NNLO calculation might resolve the discrepancy

Outline

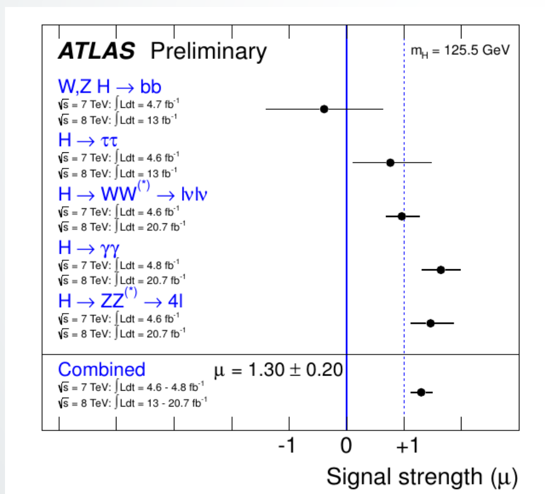
- Introduction
- Differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook

EW importance

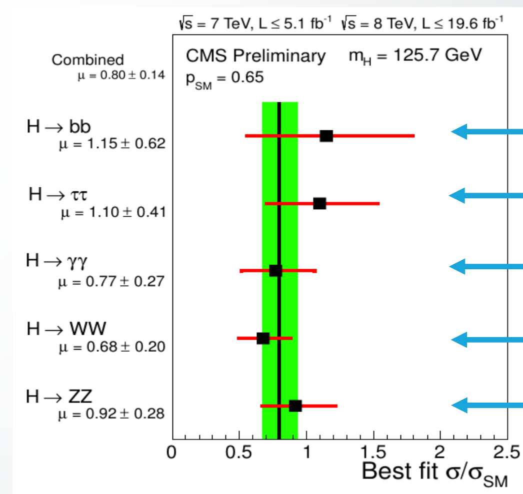
- First run LHC 2008-2013 at 7-8 TeV
- Second run to begin in 2015 at 13 TeV
- Experimental accuracy at LHC improving



HDECAY [Djouadi, Kalinowski, Spira '97]



[Atlas 2013]

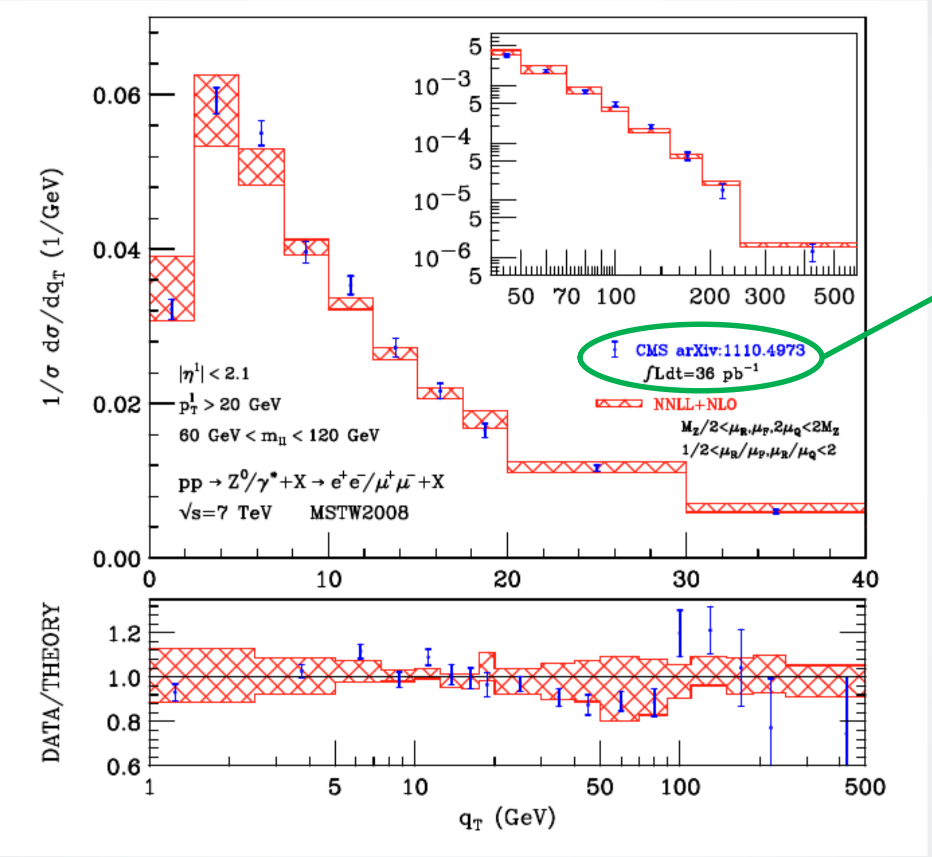


[CMS 2013]

BR (%):
60
6
0.2
20
2.5

- Uncertainty for total Higgs production: Theory ~ 10%, Exp. ~10-20%
- In particular EW processes important because of bigger signal to background ratio

DY: Theory vs Experiments



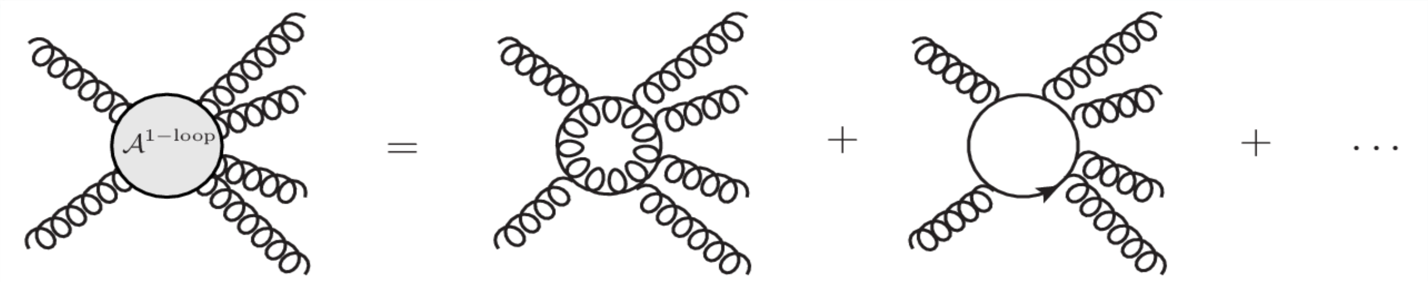
By now factor ~1000 more luminosity!

[Catani, Grazzini, Florian & Cieri]
[taken from G. Ferrera's talk 2013]

- Calculations based on perturbative expansion: $\sigma \sim \sigma_0 + \sigma_1 \alpha + \sigma_2 \alpha^2 + \cdots$
- Improved results by including resummation
- Experimental accuracy better than theoretical prediction

Example at NLO

- Imagine having to calculate a six-point gluon amplitude:

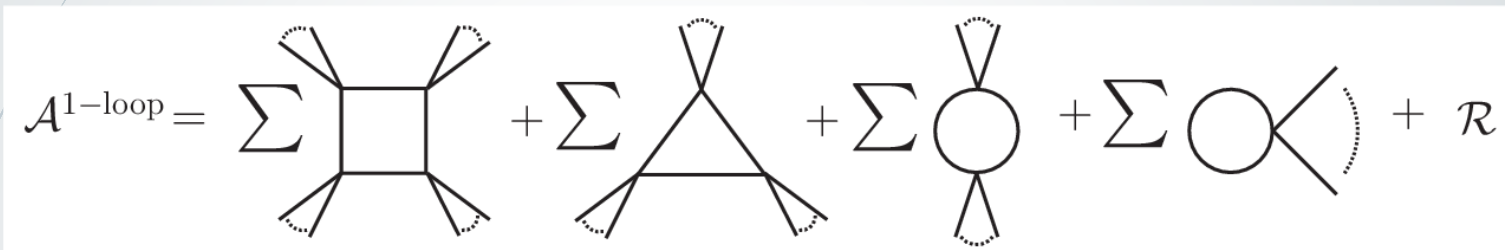


$$\mathcal{A}^{1-\mathrm{loop}} = \sum_{\mathrm{diagrams}} \int d^d k \frac{\{\gamma_\mu, \not{k}, f^{abc}, \cdots\}}{\mathrm{propagators}}$$

- Amount of diagrams grows rapidly with the amount and diversity of external particles
- Require higher-loop corrections for many processes → automation
- Do we need to calculate each diagram separately? → use unitarity

NLO revolution: efficient reduction

- Automation possible because of existence of basis of **Master Integrals** (MI)
- Valid for any *renormalizable QFT* (up to O(ϵ)):



- Traditional reduction: PV-tensor reduction [Pasarino & Veltman '79]
- MI are known ['t Hooft & Veltman '79], numerically in FF/LoopTools [Oldenborgh '90, Hahn & Victoria '98], QCDloop [Ellis & Zanderighi '07] and OneLOop [A. van Hameren '10]

