# Elliptic integrals in Feynman diagrams

**Stefan Weinzierl**

Institut für Physik, Universität Mainz

**I**: **Periodic functions and periods**

**II**: **Differential equations**

**III**: **The two-loop sun-rise diagramm**

in collaboration with
L. Adams, Ch. Bogner, S. Müller-Stach and R. Zayadeh

# Periodic functions

Let us consider a non-constant meromorphic function $f$ of a complex variable $z$.

A period $\omega$ of the function $f$ is a constant such that for all $z$:

$$f(z+\omega) = f(z)$$

The set of all periods of $f$ forms a lattice, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice,
- a double lattice.

# Examples of periodic functions

- Singly periodic function: Exponential function

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: Weierstrass's $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z}\},$$

$$\mathrm{Im}(\omega_2/\omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods $\omega_1$ and $\omega_2$.

# Inverse functions

The corresponding inverse functions are in general multivalued functions.

- For the exponential function $x = \exp(z)$ the inverse function is the logarithm

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an elliptic integral

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

# Periods as integrals over algebraic functions

In both examples the periods can be expressed as integrals involving only algebraic functions.

- Period of the exponential function:

$$2\pi i \quad = \quad 2i \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$

- Periods of Weierstrass's $\wp$-function: Assume that $g_2$ and $g_3$ are two given algebraic numbers. Then

$$\omega_1 = 2 \int_{t_1}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad \omega_2 = 2 \int_{t_3}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

  where $t_1$, $t_2$ and $t_3$ are the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$.

# Numerical periods

Kontsevich and Zagier suggested the following generalisation:

A numerical period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Remarks:

- One can replace "rational" with "algebraic".
- The set of all periods is countable.

# Feynman integrals

A Feynman graph with $m$ external lines, $n$ internal lines and $l$ loops corresponds (up to prefactors) in $D$ space-time dimensions to the Feynman integral

$$I_G = \frac{\left(\mu^2\right)^{n-lD/2}}{\Gamma(n-lD/2)} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2+m_j^2)}$$

The momenta flowing through the internal lines can be expressed through the independent loop momenta $k_1$, ..., $k_l$ and the external momenta $p_1$, ..., $p_m$ as

$$q_i = \sum_{j=1}^{l} \lambda_{ij} k_j + \sum_{j=1}^{m} \sigma_{ij} p_j, \qquad \lambda_{ij}, \sigma_{ij} \in \{-1, 0, 1\}.$$

# Feynman parametrisation

The Feynman trick:

$$\prod_{j=1}^{n} \frac{1}{P_j} = \Gamma(n) \int_{x_j \geq 0} d^n x \, \delta(1 - \sum_{j=1}^{n} x_j) \, \frac{1}{\left( \sum_{j=1}^{n} x_j P_j \right)^n}$$

We use this formula with $P_j = -q_j^2 + m_j^2$.
We can write

$$\sum_{j=1}^{n} x_j(-q_j^2 + m_j^2) = -\sum_{r=1}^{l} \sum_{s=1}^{l} k_r M_{rs} k_s + \sum_{r=1}^{l} 2k_r \cdot Q_r + J,$$

where $M$ is a $l \times l$ matrix with scalar entries and $Q$ is a $l$-vector with momenta vectors as entries.

# Feynman integrals

After Feynman parametrisation the integrals over the loop momenta $k_1$, ..., $k_l$ can be done:

$$I_G = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(l+1)D/2}}{\mathcal{F}^{n-lD/2}}, \qquad \mathcal{U} = \det(M),$$

$$\mathcal{F} = \det(M) \left( J + Q M^{-1} Q \right) / \mu^2.$$

The functions $\mathcal{U}$ and $\mathcal{F}$ are called the first and second graph polynomial.

$\mathcal{U}$ is positive definite inside the integration region and positive semi-definite on the boundary.

$\mathcal{F}$ depends on the masses $m_i^2$ and the momenta $(p_{i_1} + ... + p_{i_r})^2$. In the euclidean region $\mathcal{F}$ is also positive definite inside the integration region and positive semi-definite on the boundary.

# Feynman integrals and periods

Laurent expansion in $\varepsilon = (4-D)/2$:

$$I_G = \sum_{j=-2l}^{\infty} c_j \varepsilon^j.$$

Question: What can be said about the coefficients $c_j$ ?

