Effective methods in Differential Galois Theory and Applications in Handling (Linear) Differential Equations

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I. Apetizer - 1,2,3 examples

I. Apetizer - 1 : Closed Form Solutions

A hypergeometric equation with diedral differential Galois group :

$$L(y) := \partial^2(y) + \frac{x}{x^2-1}\partial(y) - \frac{1}{4n^2(x^2-1)}y = 0, \quad \partial = \frac{d}{dx}$$

`Closed form solutions ?`

yes : $y = \exp\left(\pm \int \frac{1}{2n} \frac{1}{\sqrt{x^2-1}} dx\right)$
Note : differential equation for $f := y^2$:

$$Sym^2(L) \quad : \quad \partial^3(f) + 3\,\frac{x}{x^2-1}\partial^2(f) + \frac{(-1+n^2)}{n^2\,(x^2-1)}\partial(f) = 0$$

Rational solution : $f = 1$ .... from which we deduce y :
this is given by the `Kovacic algorithm` .

In fact, $y = \left(x + \sqrt{x^2-1}\right)^{\frac{1}{2n}}$: algebraic `when` n `is rational.`

I. Apetizer - 2 : Bessel equations

$$L(y) := \partial^2(y) - \frac{(\sigma+1-\beta)}{\beta\, x}\partial(y) + \frac{\sigma\,(-\rho+x+1)}{x^2\beta^2} y = 0$$

Solutions computed by Maple :

$$y(x) = \_C1\, x^{\frac{\sigma+1}{2\beta}} \mathrm{BesselJ}\left(\sqrt{\frac{\sigma^2-2\sigma+1+4\sigma\rho}{\beta^2}}, 2\frac{\sqrt{\sigma}\sqrt{x}}{\beta}\right) + \_C2\, x^{\frac{\sigma+1}{2\beta}} \mathrm{BesselY}\left(\sqrt{\frac{\sigma^2-2\sigma+1+4\sigma\rho}{\beta^2}}, 2\frac{\sqrt{\sigma}\sqrt{x}}{\beta}\right)$$

`No algebraic relations` between solutions and their derivatives in general. `The differential Galois group` is (projectively) $SL(2,\mathbb{C})$.

`No rational first integral` , except when parameters satisfy ... you will soon know what.

I. 3. Is the Lorenz system rationally integrable ?

[Canalis-Durand, Ramis, Rouchon, and J.A.W, 2001]

$$\begin{cases} \frac{dx}{dt} &= -\sigma(x-y) \\ \frac{dy}{dt} &= \rho x - y - xz \\ \frac{dz}{dt} &= -\beta z + xy \end{cases}$$

Can one find a rational first integral ?

Particular solution $x_0 = y_0 = 0, \quad z_0 = exp(-\beta t)$

Variational System : ϵ-perturbation (ϵ "ideally small")

$X = X_0 + \epsilon X_1, Y = Y_0 + \epsilon Y_1, Z = Z_0 + \epsilon Z_1$ then

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - e^{-\beta t} & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

I. 3.bis. when is the Lorenz system rationally integrable ?

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - e^{-\beta\, t} & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

differential equation for x_1 ? very easy : differentiate !

$$\partial^2 + (\sigma + 1)\,\partial - \sigma\,\rho + \sigma\, e^{-\beta\, t} + \sigma$$

Now set $x = e^{-\beta\, t}$ and $\partial = d/dx$:

$$\partial^2 - \frac{(\sigma + 1 - \beta)\,\partial}{\beta\, x} + \frac{\sigma\,(-\rho + x + 1)}{x^2 \beta^2}$$

so x_1 satisfies the Bessel differential equation !

The Lorenz model is generically not rationally integrable .

Exercise : for which values of the parameters could this Lorenz system be actually rationally integrable ? ?

Outline

II. Elementary and Fundamental Algorithms

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0, \quad a_i \in C(x)$$

1. Solutions in power series.
2. Polynomial solutions
3. Rational solutions
4. View the above as "gluing" local solutions to global.
5. More gluing : exponential solutions.

