HUMBOLDT-UNIVERSITÄT ZU BERLIN

On hyperlogarithms and Feynman integrals with divergences and many scales

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Plan of the talk

Abstract

Hyperlogarithms provide a tool to carry out numerous Feynman integrals. So far, this method has been applied successfully to finite single-scale processes. However, it can be used in more general situations.
We comment on calculations of Feynman diagrams with non-trivial kinematics and further show how divergent integrals become amenable to this algorithm in dimensional regularization.

1. brief overview of the hyperlogarithm method by Francis Brown [6, 5]

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2. single-scale results [8, 2]

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2. single-scale results [8, 2]
3. examples with non-trivial kinematic dependence
4. divergences and regularization

Feynman integrals in Schwinger parameters

After replacing propagators $P_e^{-a_e} = \Gamma^{-1}(a_e) \int_0^\infty \alpha_e^{a_e-1} e^{-\alpha_e P_e} \mathrm{d}\alpha_e$ and Gauß'ian momentum integrals, Feynman integrals $\Phi(G)$ take the form

$$\Phi(G) = \left[\prod_e \int_0^\infty \frac{\alpha_e^{a_e-1} \mathrm{d}\alpha_e}{\Gamma(a_e)} \right] \frac{e^{-\frac{\varphi}{\psi}}}{\psi^{D/2}},$$

where the graph polynomials are sums over all spanning trees T and all spanning two-forests F with external momentum $q(T_1)$ entering T_1:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \quad \text{and} \quad \varphi = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e.$$

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$$\Phi(G) = \Gamma(\mathsf{sdd}) \cdot \left[\prod_e \int_0^\infty \frac{\alpha_e^{a_e - 1} \mathrm{d}\alpha_e}{\Gamma(a_e)} \right] \psi^{\mathsf{sdd} - D/2} \varphi^{-\mathsf{sdd}} \cdot \delta(H),$$

where the graph polynomials are sums over all spanning trees T and all spanning two-forests F with external momentum $q(T_1)$ entering T_1:

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$H = 1 - \sum_e \alpha_e \lambda_e$ denotes any hyperplane with $\lambda_e \geq 0$ not all zero. We choose an order $\alpha_1, \ldots, \alpha_N$ of integration and set $H = 1 - \alpha_N$.
The superficial degree of divergence is

$$\mathsf{sdd} = \sum_e a_e - \frac{D}{2} |G|,$$

using $|G|$ to denote the number of loops of G.

Hyperlogarithms for Feynman integrals, F. Brown [6, 5]

Definition

For any word $w \in \Sigma^{\times}$ in letters $\{\omega_\sigma\colon\ \sigma \in \Sigma\}$ over a finite set $0 \in \Sigma \subset \mathbb{C}$ define the *hyperlogarithms* by

$$L_{\omega_0^n}(z) := \frac{\log^n z}{n!} \quad \text{for any } n \in \mathbb{N}_0 \text{ and } \quad L_{\omega_\sigma w}(z) := \int_0^z \frac{\mathrm{d}z'}{z'-\sigma} L_w(z').$$

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Remark

These are often named *Goncharov polylogarithms* and written as

$$L_{\omega_{\sigma_1}\ldots\omega_{\sigma_n}}(z) = G\left(\sigma_1, \ldots, \sigma_n; z\right).$$

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Example (Relation to multiple polylogarithms)

For any $0 \neq \sigma \in \Sigma$ and $n \in \mathbb{N}$ note

$$L_{\omega_\sigma}(z) = \log(z-\sigma) - \log(-\sigma) = -\operatorname{Li}_1\left(\frac{z}{\sigma}\right), \quad L_{\omega_0^{n-1}\omega_\sigma}(z) = -\operatorname{Li}_n\left(\frac{z}{\sigma}\right).$$