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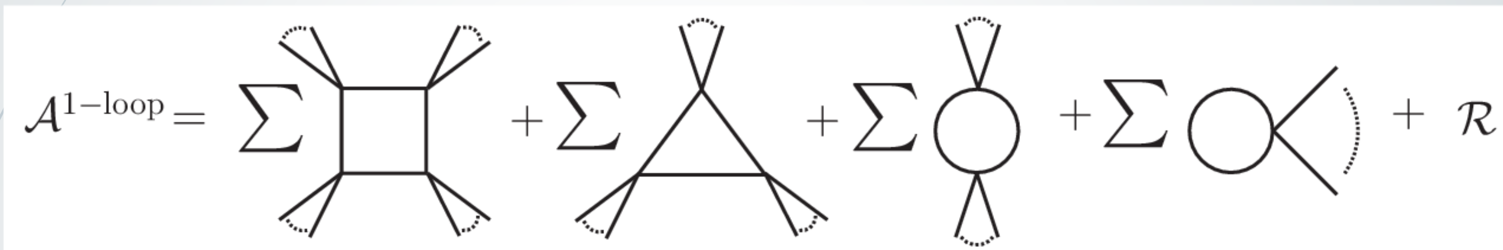

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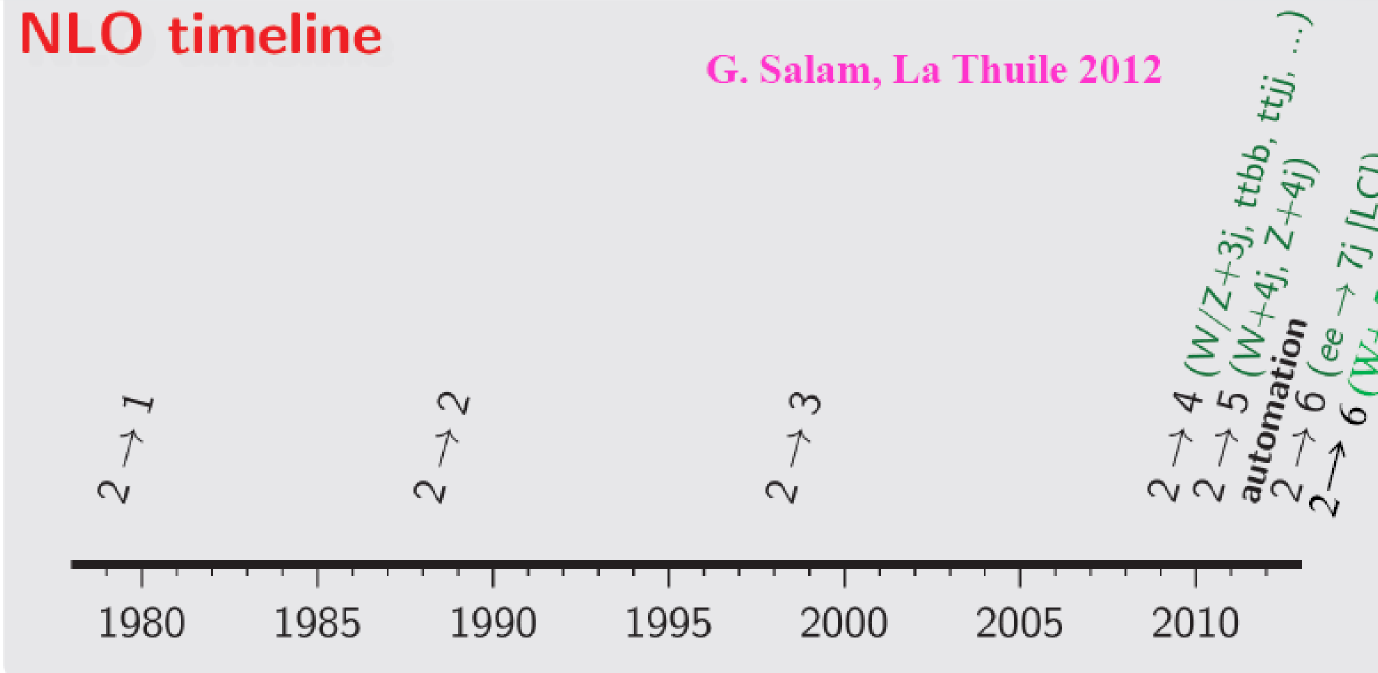
Efficient reduction of scattering amplitudes by *unitarity cuts*: $\frac{1}{p^2 - m^2} \to \delta(p^2 - m^2)$

- (Generalized) unitarity methods [Bern, Dixon, Dunbar & Kosower '94, Britto, Cachazo & Feng '04,…]
- OPP integrand reduction OPP [Ossola, Papadopoulos & Pittau '07,…]

Many numerical NLO tools: Formcalc [Hahn '99], Golem (PV) [Binoth, Cullen et al '08], Rocket [Ellis, Giele et al '09], NJet [Badger, Biederman, Uwer & Yundin '12], Blackhat [Berger, Bern, Dixon et al '12], Helac-NLO [Bevilacqua, Czakon et al '12], MadGolem, GoSam, OpenLoops, …

NLO revolution: 2 to *n* particle processes

[Slide from Z. Bern's talk at Pheno 2012]



2010: NLO $W+4j$ [BlackHat+Sherpa: Berger et al] [unitarity]
2011: NLO $WWjj$ [Rocket: Melia et al] [unitarity]
2011: NLO $Z+4j$ [BlackHat+Sherpa: Ita et al] [unitarity]
2011: NLO $4j$ [BlackHat+Sherpa: Bern et al] [Njet+Sherpa: Badger et al] [unitarity]
2011: first automation [MadNLO: Hirschi et al] [unitarity + feyn.diags]
2011: first automation [Helac NLO: Bevilacqua et al] [unitarity]
2011: first automation [GoSam: Cullen et al] [feyn.diags(+unitarity)]
2011: $e^+e^- \to 7j$ [Becker et al, leading colour] [numerical loops]
2012: NLO W+5j [BlackHat, preliminary] [published in 2013] [unitarity]
2013: NLO 5j [NJet+Sherpa: Badger et al] [unitarity]

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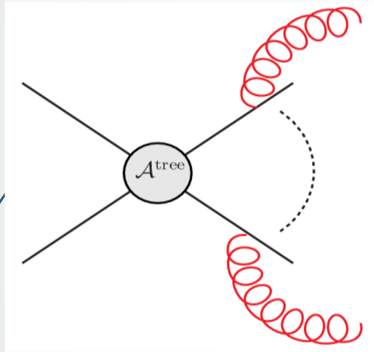
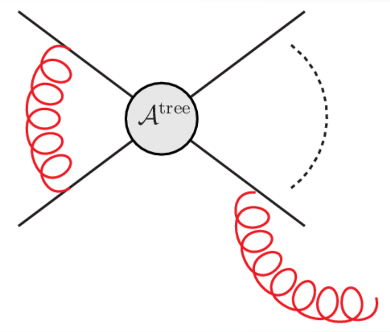
NNLO revolution?

- Next step in automation: **NNLO**

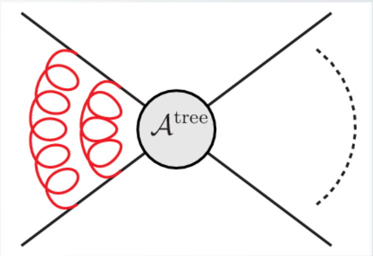
Three types of corrections:



Real-real



Real-virtual



Virtual-virtual (may also have non-planar graphs!)

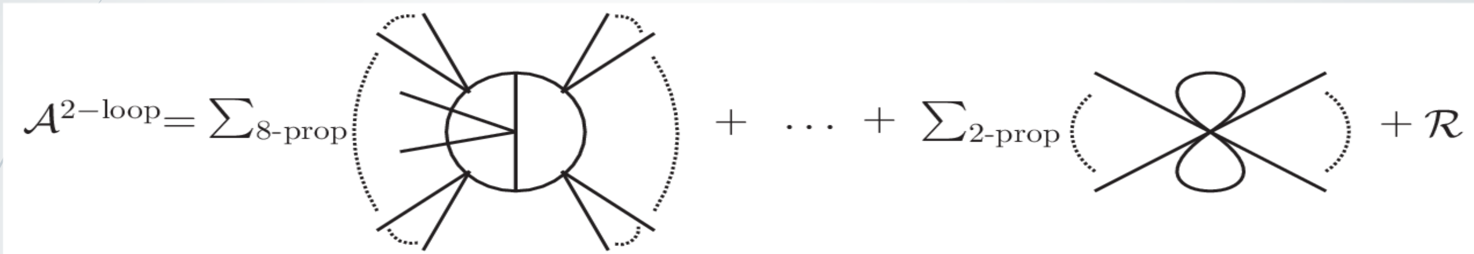
- Tree-level Feynman integrals ✓
- One-loop Feynman integrals ✓

} Basically the same as NLO

- Two-loop Feynman integrals (bottleneck) ← Virtual-virtual corrections

Two-loop overview

- A finite basis of Master Integrals exists as well at two-loops:



- Master integrals may contain loop-dependent numerators as well (*tensor integrals*)

Coherent framework for reductions for two- and higher-loop amplitudes:

- In N=4 SYM [Bern, Carrasco, Johansson et al. ’09-’12]
- Maximal unitarity cuts in general QFT’s [Johansson, Kosower, Larsen et al. ’12-’13]
- Integrand reduction with polynomial division in general QFT’s [Ossola & Mastrolia ’11, Zhang ’12, Badger, Frellesvig & Zhang ’12-’13, Mastrolia, Mirabella, Ossola & Peraro ’12-’13, Kleis, Malamos, Papadopoulos & Verheynen ’12]