**Theorem**: For rational input data in the euclidean region the coefficients $c_j$ of the Laurent expansion are numerical periods.

(Bogner, S.W., '07)

Next question: Which periods ?

# One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the spinor products and masses times two transcendental functions, whose arguments are again algebraic functions of the spinor products and the masses.

The two transcendental functions are the logarithm and the dilogarithm:

$$\begin{aligned}
\text{Li}_1(x) &= -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \\
\text{Li}_2(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2}
\end{aligned}$$

# Generalisations of the logarithm

Beyond one-loop, at least the following generalisations occur:

Polylogarithms:

$$\text{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

Multiple polylogarithms (Goncharov 1998):

$$\text{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k) = \sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

This is a nested sum:

$$\ldots \sum_{n_j=1}^{n_{j-1}-1} \frac{x_j^{n_j}}{n_j^{m_j}} \sum_{n_{j+1}=1}^{n_j-1} \ldots$$

# Iterated integrals

Define the functions $G$ by

$$G(z_1,...,z_k;y) = \int\limits_0^y \frac{dt_1}{t_1 - z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

Scaling relation:

$$G(z_1,...,z_k;y) = G(xz_1,...,xz_k;xy)$$

Short hand notation:

$$G_{m_1,...,m_k}(z_1,...,z_k;y) = G(\underbrace{0,...,0}_{m_1-1},z_1,...,z_{k-1},\underbrace{0...,0}_{m_k-1},z_k;y)$$

Conversion to multiple polylogarithms:

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1},\frac{1}{x_1 x_2},...,\frac{1}{x_1...x_k};1\right).$$

# Differential equations for Feynman integrals

If it is not feasible to compute the integral directly:

Pick one variable $t$ from the set $s_{jk}$ and $m_i^2$.

1. Find a differential equation for the Feynman integral.

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} I_G(t) = \sum_i q_i(t) I_{G_i}(t)$$

   Inhomogeneous term on the rhs consists of simpler integrals $I_{G_i}$.

   $p_j(t)$, $q_i(t)$ polynomials in $t$.

2. Solve the differential equation.

Kotikov, Remiddi, Gehrmann, Laporta

# Differential equations: The case of multiple polylogarithms

Suppose the differential operator factorises into linear factors:

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} = \left( a_r(t) \frac{d}{dt} + b_r(t) \right) \dots \left( a_2(t) \frac{d}{dt} + b_2(t) \right) \left( a_1(t) \frac{d}{dt} + b_1(t) \right)$$

Iterated first-order differential equation.

Denote homogeneous solution of the $j$-th factor by

$$\psi_j(t) = \exp\left( - \int_0^t ds \frac{b_j(s)}{a_j(s)} \right).$$

Full solution given by iterated integrals

$$I_G(t) = C_1 \psi_1(t) + C_2 \psi_1(t) \int_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} + C_3 \psi_1(t) \int_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} \int_0^{t_1} dt_2 \frac{\psi_3(t_2)}{a_2(t_2)\psi_2(t_2)} + \dots$$

Multiple polylogarithms are of this form.

# Differential equations: Beyond linear factors

Suppose the differential operator

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j}$$

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(t)\frac{d^2}{dt^2} + b_j(t)\frac{d}{dt} + c_j(t)$$

# An example from mathematics: Elliptic integral

The differential operator of the second-order differential equation

$$\left[t\left(1-t^2\right)\frac{d^2}{dt^2}+\left(1-3t^2\right)\frac{d}{dt}-t\right]f(t) \quad = \quad 0$$

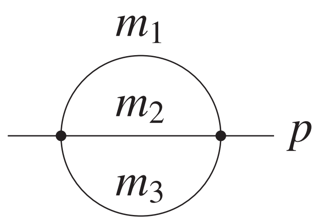
is irreducible.

The solutions of the differential equation are $K(t)$ and $K(\sqrt{1-t^2})$, where $K(t)$ is the complete elliptic integral of the first kind:

$$K(t) \quad = \quad \int_0^1 \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-t^2x^2\right)}}.$$

# An example from physics: The two-loop sunrise integral

$$S\left(p^2, m_1^2, m_2^2, m_3^2\right) \quad = $$




- Two-loop contribution to the self-energy of massive particles.
- Sub-topology for more complicated diagrams.