More generally, for $\sigma_1, \ldots, \sigma_r \in \Sigma \setminus \{0\}$ and $n_1, \ldots, n_r \in \mathbb{N}$ we have

$$L_{\omega_0^{n_r-1}\omega_{\sigma_r}\cdots\omega_0^{n_2-1}\omega_{\sigma_2}\omega_0^{n_1-1}\omega_{\sigma_1}}(z) = (-1)^r \operatorname{Li}_{n_1,\ldots,n_r}\left(\frac{\sigma_2}{\sigma_1}, \ldots, \frac{\sigma_r}{\sigma_{r-1}}, \frac{z}{\sigma_r}\right).$$

1. Algebra: $L_{\cdot} : \mathbb{Q}\langle\Sigma\rangle \longrightarrow \mathcal{O}\left(\mathbb{C} \setminus \Sigma\right)$ (extended linearly) is a morphism of algebras for the shuffle product: $L_u \cdot L_v = L_{u \sqcup\!\sqcup v}$. For example,

$$L_{\omega_\sigma} \cdot L_{\omega_{\tau_1}\omega_{\tau_2}} = L_{\omega_\sigma\omega_{\tau_1}\omega_{\tau_2}} + L_{\omega_{\tau_1}\omega_\sigma\omega_{\tau_2}} + L_{\omega_{\tau_1}\omega_{\tau_2}\omega_\sigma}.$$

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2. Grading: by *weight* $|w| := n$ for all $w = \omega_{\sigma_1} \ldots \omega_{\sigma_n} \in \Sigma^n$

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$$\int_{\gamma\star\eta} \omega_{\sigma_1}\ldots\omega_{\sigma_n} = \sum_{k=0}^{n} \int_\eta \omega_{\sigma_1}\ldots\omega_{\sigma_k} \int_\gamma \omega_{\sigma_{k+1}}\ldots\omega_{\sigma_n} \quad \text{(Chen's lemma)}$$

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$$L_w(z) = \sum_k \log^k(z-\sigma) \cdot f_w^{\sigma,k}(z). \qquad \boxed{\operatorname*{Reg}_{z\to\sigma} L_w(z) := f_w^{\sigma,0}(\sigma)}$$

By construction and use of partial fractioning, every integrand

$$f(z) \in L(\Sigma) := \mathbb{Q}\left[z, \left\{\frac{1}{z-\sigma}: \ \sigma \in \Sigma\right\}, \{L_w: \ w \in \Sigma^{\times}\}\right]$$

has a primitive $F(z)$ in the enlarged algebra

$$R(\Sigma) := \mathbb{Q}\left[\Sigma \cup \left\{\frac{1}{\sigma_i - \sigma_j}: \ \sigma_i \neq \sigma_j \text{ from } \Sigma\right\}\right] \otimes L(\Sigma).$$

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Example

$$\int \frac{L_{\omega_\sigma}(z)\mathrm{d}z}{(z-\tau)^2} = -\frac{L_{\omega_\sigma}(z)}{z-\tau} + \int \frac{\mathrm{d}z}{(z-\tau)(z-\sigma)} = -\frac{L_{\omega_\sigma}(z)}{z-\tau} + \frac{L_{\omega_\tau}(z) - L_{\omega_\sigma}(z)}{\tau - \sigma}$$

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For the following integration, we must extract the dependence of

$$\int_0^\infty f(z)\mathrm{d}z = \operatorname*{Reg}_{z\to\infty} F(z) - \operatorname*{Reg}_{z\to 0} F(0)$$

on the next Schwinger variable.

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Integration

Consider $w = \omega_{\sigma_1} \ldots \omega_{\sigma_n}$ with $\sigma_1, \ldots, \sigma_n \in \mathbb{Q}(t)$ depending rationally on a variable t (the next Schwinger parameter α_e to integrate):

$$\frac{\partial}{\partial t} L_w(z) = \int_0^z \mathrm{d}z_1 \left[\frac{\sigma_1'}{(z_1 - \sigma_1)^2} L_{\omega_{\sigma_2}\ldots}(z_1) + \frac{1}{z_1 - \sigma_1} \frac{\partial}{\partial t} L_{\omega_{\sigma_2}\ldots}(z_1) \right]$$