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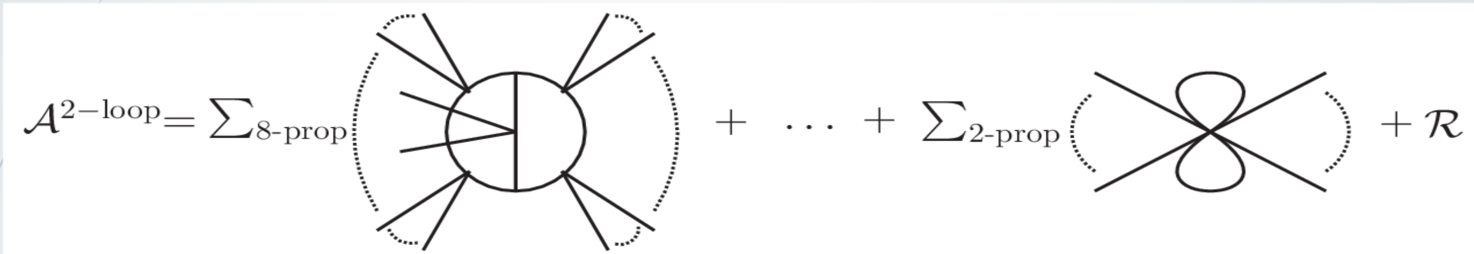

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- By now reduction substantially understood for two- and multi-loop integrals [no public code as of yet available]
- <u>**Missing ingredient: library of Master integrals (MI)**</u>
- Reduction to MI used for specific processes: *Integration by parts* (IBP) [Tkachov '81, Chetyrkin & Tkachov '81]

Reduction by IBP

[Tkachov '81, Chetyrkin & Tkachov '81]

- Fundamental theorem of calculus: given integral, by IBP get linear system of equations

$$G = \int \Big(\prod_i d^d k_i \Big) I \quad \xrightarrow{\text{IBP identities:}} \quad \int \Big(\prod_i d^d k_i \Big) \frac{\partial}{\partial k_j^\mu} \Big(v^\mu I \Big) = \text{Boundary term} \overset{DR}{=} 0$$

$$I = \frac{\text{Num}(k,p)}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \qquad D_i = c_{ijl} k_j . k_l + c_{ij} k_j . p_j + m_i^2, \qquad v \in \{k_1, \cdots, k_n, \text{external momenta}\}$$

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In practice, *generate numerator with negative indices* such that w.l.o.g.:

$$G_{a_1 \cdots a_n}(s) := \int \Big(\prod_i \frac{d^d k_i}{i\pi^{d/2}}\Big) \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}, \quad s = \{p_i.p_j\}|_{i,j}$$

IBP identities:

$$\sum_{a_1, \cdots a_n} \text{Rational}^{a_1 \cdots a_n}(s,d) G_{a_1 \cdots a_n}(s) = 0$$

Solve:

$$G_{a_1 \cdots a_n}(s) = \sum_{(b_1 \cdots b_n) \in \text{Master Integrals}} \text{Rational}_{a_1 \cdots a_n}^{b_1 \cdots b_n}(s,d) G_{b_1 \cdots b_n}(s)$$

- Systematic algorithm: [Laporta '00]. Public implementations: AIR [Anastasiou & Lazopoulos '04], FIRE [A. Smirnov '08] Reduze [Studerus '09, A. von Manteuffel & Studerus '12-13], LiteRed [Lee '12], …
- Revealing independent IBP's: ICE [P. Kant '13]

Reduction by IBP: one-loop triangle

One-loop triangle example:

$$G_{a_1a_2a_3} = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = 0$$

IBP identities: $$\int \frac{d^dk}{i\pi^{d/2}} \frac{\partial}{\partial k^\mu}\left(v^\mu \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}\right) = 0$$

Choose $v = k, p_1, p_2$ respectively →

$$\begin{aligned}
0 &\overset{v=k}{=} -a_3G_{-1+a_1,a_2,1+a_3} - a_2G_{-1+a_1,1+a_2,a_3} + (-2a_1+d-a_2-a_3)G_{a_1,a_2,a_3} + m_1a_2G_{a_1,1+a_2,a_3} \\
0 &\overset{v=p_1}{=} a_2G_{-1+a_1,1+a_2,a_3} + (a_1-a_2)G_{a_1,a_2,a_3} + a_3(G_{-1+a_1,a_2,1+a_3} - G_{a_1,-1+a_2,1+a_3} + m_2G_{a_1,a_2,1+a_3}) \\
&\quad -m_1a_2G_{a_1,1+a_2,a_3} - a_1G_{1+a_1,-1+a_2,a_3} + a_1m_1G_{1+a_1,a_2,a_3} \\
0 &\overset{v=p_2}{=} a_3G_{a_1,-1+a_2,1+a_3} + (a_2-a_3)G_{a_1,a_2,a_3} - m_2a_3G_{a_1,a_2,1+a_3} - a_2G_{a_1,1+a_2,-1+a_3} + m_2a_2G_{a_1,1+a_2,a_3} \\
&\quad +a_1(G_{1+a_1,-1+a_2,a_3} - G_{1+a_1,a_2,-1+a_3} - m_1G_{1+a_1,a_2,a_3})
\end{aligned}$$

Solve:

→ Master integrals: $\{G_{110}, G_{011}\}$

Triangle reduction by IBP: $$G_{111} = \frac{2(d-3)}{(d-4)(m_1-m_2)}(G_{011} - G_{110})$$

Methods for calculating MI

Rewriting of integrals in different representations:

- Parametric: Feynman/alpha parameters → Sector decomposition
- Mellin-Barnes [Bergere & Lam '74, Ussyukina '75, …, V. Smirnov '99, Tausk '99]

Using relations and/or cut identities:

- Dimensional shifting relations [Tarasov '96, Lee '10, Lee, V. Smirnov & A. Smirnov '10]
- Schouten identities [Remiddi & Tancredi '13]
- Integral reconstruction with cuts and using *Goncharov polylogarithms* (GP) and their properties [Abreu, Britto, Duhr & Gardi '14]

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As solutions of differential equations:

- Differentiation w.r.t. invariants [Kotikov '91, Remiddi '97, Caffo, Cryz & Remiddi '98, Gehrmann & Remiddi '00]
- Differentiation w.r.t. externally introduced parameter [Papadopoulos '14]

Many more: Dispersion relations, dualities, …

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As solutions of differential equations: Algorithmic solutions (next slides)

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Many more: Dispersion relations, dualities, …

General strategy for MI: first expand, then integrate

- For numerical purposes and phenomenology we are not interested in *exact* expressions for loop integrals, but efficient and manageable expressions
- We need ϵ *expansion* $\left(\epsilon = \frac{4-d}{2} \right)$

In practice it is easier to expand and then integrate instead of other way around:

- Make integrable by performing subtractions at integrand level
- Then expand in epsilon
- Finally integrate

$$\begin{aligned}
\int dx_1 \cdots dx_n G[\vec{x}, s, \epsilon] &= \int dx_1 \cdots dx_n G_{\text{sing}}[\vec{x}, s, \epsilon] + \int dx_1 \cdots dx_n (G[\vec{x}, s, \epsilon] - G_{\text{sing}}[\vec{x}, s, \epsilon]) \\
&= \tilde{G}_{\text{sing}}[s, \epsilon] + \sum_k \epsilon^k \int dx_1 \cdots dx_n G^{(k)}_{\text{finite}}[\vec{x}, s] \\
&= \sum_k \epsilon^k \left(\tilde{G}^{(k)}_{\text{sing}}[s] + \int dx_1 \cdots dx_n G^{(k)}_{\text{finite}}[\vec{x}, s] \right)
\end{aligned}$$

- Numerical extraction of coefficients in ϵ *expansion* are found in public codes: Secdec [Borowka, Carter & Heinrich], FIESTA [A. Smirnov, V. Smirnov & Tentyukov], sector_decomposition [Bogner & Weinzierl]

Intermission: Functional basis for (class of) MI

- The expansion in epsilon often leads to log's $\left(\cdots\right)^{a\epsilon} = 1 + a\epsilon \log\left(\cdots\right) + \frac{a^2}{2}\epsilon^2 \log^2\left(\cdots\right) + \cdots$
- Integrals <u>if parametrized correctly</u>: $\sum \int (\text{Rational function}) * \log^n\left(\cdots\right)$
- The above integrals (often) naturally lead to *Goncharov Polylogarithms* (GP) [Goncharov '98, '01, Remiddi & Vermaseren '00]:

$$\underbrace{GP(a_1, \cdots, a_n; x)}_{\text{weight n}} := \int_0^x dx' \frac{GP(a_2, \cdots, a_n; x')}{x' - a_1}, \quad GP(;x) = 1, \quad GP(\underbrace{0, \cdots, 0}_{\text{n times}}; x) = \frac{1}{n!} \log^n(x)$$

$$GP(\vec{a}; x) GP(\vec{b}; x) = \sum_{\vec{c} = \text{shuffle}\{\vec{a}, \vec{b}\}} GP(\vec{c}; x), \qquad \int_0^x dx' \text{Rational}(x') GP(a_1, \cdots, a_n; x') \stackrel{*}{=} \sum_{i=0}^{n+1} \sum_{b_0 \cdots b_i} \text{Rational}^{b_0 \cdots b_i}(x) GP(b_1, \cdots, b_i; x)$$