Solutions in power series

A nice example :

$$L(y) := 56\,y(x) - 2\,x\frac{d}{dx}y(x) + \left(1 - x^2\right)\frac{d^2}{dx^2}y(x) = 0$$

`Basic class of solutions` : formal power series.
$L(\sum_{N=0}^{\infty} u(N)x^N) = 0$ induces a `linear recursion` :

$$-(N+8)(N-7)\,u(N) + (N+1)(N+2)\,u(N+2) = 0$$

We obtain local solutions :

$$-1 + 28\,x^2 - \frac{350}{3}x^4 + O\left(x^6\right) \quad , \quad x - 9\,x^3 + \frac{99}{5}x^5 + O\left(x^6\right)$$

`Tool` : differential equations $\leftrightarrow$ recursions - and linear algebra.
This is really the basic object : *`fine algorithms are crucial`*

Polynomial solutions

$$L(y) := 56\,y\,(x) - 2\,x\frac{d}{dx}y\,(x) + \left(1 - x^2\right)\frac{d^2}{dx^2}y\,(x) = 0$$

Polynomial solution $P = \sum_{i=0}^{d} u(i)x^i$?

$$-\,(N+8)\,(N-7)\,u\,(N) + (N+1)\,(N+2)\,u\,(N+2) = 0$$

Polynomial : power series where all terms are zero after degree d..

Read the recursion for $N = d$: must have $d = 7$

Compute $u(i)$ from recursion *and* check $u(8) = u(9) = 0$.

Result : $x - 9\,x^3 + \frac{99}{5}\,x^5 - \frac{429}{35}\,x^7$

Fast algorithm : Bostan, Cluzeau and Salvy 2004 (& refs therein)

Rational Solutions

$$M(y) := -2\,\frac{\left(22\,x^2-212\,x+451\right)}{(x-4)^2(x-1)(x+1)}y + 2\,\frac{(2\,x+1)(2\,x-3)}{(x-4)(x-1)(x+1)}y' + y''$$

Singularities : $-1, 1, 4, \infty$

Rational solution must be $f(x) = \frac{P(x)}{(x-4)^{\alpha}(x-1)^{\beta}(x+1)^{\gamma}}$.

Pole order α at $x = 4$? let $T = x - 4$, and compute $L(T^{\alpha})$:

$$(\alpha+3)\,(\alpha+2)\,T^{-2} + \left(\frac{8}{15}\,\alpha + \frac{8}{5}\right)T^{-1} + O\,(1)$$

So we must have $\alpha = -3$ or $\alpha = -2$.

-3 and -2 are called the exponents of L at $x = 4$.

Exercise : show that $\beta = \gamma = 0$.

$$M\left(\frac{P(x)}{(x-4)^3}\right) = .. = L(P)$$

so rational solution : $\frac{x-9\,x^3+\frac{99}{5}\,x^5-\frac{429}{35}\,x^7}{(x-4)^3}$

Families of linear differential equations depending on param

$$\nu\,(\nu+1)\,y(x) - 2\,x\frac{d}{dx}y(x) + \left(1-x^2\right)\frac{d^2}{dx^2}y(x)$$

For which values of ν can we have a polynomial solution ?

$$-\,(N+1+\nu)\,(N-\nu)\,u(N) + (N+1)\,(N+2)\,u(N+2)$$

The above methods show that ν must be an integer.
But ... parameter ν is also the number of terms that we have to compute in a power series to decide whether this has a polynomial solution...
Fact : this problem is *undecidable* in full generality – but can often be decided, with intelligence (or good collaborators)
Note - the above polynomials are the Legendre polynomials .
Ref : PhD Delphine Boucher, Limoges, 2000

Local (formal) solutions

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0, \quad a_i \in C(x)$$

Singularities of L : poles of the a_i (and maybe ∞).

Ordinary points

At a non-singular point x_0 : basis of solutions in $C[[x_0]]$ (Cauchy).

Singular points

At a singular points, we will need more ingredients :

Regular Singular Point :

basis of local solutions of the form $(x - x_0)^{\alpha}\hat{\phi}$, with
$\phi \in C[[x - x_0]]$ or $\phi \in C[[x - x_0]][\log(x)]$.
In this case : α is called an exponent of L at x_0.

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0, \quad a_i \in C(x)$$

Regular Singular Point :

basis of local solutions of the form $(x - x_0)^{\alpha}\hat{\phi}$, with
$\phi \in C[[x - x_0]]$ or $\phi \in C[[x - x_0]][\log(x)]$.
In this case : α is called an exponent of L at x_0.

Irregular Singular Point :

Need to replace $(x - x_0)^{\alpha}$ by $\exp(\int \frac{e_{x_0}}{x - x_0} dx)$
the $e_{x_0} \in \frac{1}{(x-x_0)^{\frac{1}{r}}}[\frac{1}{(x-x_0)^{\frac{1}{r}}}]$ are generalized exponents and r is the *ramification index* at the singularity

Practical computation :

algorithmic, technicalities remain for parametrized cases.

Local solutions are the founding object of all further algorithms.