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&= \frac{-\sigma_1'}{z-\sigma_1} L_{\omega_{\sigma_2}\ldots}(z) + \int_0^z \frac{\mathrm{d} z_1}{z_1-\sigma_1} \left[\frac{\partial}{\partial t} + \sigma_1' \frac{\partial}{\partial z_1} \right] L_{\omega_{\sigma_2}\ldots}(z_1) \\
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\end{aligned}$$

Consider $w = \omega_{\sigma_1} \ldots \omega_{\sigma_n}$ with $\sigma_1, \ldots, \sigma_n \in \mathbb{Q}(t)$ depending rationally on a variable t (the next Schwinger parameter α_e to integrate):

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&= \frac{-\sigma_1'}{z-\sigma_1} L_{\omega_{\sigma_2}\ldots}(z) + \sum_{i=1}^{n-1} \frac{(\sigma_i - \sigma_{i+1})'}{\sigma_i - \sigma_{i+1}} L_{\ldots\not{\omega}_{\sigma_{i+1}}\ldots - \ldots\not{\omega}_{\sigma_i}\ldots}(z) \\
&\quad - \frac{\sigma_n'}{\sigma_n} L_{\ldots\not{\omega}_{\sigma_n}}(z)
\end{aligned}$$

From $\frac{\partial}{\partial t}\operatorname{Reg}_{z\to\infty} L_w(z) = \operatorname{Reg}_{z\to\infty}\frac{\partial}{\partial t} L_w(z)$ we deduce

Proposition

Let $w = \omega_{\sigma_1}\ldots\omega_{\sigma_n}$ where $\sigma_1(t),\ldots\sigma_n(t) \in \mathbb{Q}(t)$. Then

$$\operatorname*{Reg}_{z\to\infty} L_w(z) = L_u(t)$$

for a unique word $u \in \mathbb{Q}\langle\Sigma_t\rangle \otimes \operatorname{Reg}_{t\to 0}\operatorname{Reg}_{z\to\infty} L(\Sigma)$. The alphabet Σ_t contains all zeros and poles of

$$\sigma_i(t) - \sigma_j(t) = c_{i,j}\cdot\prod_{\nu\in\Sigma_t}(t-\nu)^{\lambda_{i,j}^{(\nu)}} \quad \textit{for} \quad \lambda_{i,j}^{(\nu)} \in \mathbb{Z} \quad \textit{and all} \quad \sigma_i,\sigma_j \in \Sigma.$$

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Example ($\operatorname{Reg}_{z\to\infty} L_{\omega_{-t}\omega_{-1}}(z)$)

$$\frac{\partial}{\partial t} \operatorname*{Reg}_{z\to\infty} L_{\omega_{-t}\omega_{-1}}(z) = \left[\frac{\partial}{\partial t}\log(t-1)\right] \cdot \operatorname*{Reg}_{z\to\infty} L_{\omega_{-t}-\omega_{-1}}(z)$$

$$= \frac{1}{t-1} \lim_{z\to\infty} \log\frac{(z+t)}{t(z+1)} = -\frac{1}{t-1} L_{\omega_0}(t)$$

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Example ($\operatorname{Reg}_{z\to\infty} L_{\omega_{-t}\omega_{-1}}(z) = C - L_{\omega_1\omega_0}(t)$)

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$$= \frac{1}{t-1} \lim_{z\to\infty} \log\frac{(z+t)}{t(z+1)} = -\frac{1}{t-1} L_{\omega_0}(t)$$

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Regularized limits

We further need to obtain the constants of integration like

$$C = \operatorname*{Reg}_{t\to 0} \circ \operatorname*{Reg}_{z\to\infty} L_{\omega_{-t}\omega_{-1}}(z) = \operatorname*{Reg}_{z\to\infty} L_{\omega_0\omega_{-1}}(z) = \zeta(2).$$