# The two-loop sunrise integral: Prior art

Integration-by-parts identities allow to derive a coupled system of 4 first-order differential equations for $S$ and $S_1$, $S_2$, $S_3$, where

$$S_i = \frac{\partial}{\partial m_i^2} S$$

(Caffo, Czyz, Laporta, Remiddi, 1998).

This system reduces to a single second-order differential equation in the case of equal masses $m_1 = m_2 = m_3$.

Dimensional recurrence relations relate integrals in $D = 4$ dimensions and $D = 2$ dimensions

(Tarasov, 1996, Baikov, 1997, Lee, 2010).

Analytic result known in the equal mass case, result involves elliptic integrals

(Laporta, Remiddi, 2004).

# The two-loop sunrise integral

Is the system of 4 coupled first-order differential equations generic for the unequal mass case or can we do better ?

Yes, we can !
Also in the unequal mass case there is a single second-order differential equation.

The second-order differential equation follows from algebraic geometry.

(S. Müller-Stach, S.W., R. Zayadeh, arXiv:1112:4360)

# Algebraic geometry

Algebraic geometry studies the zero sets of polynomials.

Example:

$$x_1 x_2 + x_2 x_3 + x_3 x_1 \quad = \quad 0.$$

This is actually an equation in projective space $\mathbb{P}^2$.

Study integrals where polynomials appear in the denominator:

$$\int d^3x \, \delta\left(1 - \sum_{i=1}^{3} x_3\right) \frac{1}{x_1 x_2 + x_2 x_3 + x_3 x_1}$$

What happens in the points $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ ?

# Abstract periods

Input:

- $X$ a smooth algebraic variety of dimension $n$ defined over $\mathbb{Q}$,
- $D \subset X$ a divisor with normal crossings (i.e. a subvariety of dimension $n-1$, which looks locally like a union of coordinate hyperplanes),
- $\omega$ an algebraic differential form on $X$ of degree $n$,
- $\sigma$ a singular $n$-chain on the complex manifold $X(\mathbb{C})$ with boundary on the divisor $D(\mathbb{C})$.

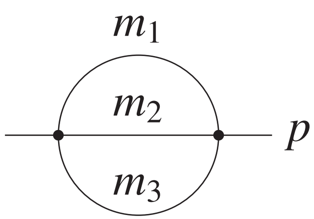
To each quadruple $(X, D, \omega, \sigma)$ associate the period

$$P(X, D, \omega, \sigma) \quad = \quad \int_{\sigma} \omega.$$

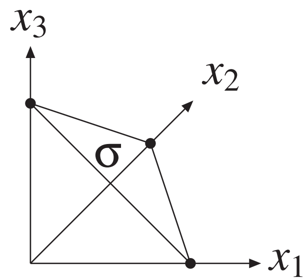
# The two-loop sunrise integral

The two-loop sunrise integral with unequal masses in two-dimensions ($t = p^2$):

$$S(t) \quad =$$




$$= \int_\sigma \frac{\omega}{\mathcal{F}},$$




$$\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2,$$

$$\mathcal{F} = -x_1 x_2 x_3 t + \left( x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2 \right) \left( x_1 x_2 + x_2 x_3 + x_3 x_1 \right)$$

Algebraic geometry studies the zero sets of polynomials.

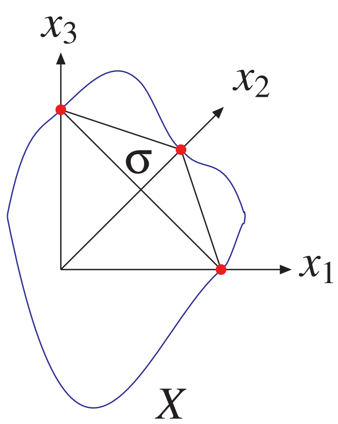
In this case look at the set $\mathcal{F} = 0$.

# The two-loop sunrise integral

From the point of view of algebraic geometry there are two objects of interest:

- the domain of integration $\sigma$,
- the zero set $X$ of $\mathcal{F} = 0$.