*Assuming convergence of integral, i.e. after subtracting singularities

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One-loop triangle example in Feynman parameters:

$$G_{111} = \int\frac{d^dk}{i\pi^{d/2}}\frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2} = -\frac{(-m_1)^{-\epsilon}-(-m_2)^{-\epsilon}}{m_1-m_2}\frac{\Gamma[1+\epsilon]}{\epsilon}\int_0^1 x^{-1-\epsilon}(1-x)^{-\epsilon}dx$$

$$\begin{aligned}\int_0^1 x^{-1-\epsilon}(1-x)^{-\epsilon}dx &= \int_0^1 x^{-1-\epsilon}dx + \int_0^1(x^{-1-\epsilon}(1-x)^{-\epsilon}-x^{-1-\epsilon})dx \\ &= \frac{1}{-\epsilon} + \Big(-\epsilon\int_0^1 dx\frac{GP(1;x)}{x} + \epsilon^2\int_0^1 dx\frac{GP(0,1;x)+GP(1,0;x)+GP(1,1;x)}{x}\Big) \\ &= \frac{1}{-\epsilon} + \Big(-\epsilon GP(0,1;1) + \epsilon^2(GP(0,0,1;1)+GP(0,1,0;1)+GP(0,1,1;1)\Big)\end{aligned}$$

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$$\begin{aligned}\int_0^1 x^{-1-\epsilon}(1-x)^{-\epsilon}dx &= \int_0^1 x^{-1-\epsilon}dx + \int_0^1(x^{-1-\epsilon}(1-x)^{-\epsilon} - x^{-1-\epsilon})dx \\ &= \frac{1}{-\epsilon} + \Big(-\epsilon\int_0^1 dx\frac{GP(1;x)}{x} + \epsilon^2\int_0^1 dx\frac{GP(0,1;x)+GP(1,0;x)+GP(1,1;x)}{x}\Big) \\ &= \frac{1}{-\epsilon} + \Big(-\epsilon GP(0,1;1) + \epsilon^2(GP(0,0,1;1)+GP(0,1,0;1)+GP(0,1,1;1)\Big)\end{aligned}$$

GP's are fundamental building blocks for many MI

*Assuming convergence of integral, i.e. after subtracting singularities

→ DE method takes advantage of this fact

Outline

- Introduction
- Differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook

General loop integrals

[Kotikov '91, Remiddi '97, Caffo, Cryz & Remiddi '98, Gehrmann & Remiddi '00]

- Assume one is interested in a multi-loop Feynman integral:

$$G_{a_1\cdots a_n}(\tilde{s}) := \int \Big(\prod_i \frac{d^d k_i}{i\pi^{d/2}}\Big) \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}, \quad D_i = c_{ijl} k_j.k_l + c_{ij} k_j.p_j + m_i^2, \;\; \tilde{s} = \{\tilde{s}_1, \tilde{s}_2 \cdots\}$$

- Define denominators such as to be able to generate all (irreducible) scalar products
- Numerators may be included as negative indices $a_i < 0$
- Invariants $\tilde{s}$ may be functions of scalar products: $\tilde{s} = \{f_1(p_i.p_j), f_2(p_i.p_j)\ldots\}$

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- Differentiate w.r.t. external momenta:
- Use IBP identities to *reduce* r.h.s. to MI:

$$\begin{aligned} p_i.\frac{\partial}{\partial p_j} G_{a_1\cdots a_n}(\tilde{s}) &= \int \Big(\prod_i \frac{d^d k_i}{i\pi^{d/2}}\Big) p_i.\frac{\partial}{\partial p_j} \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}} \\ &= \sum_{b_1,\cdots b_n} c_{a_1\cdots a_n}^{b_1\cdots b_n}(\tilde{s}) G_{b_1\cdots b_n}(\tilde{s}) \\ &\overset{IBP}{=} \sum_{(b_1\cdots b_n)\in \text{Master Integrals}} \tilde{c}_{a_1\cdots a_n}^{b_1\cdots b_n}(\tilde{s}) G_{b_1\cdots b_n}(\tilde{s}) \end{aligned}$$

- If all MI on r.h.s. known, solve the above *differential equation* (DE) to get required $G_{a_1\ldots a_n}$

DE for MI

[Kotikov '91, Remiddi '97, Caffo, Cryz & Remiddi '98, Gehrmann & Remiddi '00]

- DE method can be most efficiently used to calculate MI themselves
- In that case the r.h.s. of previous DE may contain MI $G_{a_1 \dots a_n}$ itself
- Perform differentiation on every MI to get a *matrix equation*:

$$\vec{G}^{MI} = \{G_{b_1,\cdots b_n} | (b_1, \cdots b_n) \in \text{Master Integrals}\} =: \{G_1, \cdots, G_m\}$$

$$p_i.\frac{\partial}{\partial p_j} \vec{G}^{MI}(\tilde{s}, \epsilon) = \overline{\overline{M}}(\tilde{s}, \epsilon, i, j).\vec{G}^{MI}(\tilde{s}, \epsilon)$$

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$$p_i \cdot \frac{\partial}{\partial p_j} \vec{G}^{MI}(\tilde{s}, \epsilon) = \overline{\overline{M}}(\tilde{s}, \epsilon, i, j).\vec{G}^{MI}(\tilde{s}, \epsilon)$$

- Solve above linear system of DE to get all MI in one go
- In practice first need to change to a DE in differentiation w.r.t. invariants $\tilde{s}$:

$$p_i \cdot \frac{\partial}{\partial p_j} \mathrm{F}(\tilde{s}) = \sum_k p_i \cdot \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} \mathrm{F}(\tilde{s})$$

$$\rightarrow \quad \boxed{\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s}, \epsilon) = \overline{\overline{M}}_k(\tilde{s}, \epsilon).\vec{G}^{MI}(\tilde{s}, \epsilon)}$$

- Boundary condition $\vec{G}^{MI}(\tilde{s}_k = \tilde{s}_{k,0})$ may be found (among other ways) by plugging in special kinematical values for invariants (if regular), via symmetries, via analytical/regularity constraints, solving DE's in the other invariants or asymptotic expansion

Uniform weight solution

- In general matrix in DE is dependent on ϵ:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- **Conjecture:** possible to make a rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\boxed{\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)}$$

[Henn '13]

- Explicitly shown to be true for many examples [Henn '13, Henn, Smirnov et al '13-'14]
- **If** set of invariants $\tilde{s} = \{f(p_i.p_j)\}$ chosen correctly: $\overline{\overline{M}}_k(\tilde{s}) = \sum_{\text{poles } \tilde{s}_k^{(0)}} \frac{\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}}{(\tilde{s}_k - \tilde{s}_k^{(0)})}$
- Solution is *uniform* in *weight* of GP's:

$$\vec{G}^{MI}(\tilde{s},\epsilon) = Pe^{\epsilon\int_{C[0,\tilde{s}]}\overline{\overline{M}}_k(\tilde{s}'_k)}\vec{G}^{MI}(0,\epsilon) = (\mathbf{1} + \epsilon\int_0^{\tilde{s}_k}\overline{\overline{M}}_k(\tilde{s}'_k) + \cdots)\underbrace{\vec{G}^{MI}(0,\epsilon)}_{\vec{G}_0^{MI}+\epsilon\vec{G}_1^{MI}+\cdots}$$

$$= \underbrace{\vec{G}_0^{MI}}_{\text{weight i}} + \epsilon\underbrace{(\underbrace{\vec{G}_1^{MI}}_{\text{weight i+1}} + \sum_{\text{poles } \tilde{s}_k^{(0)}} \overbrace{\left(\int_0^{\tilde{s}_k}\frac{d\tilde{s}'_k}{(\tilde{s}'_k - \tilde{s}_k^{(0)})}\right)}^{GP(\tilde{s}_k^{(0)};\tilde{s}_k)}\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}.\underbrace{\vec{G}_0^{MI}}_{\text{weight i}})}_{\text{weight i+1}} + \cdots$$

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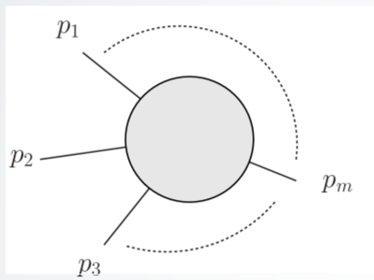
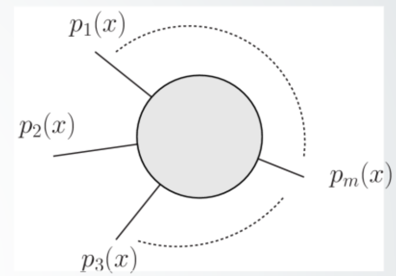
x-Parametrization

[Papadopoulos '14]

- Method up till now only developed for *massless internal lines* (assume in following)
- Introduce extra parameter *x* in the denominators of loop integral
- *x*-parameter describes off-shellness of external legs



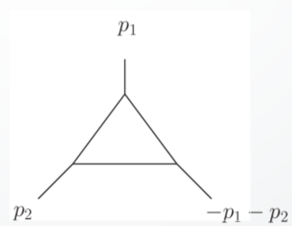
x-parametrize →



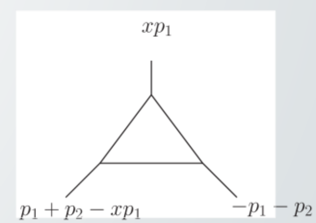
$$G_{a_1\cdots a_n}(s,\epsilon) = \int \Big(\prod_i d^d k_i\Big) \frac{1}{D_1^{2a_1}(k,p)\cdots D_n^{2a_n}(k,p)} \longrightarrow G_{a_1\cdots a_n}(x,s,\epsilon) = \int \Big(\prod_i d^d k_i\Big) \frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$p_i(x) = p_i + (1-x)q_i, \quad \sum_i q_i = 0, \quad D_i(k,p) = c_{ij}k_j + d_{ij}p_j, \quad s = \{p_i.p_j\}|_{i,j}$$