Proposition

To any $w = \omega_{\sigma_1} \ldots \omega_{\sigma_n}$ *and* $\sigma_1, \ldots, \sigma_n \in \mathbb{Q}(t)$ *exists* $u \in \mathbb{Q}\langle\widetilde{\Sigma}\rangle$ *such that*

$$\operatorname*{Reg}_{t\to 0} \circ \operatorname*{Reg}_{z\to\infty} L_{w(t)}(z) = \operatorname*{Reg}_{z\to\infty} L_u(z).$$

The alphabet $\widetilde{\Sigma}$ *contains the leading terms from* $\sigma_i(t) = t^{\lambda_i} \cdot [\widetilde{\sigma}_i + \mathcal{O}(t)]$*:*

$$\widetilde{\Sigma} = \operatorname*{Reg}_{t\to 0} \Sigma := \{0\} \,\dot{\cup}\, \{\widetilde{\sigma}_i \colon\ 0 \neq \sigma_i \in \Sigma\}.$$

The resulting integration algorithm is completely algebraic, i.e. no numeric evaluations or external input of boundary values are needed.

Linear reducibility

This method applies only if for all $n < N$, the partial integrals

$$f_n := \left[\prod_{e=1}^{n-1} \int_0^\infty \mathrm{d}\alpha_e \right] f_0 \quad \text{lie in} \quad L(\Sigma_n)\,(\alpha_n) \tag{1.1}$$

for some sets $\Sigma_n \subset \mathbb{Q}(\alpha_{n+1}, \ldots, \alpha_N)$. In particular, at the next stage the differences $\sigma_i - \sigma_j$ have to factor linearly in α_{n+1} for all $\sigma_i, \sigma_j \in \Sigma_n$.

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Polynomial reduction

A simple algorithm providing sufficient conditions for linear reducibility is given by Stembridge's procedure as explained in [6]. Much more powerful is the method of compatibility graphs developed in [5].

Single-scale integrals

Massless propagators: Coefficients of ε-expansions

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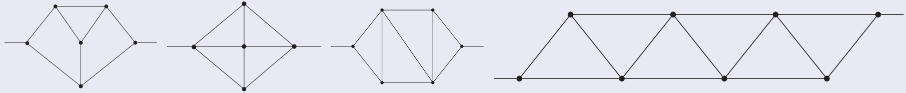
Theorem (positive matrix graphs [5])

All graphs G of vertex-width $\mathrm{vw}(G) \leq 3$ *are linearly reducible and their periods lie in* $\mathcal{Z} := \mathrm{lin}_{\mathbb{Q}} \left\{ \zeta(n_1, \ldots, n_r) \colon\ n_1, \ldots, n_r \in \mathbb{N} \text{ and } n_r > 1 \right\}$.



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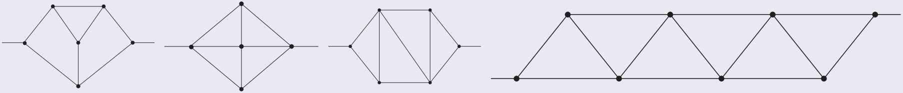
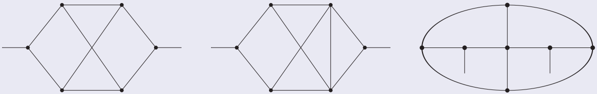


Theorem (vacuum graphs with four or five loops [8])

All massless propagators up to four loops are linearly reducible. Their periods are constrained to alternating Euler sums $\mathcal{Z}_2$:

$$\mathcal{Z}_\mu := \mathrm{lin}_{\mathbb{Q}} \left\{\mathrm{Li}_{n_1,\ldots,n_r}(\sigma_1, \ldots, \sigma_r)\colon\ n_i \in \mathbb{N}, \sigma_i^\mu = 1 \forall i \text{ and } (n_r, \sigma_r) \neq (1,1)\right\}.$$