$X$ and $\sigma$ intersect at three points:



# The motive

$P$: Blow-up of $\mathbb{P}^2$ in the three points, where $X$ intersects $\sigma$.

$Y$: Strict transform of the zero set $X$ of $\mathcal{F} = 0$.

$B$: Total transform of $\{x_1 x_2 x_3 = 0\}$.

Mixed Hodge structure:

$$H^2(P \backslash Y, B \backslash B \cap Y)$$

(S. Bloch, H. Esnault, D. Kreimer, 2006)

We need to analyse $H^2(P \backslash Y, B \backslash B \cap Y)$.
We can show that essential information is given by $H^1(X)$.

# The elliptic curve

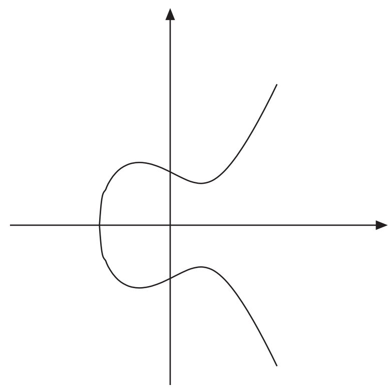
Algebraic variety $X$ defined by the polynomial in the denominator:

$$-x_1x_2x_3t + \left(x_1m_1^2 + x_2m_2^2 + x_3m_3^2\right)\left(x_1x_2 + x_2x_3 + x_3x_1\right) = 0.$$

This defines an elliptic curve.

Change of coordinates $\rightarrow$ Weierstrass normal form

$$y^2z - x^3 - a_2(t)xz^2 - a_3(t)z^3 = 0.$$



$y^2 = x^3 - x + 1$

In the chart $z = 1$ this reduces to

$$y^2 - x^3 - a_2(t)x - a_3(t) = 0.$$

The curve varies with $t$.

# The elliptic curve

In the Weierstrass normal form $H^1(X)$ is generated by

$$\eta = \frac{dx}{y} \quad \text{and} \quad \dot{\eta} = \frac{d}{dt}\eta.$$

$\ddot{\eta} = \frac{d^2}{dt^2}\eta$ must be a linear combination of $\eta$ and $\dot{\eta}$:

$$p_0(t)\ddot{\eta} + p_1(t)\dot{\eta} + p_2(t)\eta \quad = \quad 0.$$

Picard-Fuchs operator:

$$L^{(2)} \quad = \quad p_0(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_2(t)$$

# The second-order differential equation

We can show that applying the Picard-Fuchs operator to the integrand gives an exact form:

$$L^{(2)}\left(\frac{\omega}{\mathcal{F}}\right) = d\beta$$

Integrating over $\sigma$ and using Stokes yields (integration of $\beta$ over $\partial\sigma$ is elementary):

$$L^{(2)}S(t) = \int_{\sigma} d\beta = \int_{\partial\sigma} \beta = p_3(t)$$

Differential equation:

$$\left[p_0(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_2(t)\right] S(t) = p_3(t)$$

$p_0$, $p_1$, $p_2$ and $p_3$ are polynomials in $t$.

# Solution of the second-order differential equation for the sunrise graph

$$\left[ p_0(t) \frac{d^2}{dt^2} + p_1(t) \frac{d}{dt} + p_2(t) \right] S(t) = p_3(t)$$

Let $\psi_1(t)$ and $\psi_2(t)$ be solutions of the corresponding homogeneous equation

$$\left[ p_0(t) \frac{d^2}{dt^2} + p_1(t) \frac{d}{dt} + p_2(t) \right] \psi_i(t) = 0$$

Variation of the constants:

$$S(t) = C_1 \psi_1(t) + C_2 \psi_2(t) + \int\limits_0^t dt_1 \frac{p_3(t_1)}{p_0(t_1) W(t_1)} \left[ -\psi_1(t) \psi_2(t_1) + \psi_2(t) \psi_1(t_1) \right]$$

$W(t)$: Wronski determinant

# Periods of an elliptic curve

Consider the elliptic curve

$$y^2 = (x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

with

$$x_1 = \frac{(m_1-m_2)^2}{\mu^2}, \quad x_2 = \frac{\left(m_3-\sqrt{t}\right)^2}{\mu^2}, \quad x_3 = \frac{\left(m_3+\sqrt{t}\right)^2}{\mu^2}, \quad x_4 = \frac{(m_1+m_2)^2}{\mu^2}.$$

Bauberger, Böhm, Berends, Buza, '94

Holomorphic one-form is $\frac{dx}{y}$, associated periods are

$$\psi_1(t) = 2\int_{x_2}^{x_3} \frac{dx}{y}, \qquad \psi_2(t) = 2\int_{x_4}^{x_3} \frac{dx}{y}.$$

These periods are the solutions of the homogeneous differential equation.