One-loop triangle example:



→



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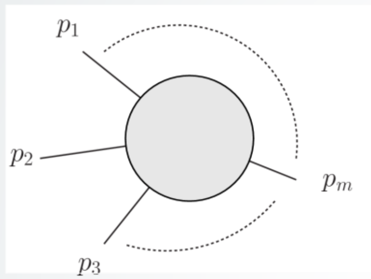
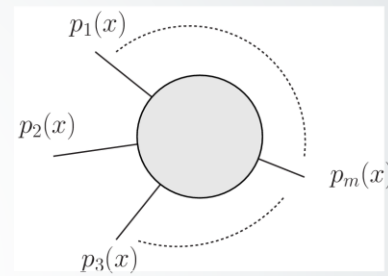


x-parametrize



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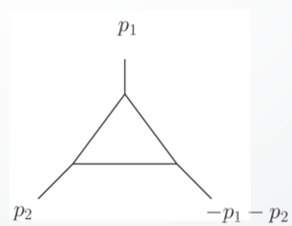
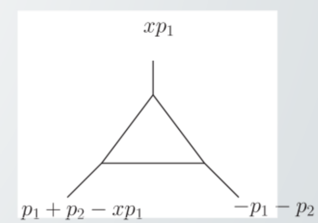




- Take derivative of integral *G* w.r.t. *x*-parameter instead of w.r.t. invariants and reduce r.h.s. by IBP identities

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \overset{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon), \;\; s = \{p_i.p_j\}|_{i,j}$$

Bottom-up approach

- Notation: upper index "(m)" in integrals $G^{(m)}_{\{a_1 \ldots a_n\}}$ denotes amount of positive indices, i.e. amount of denominators/propagators

$$G^{(m)}_{a_1 \cdots a_n} = \int \left(\prod_i d^d k_i \right) \underbrace{\frac{1}{D_1^{2a_1}(k,p) \cdots D_n^{2a_n}(k,p)}}_{m \text{ propagators, (positive indices) } a_i}$$

- In practice individual DE's of MI are of the form:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1 \cdots a_n}(x,s,\epsilon) = \sum_{m'=m_0}^{m} \sum_{b_1, \cdots b_n} \text{Rational}^{b_1, \cdots b_n}_{a_1 \cdots a_n}(x,s,\epsilon) G^{(m')}_{b_1 \cdots b_n}(x,s,\epsilon)$$

Bottom-up:

- Solve first for all MI with least amount of denominators m_0 (these are often already known to all orders in ϵ or often calculable with other methods)
- After solving all MI with m denominators ($m \geq m_0$), solve all MI with $m+1$ denominators
- In practice: $G^{(m_0)}_{a_1 \cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left(\sum \text{Rational}(x) GP(\cdots ; x) \right)$

Choice of x-parametrization

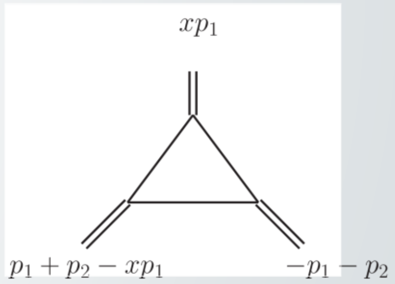
main criteria for choice of x-parametrization: *constant term* ($\epsilon = 0$) of *residues of homogeneous term* for *every* DE needs to be an *integer:*

$$\frac{\partial}{\partial x} G^{(m)}_{a_1 \cdots a_n}(x,\epsilon) = H(x,\epsilon) G^{(m)}_{a_1 \cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon} \Big(\sum \text{Rational}(x) GP(\cdots;x) \Big)$$

$$= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1 \cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon} \Big(\sum \text{Rational}(x) GP(\cdots;x) \Big) \longrightarrow$$

$$\frac{\partial}{\partial x}\left(M(x,\epsilon) G^{(m)}_{a_1 \cdots a_n}(x,\epsilon)\right) = M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon} \Big(\sum \text{Rational}(x) GP(\cdots;x) \Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}$$

For all MI that we have calculated, the criteria could be easily met. Often it was enough to choose the external legs such that the corresponding massive MI triangles (found by pinching external legs) are as follows:



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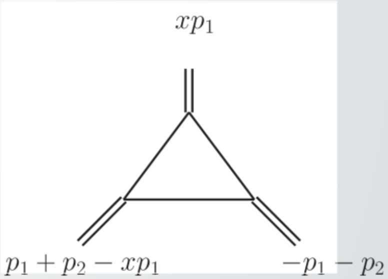


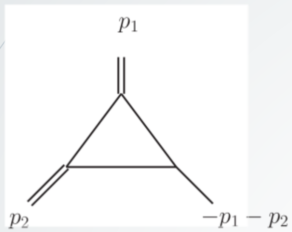
Boundary condition:

- Boundary condition almost always zero in bottom-up approach
- Except in three cases, all loop integrals we have come across: $(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} = 0$

  → Not well understood yet why this is so!

- If not zero, boundary condition $(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0}$ may be found (in principle) by plugging in special values for x, via analytical/regularity constraints, asymptotic expansion in $x \to 0$ or some modular transformation like $x \to 1/x$

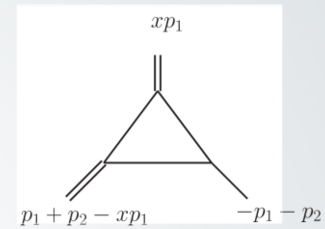
Example: one-loop triangle



$$G_{111}(m_1, m_2) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2}$$

$$p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = 0$$

Parametrize, the off-shellness with x



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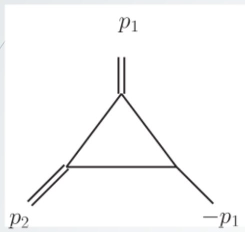

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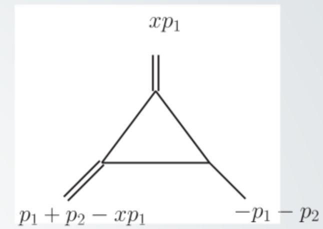
Parametrize, the off-shellness with *x*



$$G_{111}(m_1, x) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2(k+xp_1)^2(k+p_1+p_2)^2}$$

$$p_1^2 = m_1, p_2^2 = 0, (p_1+p_2-xp_1)^2 \neq 0, (p_1+p_2)^2 = 0$$

- Differentiate to *x* and use IBP to reduce:

$$\frac{\partial}{\partial x} G_{111}(x) = \frac{-x^{-2-\epsilon}}{\epsilon^2 m_1}((-m_1 - i.0)^{-\epsilon}(1+2\epsilon)x^{-\epsilon} - (m_1 - i.0)^{-\epsilon}(1-x)^{-1-\epsilon}(1+\epsilon - x(1+2\epsilon)))$$

- Subtracting the singularities and expanding the finite part leads to:

$$\begin{aligned} G_{111}(x) &= G_{111}(0) + \int_0^x dx' \frac{-x'^{-2-\epsilon}}{\epsilon^2 m_1}((-m_1 - i.0)^{-\epsilon}(1+2\epsilon)x'^{-\epsilon} - (m_1 - i.0)^{-\epsilon}(1-x')^{-1-\epsilon}(1+\epsilon - x'(1+2\epsilon))) \\ &= \underbrace{G_{111}(0)}_{=0} + \frac{-(m_1 - i.0)^{-\epsilon}x^{-\epsilon} + (-m_1 - i.0)^{-\epsilon}x^{-2\epsilon}}{m_1 x \epsilon^2} + \frac{(m_1 - i.0)^{-\epsilon}(-x^{-\epsilon} + (x + GP(1;x)))}{m_1 x \epsilon} + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Agrees with expansion of exact solution: $G_{111}(m_1 * x^2, m_2 = (-m_1)x(1-x)) = \frac{c_\Gamma(\epsilon)}{\epsilon^2} \frac{(-m_1 x^2)^{-\epsilon} - (-(-m_1)x(1-x))^{-\epsilon}}{m_1 x^2 - (-m_1)x(1-x)}$

Comparison of DE methods

Traditional DE method:

- Choose $\tilde{s} = \{f(p_i.p_j)\}$ and use chain rule to relate differentials of (independent) momenta and invariants:

$$p_i.\frac{\partial}{\partial p_j}\mathrm{F}(\tilde{s}) = \sum_k p_i.\frac{\partial \tilde{s}_k}{\partial p_j}\frac{\partial}{\partial \tilde{s}_k}\mathrm{F}(\tilde{s})$$

- Solve above linear equations:

$$\frac{\partial}{\partial \tilde{s}_k} = g_k(\{p_i.\frac{\partial}{\partial p_j}\})$$

- Differentiate w.r.t. invariant(s) $\tilde{s}_k$:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = g_k(\{p_i.\frac{\partial}{\partial p_j}\})\vec{G}^{MI}(\tilde{s},\epsilon) \overset{IBP}{=} \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- Make rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)$$ [Henn '13]

- Solve perturbatively in ϵ to get GP's if $\tilde{s} = \{f(p_i.p_j)\}$ chosen properly
- Solve DE of different $\tilde{s}_k$, to capture boundary condition

Simplified DE method:

- Introduce external parameter x to capture off-shellness of external momenta:

$$G_{a_1\cdots a_n}(s,\epsilon) = \int \left(\prod_i d^d k_i\right)\frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$p_i(x) = p_i + (1-x)q_i, \quad \sum_i q_i = 0, \quad s = \{p_i.p_j\}|_{i,j}$$

- Parametrization: pinched massive triangles should have legs (not fully constraining):

$$q_1(x) = xp', q_2(x) = p'' - xp', \;\; p'^2 = m_1, p''^2 = m_3$$

- Differentiate w.r.t. parameter x:

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \overset{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon)$$

- Check if *constant term* ($\epsilon = 0$) *of residues of homogeneous term* for *every* DE is an *integer*: 1) if yes, solve DE by "bottom-up" approach to express in GP's; 2) if no, change parametrization and check DE again
- Boundary term almost always captured, if not: try $x \to 1/x$ or asymptotic expnansion

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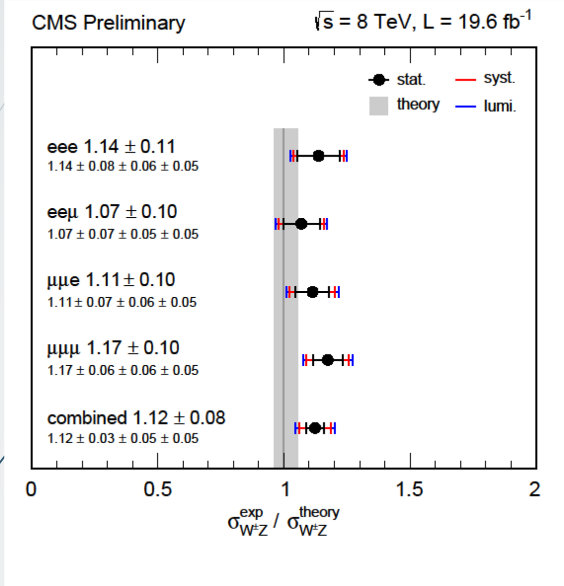
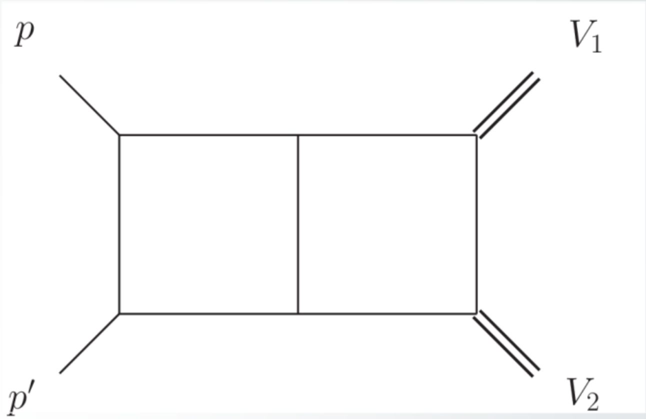
Two-loop planar double-box



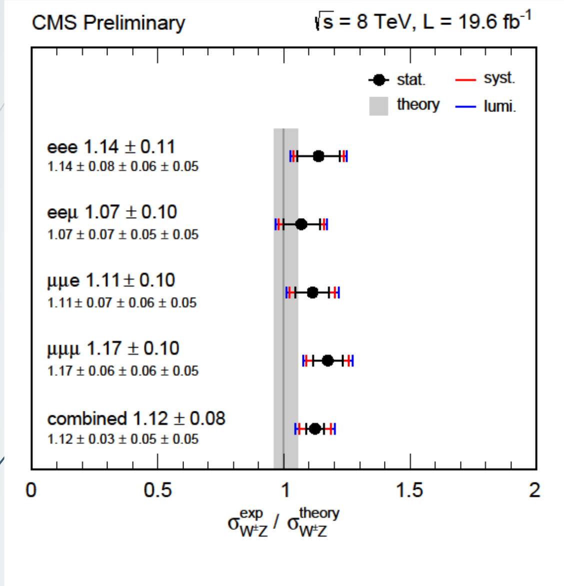
Example of planar diagrams:



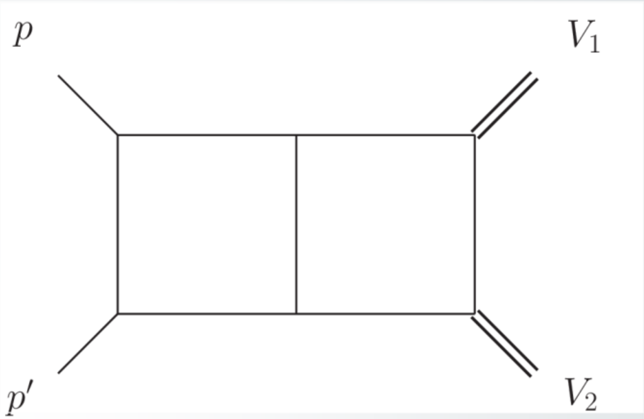
$$pp' \to V_1 V_2, \quad m_{V_1} \neq m_{V_2} \neq 0$$

Require 4-point two-loop MI with 2 off-shell legs and massless internal legs (at LHC light-flavor quarks are massless to good degree): diboson production

Two-loop planar double-box



Example of planar diagrams:



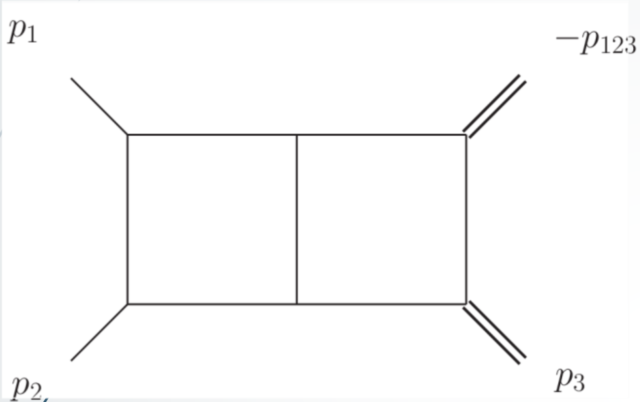
$$pp' \to V_1 V_2, \quad m_{V_1} \neq m_{V_2} \neq 0$$

Require 4-point two-loop MI with 2 off-shell legs and massless internal legs (at LHC light-flavor quarks are massless to good degree): diboson production

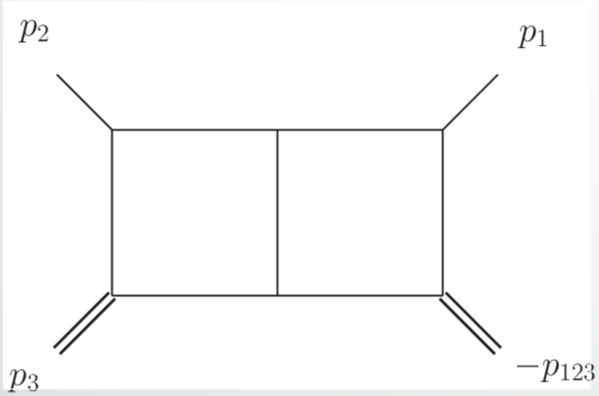
- On-shell legs: $q_1^2 = \cdots = q_4^2 = 0$ [planar: V. Smirnov '99, V. Smirnov & Veretin '99, non-planar: Tausk '99, Anastasiou, Gehrmann, Oleari, Remiddi & Tausk '00]
- One off-shell leg (pl.+non-pl.): $q_1^2 = q^2,\ q_2^2 = q_3^2 = q_4^2 = 0$ [Gehrmann & Remiddi '00-'01]
- Two off-shell legs with same masses: $q_1^2 = q_2^2 = q^2,\ q_3^2 = q_4^2 = 0$ [planar: Gehrmann, Tancredi & Weihs '13, non-planar: Gehrman, Manteuffel, Tancredi & Weihs '14]
- Two off-shell legs with different masses: $q_1^2 \neq 0, q_2^2 \neq 0,\ q_3^2 = q_4^2 = 0$ [planar: Henn, Melnikov & Smirnov '14, non-planar: Caola, Henn, Melnikov & Smirnov '14]