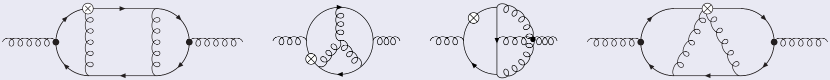


Single-scale integrals

Massive on-shell operator elements

Theorem ([2, 3, 1])

The following on-shell ($p^2 = 0$) propagators with equally massive fermions are linearly reducible:



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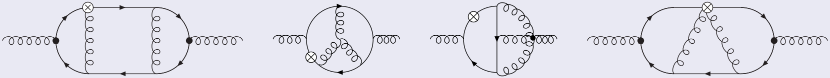
$\hat{I}_4 =$ 

In fact, the calculation used a two-scale setup to get all Mellin moments:

$$\tilde{I}_4(x) := \int \frac{\Omega}{\varphi^2} \sum_{0=j\leq N}^{\infty} x^N \cdot \frac{T_{4a}^{N-j} T_{4b}^{j}}{\psi^N} = \int \frac{\Omega}{(\psi - xT_{4a})(\psi - xT_{4b})M^2},$$

where $T_{4a}, T_{4b}, \psi \in \mathbb{Z}[\alpha_1, \ldots, \alpha_8]$ are cubic polynomials and $M := \sum_{m_e \neq 0} \alpha_e$.

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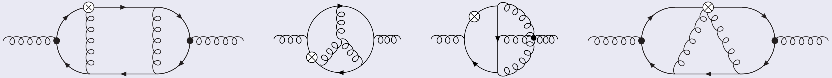
$\hat{I}_4 =$ 

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$$\tilde{I}_4(x) = \left[-\frac{1+x}{x} L_{\omega_{-1}} - \frac{1-2x-x^2}{(1-x)x^3} L_{\omega_0\omega_{-1}} + \cdots \right] \zeta(3) + 3\frac{1+x}{2x^3} L_{\omega_{-1}\omega_0\omega_0\omega_1} + \cdots$$
$$\in L\left(\left\{-1, 0, \tfrac{1}{2}, 1\right\}\right) \otimes \operatorname{lin}_{\mathbb{Q}} \left\{1, \zeta(3)\right\}.$$

Non-trivial kinematics: 2 variables

Massless on-shell four-point graphs

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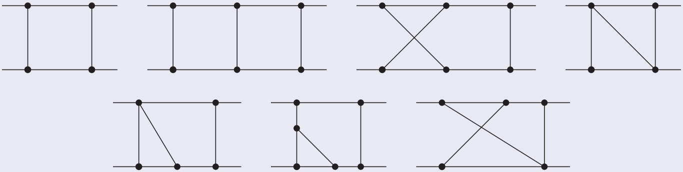
Massless on-shell four-point graphs

Theorem ([4])

All massless four-point on-shell graphs ($p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$) with at most two loops are linearly reducible. In particular these include



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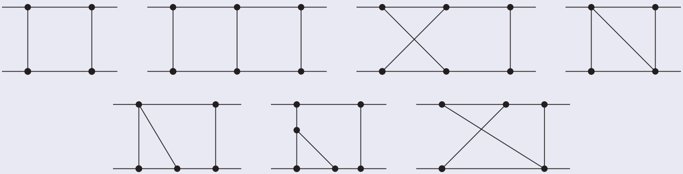
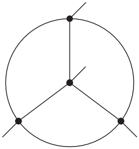


At three loops, there are non-reducible graphs characterized by forbidden minors like



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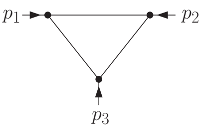
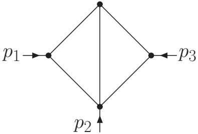
$G_1 :=$ 

and $G_2 :=$ 

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Non-trivial kinematics: 3 variables