Exponential solutions

Def : y is an exponential solution of $L(y) = 0$ if $r := y'/y \in \overline{\mathbb{Q}}(x)$: $y = \exp(\int r)$.
Lemma : Let y exponential solution, $y'/y = r \in \overline{\mathbb{Q}}(x)$. For all singular points x_i, there exists $e_{x_i} \in \overline{\mathbb{Q}}[t_i^{-1}]$ such that $r = S + \frac{Q'}{Q}$ where

$$S = \sum \frac{e_{x_i}}{t_i} - t_\infty e_\infty^*, \qquad Q \in \overline{\mathbb{Q}}[x]$$

and

$$(\text{Fuchs' relation}) : \deg(Q) + \sum \mathrm{Const}(e_{x_i}) = 0$$

"Beke" Algorithm : for each combination of the e_{x_i}, check if there is a polynomial Q such that the above holds (linear algebra).
Drawbacks : exponential number of combinations to be checked, complicated definition field for the chosen combination.
Best : Cluzeau et van Hoeij, 04 : reduce mod p to decide the "good" combinations !

III. Differential Galois Group

Let k be a `differential field` :
$k = C(x), C((x)), C(x, exp(x)), C(x, \sqrt{x}, exp(\sqrt{x}), \ldots.$

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0$$

All that follows extends *mutatis mutandis* to linear differential systems

$$Y' = A(x)Y, \quad A \in \mathcal{M}_n(k)$$

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0$$

Definition

A differential field extension $K \supset k$ is called a *Picard-Vessiot extension* of k (for $L(y) = 0$) if

1. $K = k(y_1, y_1', \ldots, y_i^{(j)}, \ldots, y_n^{(n-1)})$, where the y_i are a basis of solutions of $L(y) = 0$ (i.e K is the differential field generated [1] by the solutions of L.
2. K and k have the same field of constants.

Construction : assume 0 not singular. Pick series solutions $y_1, \ldots, y_n$. Let I ideal of polynomials in $k[X_{i,j}]$ s.t $P(y_i^{(j)}) = 0$. Then $k[X_{i,j}]/I$ is Picard-Vessiot Ring.

I is the ideal of relations

$V(L) := Sol_K(L)$ is a C-vector space of dimension n.

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1. Note that as $L(y_i) = 0$, we have $y_i^{(n)}$ and the higher derivatives in K, which really makes it a differential field

L a differential operator, $K = PV(L)$ Picard-Vessiot Extension.
$V(L) = Span(y_1, \ldots, y_n)$.

Definition

We call a *differential k-automorphism* of K an automorphism g of K which leaves k fixed et which commutes with the derivation,i.e :

1. $\forall y \in K,\ g(y)' = g(y')$
2. $\forall y \in k,\ g(y) = y$

The *differential Galois group* $G = Gal(L) = Gal_\partial(K/k)$ is the group of differential k-automorphisms of K.

$$g \in Gal(L), \quad L(y) = 0 \quad \longrightarrow L(g(y)) = 0 \quad \Rightarrow g(y_i) = \sum_j c_{i,j} y_j$$

Essential Property 1 :

Faithful representation of $Gal(L)$ as a group of matrices

$Gal(L)$ is a linear algebraic group

Exercise : the Galois group of a logarithm

Let $\mathcal{L} := \log(x)$ in a Picard-Vessiot extension $K \supset C(x)$.
Let $g \in Gal(K/C(x))$.

$g(\mathcal{L})$?

$g(\mathcal{L})' = g(\mathcal{L}') = g(1/x) = 1/x = \mathcal{L}'$ so there exists a constant $c_g \in C$ s.t $g(\mathcal{L}) = \mathcal{L} + c_g$

$$y'' + \frac{1}{x}y' = 0.$$

Solutions are 1 and $\mathcal{L}$. The Galois group is :

$$G = \left\{ \begin{pmatrix} 1 & c_g \\ 0 & 1 \end{pmatrix}, \quad c_g \in C \right\}$$

`Compare with the monodromy :` $\log(xe^{2i\pi}) = \log(x) + 2i\pi$

Some classical examples of linear algebraic groups.

1. $GL(n, C)$ and $SL(n, C)$ (defined by $\det(g) = 1$).
2. The group of upper triangular matrices T (defined by $T_{i,j} = 0$ for $j < i$).
3. Let I_n denote the identity matrix of size n and the standard symplectic matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

   The set of matrices M that satisfy ${}^tM.J.M = J$ (this relation induces a finite set of polynomial relations on the entries of M) is called the *Symplectic* group $Sp(2n, C)$ and will be central in the applications to symplectic mechanics.
4. Any finite group of matrices (check this !)