Double planar box: IBP families

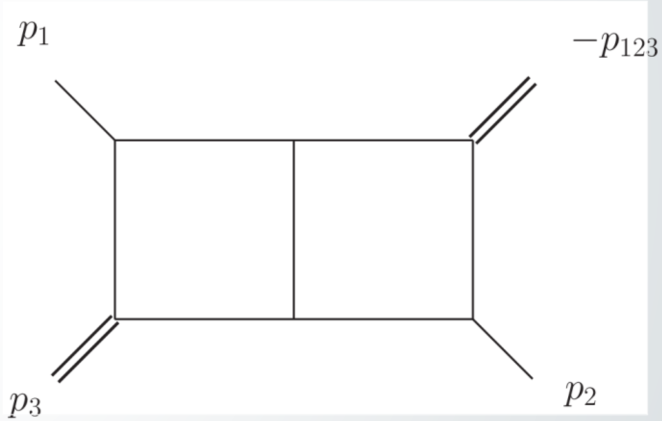
- MI form 3 families of coupled DE's



1st Adjacent mass topology → 31 MI

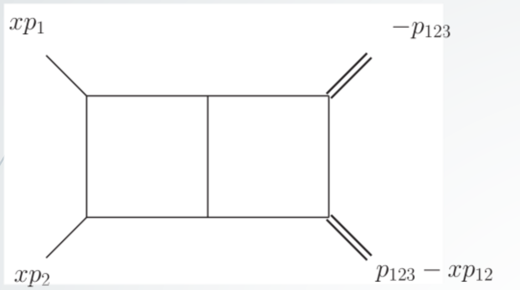


2nd Adjacent mass topology → 29 MI



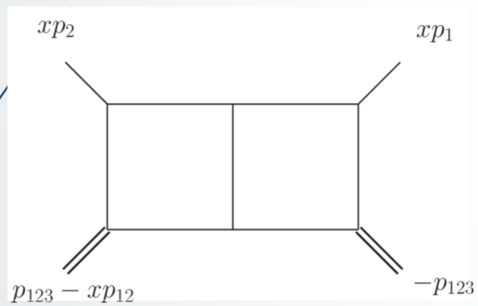
Opposite mass topology → 28 MI

Double planar box: Parametrization



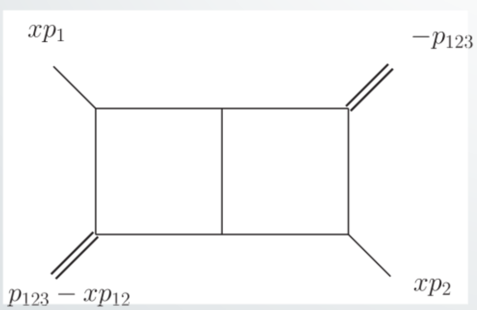
1st Adjacent mass topology → 31 MI

$$G^{(1)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}} \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$



2nd Adjacent mass topology → 29 MI

$$G^{(2)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}} \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$



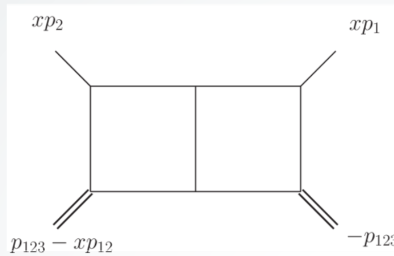
Opposite mass topology → 28 MI

$$G^{(3)}_{a_1\cdots a_9}(x) := \int \frac{d^dk_1}{i\pi^{d/2}} \frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}$$

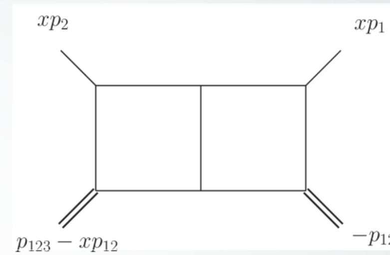
Bottom-up approach

- The DE's of 2nd family :

7-denominators: $\frac{\partial}{\partial x}$

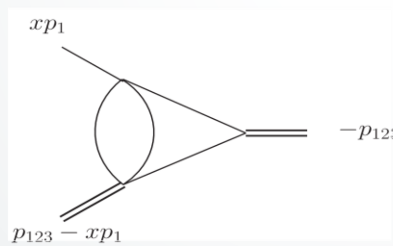


$=$

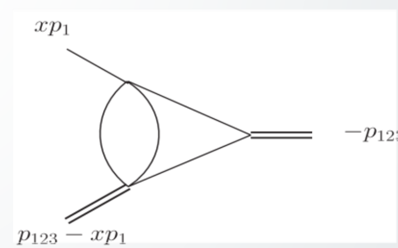


$+ \underbrace{\cdots}_{\leq 7\text{-demominators}}$

4-denominators: $\frac{\partial}{\partial x}$

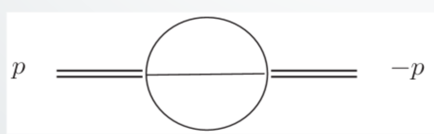


$=$



$+ \underbrace{\cdots}_{\leq 4\text{-demominators}}$

3-denominators:

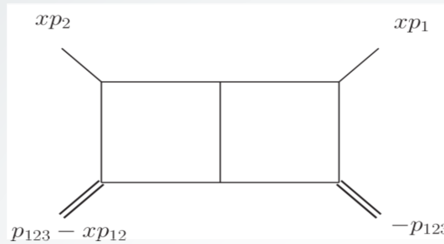


$= \text{constant} \times (-p^2)^{1-2\epsilon}$

- The 3-denominator MI are trivially known sunset diagrams
- We solve first the 4-denominator MI, then the 5-denominator MI etc.

Solutions in GP

$$G^{(2)}_{011111011}(x) =$$



$$s_{12}=p_{12}^2,\ s_{23}=p_{23}^2,\ m_4=p_{123}^2$$

solution of DE →

$$\begin{aligned}
G^{(2)}_{011111011}(x) = {} & \frac{A_3(\epsilon)}{x^2 s_{12}(-m_4+x(m_4-s_{23}))^2}\Bigg(\frac{-1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left(-GP\left(\frac{m_4}{s_{12}};x\right)+2GP\left(\frac{m_4}{m_4-s_{23}};x\right)+2GP(0;x)-GP(1;x)+\log(-s_{12})+\frac{9}{4}\right) \\
& +\frac{1}{4\epsilon^2}\Bigg(18GP\left(\frac{m_4}{s_{12}};x\right)-36GP\left(\frac{m_4}{m_4-s_{23}};x\right)-8GP\left(0,\frac{m_4}{s_{12}};x\right)+16GP\left(0,\frac{m_4}{m_4-s_{23}};x\right)+8GP\left(\frac{s_{23}}{s_{12}}+1,\frac{m_4}{m_4-s_{23}};x\right) \\
& +8GP\left(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}}+1;x\right)-8GP\left(\frac{m_4}{s_{12}},\frac{m_4}{m_4-s_{23}};x\right)+8GP\left(\frac{m_4}{m_4-s_{23}},1;x\right)+4\Bigg(-2GP\left(\frac{s_{23}}{s_{12}}+1;x\right)GP\left(\frac{m_4}{s_{12}};x\right) \\
& +2GP\left(\frac{m_4}{m_4-s_{23}};x\right)\left(2GP\left(\frac{m_4}{s_{12}};x\right)-2GP\left(\frac{m_4}{m_4-s_{23}};x\right)+GP(1;x)\right)+GP(0;x)\Bigg(4GP\left(\frac{m_4}{s_{12}};x\right)-8GP\left(\frac{m_4}{m_4-s_{23}};x\right) \\
& +4GP(1;x)-4\log(-s_{12})-9\Bigg)+2\log(-s_{12})\left(GP\left(\frac{m_4}{s_{12}};x\right)-2GP\left(\frac{m_4}{m_4-s_{23}};x\right)+GP(1;x)\right)-4GP(0;x)^2-\log^2(-s_{12})\Bigg) \\
& -8GP\left(\frac{s_{23}}{s_{12}}+1,1;x\right)+18GP(1;x)-8GP(0,1;x)-18\log(-s_{12})-9\Bigg)+\frac{1}{\epsilon}(\cdots) \\
& +\Bigg(-3GP\left(0,\frac{m_4}{s_{12}};x\right)^2-18GP\left(0,\frac{m_4}{m_4-s_{23}};x\right)^2-GP\left(\frac{s_{23}}{s_{12}}+1,\frac{m_4}{m_4-s_{23}};x\right)^2-GP\left(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}}+1;x\right)^2+GP\left(\frac{m_4}{s_{12}},\frac{m_4}{m_4-s_{23}};x\right)^2 \\
& +GP\left(\frac{m_4}{m_4-s_{23}},1;x\right)^2-2\Bigg(4GP\left(0,0,0,\frac{m_4}{s_{12}};x\right)-8GP\left(0,0,0,\frac{m_4}{m_4-s_{23}};x\right)-GP\left(0,0,1,\frac{m_4}{s_{12}};x\right)+7GP\left(0,0,1,\frac{m_4}{m_4-s_{23}};x\right) \\
& +6\Bigg(GP\left(0,0,1,\frac{m_4 s_{12}-\sqrt{m_4 s_{12} s_{23}\left(-m_4+s_{12}+s_{23}\right)}}{s_{12}\left(m_4-s_{23}\right)};x\right)+GP\left(0,0,1,\frac{m_4 s_{12}+\sqrt{m_4 s_{12} s_{23}\left(-m_4+s_{12}+s_{23}\right)}}{s_{12}\left(m_4-s_{23}\right)};x\right)\Bigg)\Bigg) \\
& -10GP\left(0,0,1,\frac{s_{23}}{s_{12}}+1;x\right)+4GP(0,0,0,1;x)-GP(0,0,1,1;x)\Bigg)-GP\left(\frac{s_{23}}{s_{12}}+1,1;x\right)^2-3GP(0,1;x)^2+\cdots\Bigg)\Bigg)
\end{aligned}$$

ϵ^0 terms

Solutions in GP

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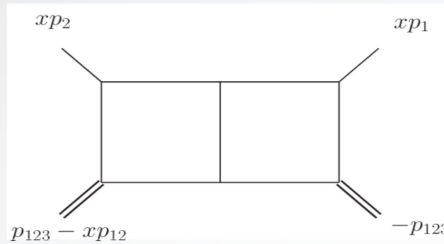