L.Adams, Ch. Bogner, S.W., arXiv:1302.7004

# The homogeneous solutions

$$\psi_1(t) = \frac{4}{\sqrt{X_3(t)}} K(k(t)), \qquad \psi_2(t) = \frac{4i}{\sqrt{X_3(t)}} K(k'(t)).$$

Elliptic integral of the first kind:

$$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}.$$

The modulus $k(t)$ and the complementary modulus $k'(t)$ are defined by

$$k(t) = \sqrt{\frac{X_1(t)}{X_3(t)}}, \qquad k'(t) = \sqrt{\frac{X_2(t)}{X_3(t)}}.$$

$$X_1(t) = 16 m_1 m_2 m_3 \sqrt{t}/\mu^4,$$

$$X_2(t) = \left(\mu_1 - \sqrt{t}\right)\left(\mu_2 - \sqrt{t}\right)\left(\mu_3 - \sqrt{t}\right)\left(\mu_4 + \sqrt{t}\right)/\mu^4,$$

$$X_3(t) = \left(\mu_1 + \sqrt{t}\right)\left(\mu_2 + \sqrt{t}\right)\left(\mu_3 + \sqrt{t}\right)\left(\mu_4 - \sqrt{t}\right)/\mu^4.$$

$\mu_1$, $\mu_2$, $\mu_3$ pseudo-thresholds, $\mu_4$ threshold.

# The full result

$$S(t) = \frac{1}{\pi}\left[\mathrm{Cl}_2(\alpha_1) + \mathrm{Cl}_2(\alpha_2) + \mathrm{Cl}_2(\alpha_3)\right]\psi_1(t)$$
$$+\frac{1}{i\pi\mu^2}\int_0^t dt_1 \left\{\eta_1(t_1) - \frac{b_1(m_1,m_2,m_3)t_1 - b_0(m_1,m_2,m_3)}{3\mu^4 X_2(t_1)}\left[\eta_2(t_1) - \eta_1(t_1)\right]\right\},$$

$$b_i(m_1,m_2,m_3) = d_i(m_1,m_2,m_3)\ln\frac{m_1^2}{\mu^2} + d_i(m_2,m_3,m_1)\ln\frac{m_2^2}{\mu^2} + d_i(m_3,m_1,m_2)\ln\frac{m_3^2}{\mu^2},$$

$$d_1(m_1,m_2,m_3) = 2m_1^2 - m_2^2 - m_3^2, \qquad d_0(m_1,m_2,m_3) = 2m_1^4 - m_2^4 - m_3^4 - m_1^2m_2^2 - m_1^2m_3^2 + 2m_2^2m_3^2,$$

$$\eta_1(t_1) = \psi_2(t)\psi_1(t_1) - \psi_1(t)\psi_2(t_1), \qquad \eta_2(t_1) = \psi_2(t)\phi_1(t_1) - \psi_1(t)\phi_2(t_1),$$

$$\psi_1(t) = \frac{4}{\sqrt{X_3(t)}}K(k(t)), \qquad \psi_2(t) = \frac{4i}{\sqrt{X_3(t)}}K(k'(t)),$$

$$\phi_1(t) = \frac{4}{\sqrt{X_3(t)}}\left[K(k(t)) - E(k(t))\right], \qquad \phi_2(t) = \frac{4i}{\sqrt{X_3(t)}}E(k'(t)).$$

# The equal mass case

- In the equal mass case the result reduces to the previously known one.

  Laporta, Remiddi, '04

- The equal mass result can be expressed in terms of elliptic polylogarithms.

  Bloch, Vanhove, '13

# Summary

Feynman integrals beyond multiple polylogarithms:

- Algebraic geometry, periods,
- Differential equations
- Elliptic integrals