G is an equidimensional variety ;
G° : component containing the identity
Tangent plane at the identity : Lie algebra $\mathfrak{g}$

IV. Normality : What the Galois Group Measures

Essential Property 2 :

Theorem (Galois normality)

Let K denote a Picard-Vessiot extension of k, let G be its differential Galois group, and let $z \in K$. Then :

$$z \in k \iff \forall g \in G,\ g(z) = z.$$

Normality : caracterizing algebraic elements

Normality : $z \in k \iff \forall g \in G,\ g(z) = z.$

Theorem

Let $z \in K$. Then z is algebraic of degree m over k if and only if $\mathrm{Orb}_G(z)$ has exactly m elements.
All solutions of $L(y) = 0$ are algebraic if and only if G is a finite group.

Sketch : z algebraic iff $P = \prod_{g \in G} (Y - g(z))$ has coefficients in k.
Generalization : The dimension of $\mathrm{Orb}_G(z)$ mesures the differential order of z (Katz).

Normality : caracterizing exponential elements

Normality : $z \in k \iff \forall g \in G,\ g(z) = z.$

Theorem

An non-zero element z of K is exponential over k if and only if, for all $g \in G$, there exists a constant $c_g \in C$ such that $g(y) = c_g.y$.

V. Order 2 Equations : the Kovacic Algorithms

$$L(y) := y'' + a_1(x)y' + a_0(x)y = 0$$

Closed form solutions ? Galois group ?

Assume that $\exists f \in k, a_1 = f'/f$, so that $Gal(L) \subset SL(2, C)$.

Kovacic 77-86 ;
Baldassari & Dwork 79
Singer 81
Duval & Loday-Richaud 91
Ulmer & Singer 93 ;
Ulmer & Weil 95 ; Fakler 97 ;
Berkenbosch & van Hoeij & Weil 02-05
19th century : Klein, Fuchs, Pepin, Vessiot, Marotte, etc.

Case 1 : Reducible case

Definition

Let G be a linear group acting on a vector space. We say that (the action of) G is *reducible* if there exists a non-trivial subspace $W \subset V$ such that $G(W) \subset W$.

In our case : subspace of dimension 1 $\leftrightarrow$ exponential solution.

Case 2 : Imprimitive case (definition)

Definition

Let G be an irreducible group acting on a vector space V. We say that G is *imprimitive* if there exist subspaces V_i such that $V = V_1 \oplus \ldots \oplus V_r$ and G permutes transitively the V_i :

$$\forall i = 1, \ldots, r \; \forall g \in G, \qquad \exists j \in \{1, \ldots, r\} : \; g(V_i) = V_j.$$

In our case, we must have $r = 2$ and $\dim(V_1) = \dim(V_2) = 1$. The matrices have the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \quad \text{with } \; a, b \in C^*.$$

Case 2 : Imprimitive case (algorithm)

Lemma

Assume that G is irreducible. Then :
G is imprimitive ..

`if and only if`

the Riccati equation has an algebraic solution of degree 2

`if and only if`

G has a semi-invariant of degree 2.

How we detect this situation :
semi-invariant of degree 2 $\leftrightarrow$ exponential (radical) solution of $Sym^2(L)$.
Let's get back to the hypergeometric example of slide 3..

Case 3 : Primitive case

`Remaining possibilities :`

3 exceptional `finite groups` (tetraedral, octaedral, icosaedral), caracterized by their *invariants* : see notes

OR

The Galois group is $SL(2, C)$: no closed form solution. Let's get back to the Bessel example of slide 4 ..

VI. *D*-finiteness, Chevalley approach and Tannakian Corresp

Underlying philosophy to what we have seen :
Assume we want to study some algebraic property $\mathcal{P}$ of solutions

1. Transform it into a `group` property
2. Transform this to a `representation` property
3. `Chevalley` : a representation property is characterized by the fact that a line is fixed under the group in some tensorial construction $\mathcal{C}(V(L))$
4. Associate to $\mathcal{C}(V(L)$ a linear differential equation $\mathcal{C}(L)$: `Tannakian` equivalence
5. $\longrightarrow$ `rational solutions of tensor constructions`
   `Example : factoring`

To Summarize

1. Basic Algorithms : local analysis (power series) and global recombining (polynomials, rational functions, exponentials)
2. A differential polynomial in D-finite objects is D-finite.
3. `Galois = classifying object : you do not see the Galois group but you see its effects`
   Relate "Algebraic relations among solutions" to "rational (exponential) solutions in tensor constructions".
4. Available in Maple
   - DETOOLS package
   - Packages from the Limoges C.A group (rational solutions, tensor constructions, local solutions) : see my web page `http://unil.im/jaw`