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&+8GP\Big(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}}+1;x\Big)-8GP\Big(\frac{m_4}{s_{12}},\frac{m_4}{m_4-s_{23}};x\Big)+8GP\Big(\frac{m_4}{m_4-s_{23}},1;x\Big)+4\Big(-2GP\Big(\frac{s_{23}}{s_{12}}+1;x\Big)GP\Big(\frac{m_4}{s_{12}};x\Big)\\
&+2GP\Big(\frac{m_4}{m_4-s_{23}};x\Big)\Big(2GP\Big(\frac{m_4}{s_{12}};x\Big)-2GP\Big(\frac{m_4}{m_4-s_{23}};x\Big)+GP(1;x)\Big)+GP(0;x)\Big(4GP\Big(\frac{m_4}{s_{12}};x\Big)-8GP\Big(\frac{m_4}{m_4-s_{23}};x\Big)\\
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&-8GP\Big(\frac{s_{23}}{s_{12}}+1,1;x\Big)+18GP(1;x)-8GP(0,1;x)-18\log(-s_{12})-9\Big)+\frac{1}{\epsilon}\Big(\cdots\Big)\\
&+\Big(-3GP\Big(0,\frac{m_4}{s_{12}};x\Big)^2-18GP\Big(0,\frac{m_4}{m_4-s_{23}};x\Big)^2-GP\Big(\frac{s_{23}}{s_{12}}+1,\frac{m_4}{m_4-s_{23}};x\Big)^2-GP\Big(\frac{m_4}{s_{12}},\frac{s_{23}}{s_{12}}+1;x\Big)^2+GP\Big(\frac{m_4}{s_{12}},\frac{m_4}{m_4-s_{23}};x\Big)^2\\
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&-10GP\Big(0,0,1,\frac{s_{23}}{s_{12}}+1;x\Big)+4GP(0,0,0,1;x)-GP(0,0,1,1;x)\Big)-GP\Big(\frac{s_{23}}{s_{12}}+1,1;x\Big)^2-3GP(0,1;x)^2+\cdots\Big)\Bigg)
\end{aligned}
$$

ϵ^0 terms

- Numerical agreement in *Euclidean region* found with Secdec [Borowka, Carter & Heinrich]:

$$G^{(2)}_{011111011}(x=1/3, s_{12}=-2, s_{23}=-5, m_4=-9) = -\frac{0.0191399}{\epsilon^4}-\frac{0.0292887}{\epsilon^3}+\frac{0.0239971}{\epsilon^2}+\frac{0.340233}{\epsilon}+0.870356+\mathcal{O}(\epsilon)$$

Outline

- Introduction
- Differential equations method to integration
- Simplified differential equations method
- Application
- Summary and outlook

Summary

- In LHC era multi-loop calculations are compulsory
- Two-loop automation is the next step: reduction substantially understood, library of MI mandatory but still missing
- Functional basis for large class of MI: *Goncharov polylogarithms*
- DE method is very fruitful for deriving MI in terms of GP
- Simplified DE method [Papadopoulos '14] captures GP solution naturally, boundary constraints taken into account, very algorithmic

- Recent application: planar double box

Outlook

- Application to non-planar graphs
- Application/extension to diagrams with massive propagators

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Outlook

Thank you very much!

- Application to non-planar graphs
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Backup slides

GP-structure of solution

- Assume for $m' < m$ denominators: $G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$

- For simplicity we assume here a non-coupled DE for a MI with m denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1,\cdots b_n} \text{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

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dependence on invariants s suppressed

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \longrightarrow \\
\frac{\partial}{\partial x}\big(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\big) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

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dependence on invariants s suppressed →

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\end{aligned}$$

- Formal solution:

$$\begin{aligned}
M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l}\prod_{\text{poles } x^{(0)}} \int_0^x dx' \left(x'^{-n+l\epsilon}(x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\right)\left(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\right) \\
&= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{\tilde{n},l}\int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon) + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')
\end{aligned}$$

GP-structure of solution

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$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1,\cdots b_n} \text{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

dependence on invariants s suppressed

$$\begin{aligned} \frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right) \\ &= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right) \longrightarrow \\ \frac{\partial}{\partial x}\left(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\right) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)} \end{aligned}$$

- Formal solution:

$$\begin{aligned} M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l} \prod_{\text{poles } x^{(0)}} \int_0^x dx' \left(x'^{-n+l\epsilon} (x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\right)\left(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\right) \\ &= \underbrace{(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0}}_{\text{boundary condition}} + \sum_{\tilde{n},l} \underbrace{\int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon)}_{x^{-\tilde{n}+l\epsilon+1}\tilde{I}_{\tilde{n},l}(\epsilon)} + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \underbrace{\sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')}_{\sum \text{Rational}_k(x) GP(\cdots;x) \text{ if } r_{x^{(0)}} \in \mathbb{Z}} \end{aligned}$$

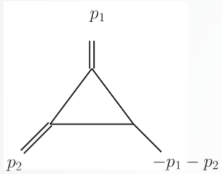
→ MI expressible in GP's: $G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right)$

Fine print for coupled DE's: if the non-diagonal piece of $\epsilon = 0$ term of *matrix H* is nilpotent (e.g. triangular) and if diagonal elements of *matrices* $r_{x^{(0)}}$ are integers, then above "GP-argument" is still valid

Example of tradition DE method: one-loop triangle (1/2)

- Consider again one-loop triangles with 2 massive legs and massless propagators:

$$G_{a_1a_2a_3}(\tilde{s}) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = m_3 = 0$$

$G_{111} =$ 

- General function:

$$p_i \cdot \frac{\partial}{\partial p_j} \mathrm{F}(m_1, m_2, m_3) = \sum_{k=1}^{3} p_i \cdot \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} \mathrm{F}(m_1, m_2, m_3), \quad i, j \in \{1, 2\}$$

$$\tilde{s}_1 = p_1^2 = m_1, \tilde{s}_2 = p_2^2 = m_2, \tilde{s}_3 = (p_1+p_2)^2 = m_3$$

- Four linear equations, of which three independent because of invariance under Lorentz transformation [Remiddi & Gehrmann '00], in three unknowns: $\{\frac{\partial}{\partial m_1}, \frac{\partial}{\partial m_2}, \frac{\partial}{\partial m_3}\}$

- Solve linear equations: $\frac{\partial}{\partial m_k} = g_k(p_1 \cdot \frac{\partial}{\partial p_1}, p_2 \cdot \frac{\partial}{\partial p_2}, p_2 \cdot \frac{\partial}{\partial p_1}), \quad k = 1, 2, 3$

$$\frac{\partial}{\partial m_1} G_{111} = \frac{1-2\epsilon}{\epsilon(m_1-m_2)^2}(G_{011} - (1+\epsilon(1-\frac{m_2}{m_1}))G_{110}), \quad \frac{\partial}{\partial m_2} G_{111} = \frac{\partial}{\partial m_1} G_{111}\ (m_1 \leftrightarrow m_2, G_{011} \leftrightarrow G_{110})$$

Example of tradition DE method: one-loop triangle (2/2)

$$\frac{\partial}{\partial m_1}G_{111} = \frac{1}{\epsilon^2(m_1-m_2)^2}((-m_2)^{-\epsilon}+(-m_1)^{-\epsilon}(1+\epsilon)-\epsilon m_2(-m_1)^{-1-\epsilon}) =: F[m_1,m_2], \quad \frac{\partial}{\partial m_2}G_{111} = F[m_2,m_1]$$

- Solve by usual subtraction procedure: $F_{\text{sing}}[m_1,m_2] = \frac{-1}{\epsilon m_2}(-m_1)^{-1-\epsilon}$

$$\begin{aligned} G_{111}(m_1,m_2) &= G_{111}(0,m_2) + \int_0^{m_1} F_{\text{sing}}[m_1',m_2] + \int_0^{m_1}(F[m_1',m_2]-F_{\text{sing}}[m_1',m_2]) \\ &= G_{111}(0,m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \int_0^{m_1}\Big(\frac{(1-(-m_2)^{-\epsilon})GP(;-m_1')}{\epsilon^2(m_2-m_1')^2} - \frac{(m_2-m_1')GP(;-m_1')+m_2GP(0;-m_1')}{\epsilon m_2(m_2-m_1')^2} + \mathcal{O}(\epsilon^0)\Big) \\ &= G_{111}(0,m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \Big(\frac{m_1(1-(-m_2)^{-\epsilon})}{\epsilon^2 m_2(m_1-m_2)} + \frac{m_1 GP(0;-m_1)}{\epsilon m_2(m_2-m_1)}\Big) + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Boundary condition follows by plugging in above solution in $\frac{\partial}{\partial m_2}G_{111} = F[m_2,m_1]$

$$\frac{\partial}{\partial m_2}G_{111}(0,m_2) = \frac{(1+\epsilon)}{\epsilon^2}(-m_2)^{-2-\epsilon} \to G_{111}(0,m_2) = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2} + \underbrace{G_{111}(0,0)}_{\text{scaleless}=0} = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2}$$

- Agrees with exact solution: $G_{111} = \frac{c_\Gamma(\epsilon)}{\epsilon^2}\frac{(-m_1)^{-\epsilon}-(-m_2)^{-\epsilon}}{m_1-m_2} = \frac{c_\Gamma(\epsilon)}{m_1-m_2}\Big(-\frac{1}{\epsilon}\log(\frac{-m_1}{-m_2}) + \mathcal{O}(\epsilon^0)\Big)$

Open questions

- Is there a way to pre-empt the choice of x-parametrization without having to calculate the DE?
- Why are the boundary conditions (almost always) naturally taken into account?
- How do the DE in the x-parametrization method relate exactly to those in the traditional DE method?
- How to easily extend parameter x to whole real axis and extend the invariants to the *physical region*?