Examples: Off-shell massless vertices

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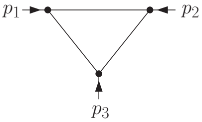
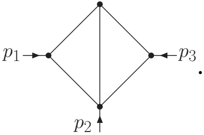
$G_1 :=$ 

and $G_2 :=$ 

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To rationalize the square-root of the Källén function we parametrize as

$$p_2^2 = p_1^2 \cdot z\bar{z} \quad \text{and} \quad p_3^2 = p_1^2 \cdot (1-z)(1-\bar{z}), \quad \text{such that}$$

$$z - \bar{z} = \sqrt{p_1^2 + p_2^2 + p_3^2 - 2p_1p_2 - 2p_1p_3 - 2p_2p_3}.$$

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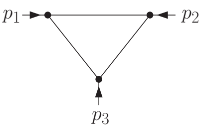
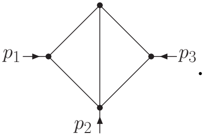
$G_1 :=$ 

and $G_2 :=$ 

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Then the triangle G_1 expands near $D = 4 - 2\varepsilon$ with $a_1 = a_2 = a_3 = 1$ as

$$\Phi(G_1) = \frac{\Gamma(1+\varepsilon)}{z-\bar{z}} \cdot p_1^{-2(1+\varepsilon)} \cdot \sum_{n=0}^{\infty} f_n(z,\bar{z})\varepsilon^n, \quad \text{at leading order}$$

$$f_0 = 4i\Im\left\{\operatorname{Li}_2(z) + \ln|z| \cdot \ln(1-z)\right\} \quad \text{(Bloch-Wigner dilogarithm).}$$

Non-trivial kinematics: 3 variables

Examples: Off-shell massless vertices

Similarly we can compute all f_n, for example the contribution $\propto \varepsilon$ is

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The functions f_n have symbols with letters in $\{z,\bar{z},1-z,1-\bar{z},z-\bar{z}\}$ and are generalized single-valued multiple polylogarithms (SVMP) [9].

Non-trivial kinematics: 3 variables

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The functions f_n have symbols with letters in $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}\}$ and are generalized single-valued multiple polylogarithms (SVMP) [9].
In the same way the double-triangle G_2 can be expanded as

$$\Phi\left(G_2\right) = \frac{\Gamma(1+2\varepsilon)}{z-\bar{z}} \cdot p_1^{-2(1+2\varepsilon)} \cdot \sum_{n=0}^{\infty} f_n(z,\bar{z})\varepsilon^n, \quad \text{at leading order}$$

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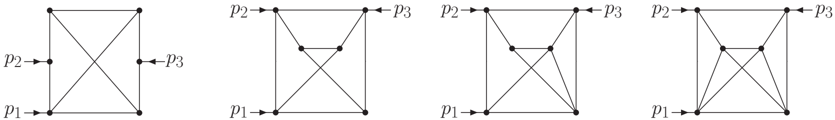
Non-trivial kinematics: 3 variables

Examples: Off-shell massless vertices and conformal four-point integrals

Many massless off-shell vertices are linearly reducible, for example



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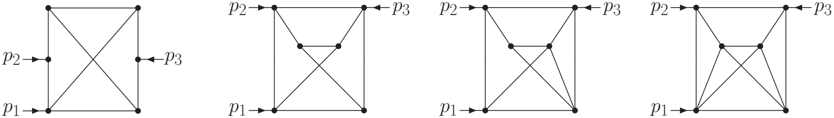
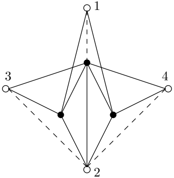
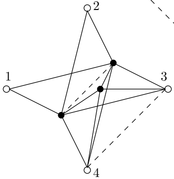


The same functions with $z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $(1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ describe conformal four-point integrals [7]. Many of them are linearly reducible, e.g.



$$E_{12;34} = \text{(graph)} = \frac{x_{23}^2 x_{24}^2}{\pi^6} \int_{\mathbb{R}^{12}} \frac{\mathrm{d}^4 x_5\, \mathrm{d}^4 x_6\, \mathrm{d}^4 x_7 \cdot x_{16}^2}{(x_{15}^2 x_{25}^2 x_{35}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{47}^2)}$$



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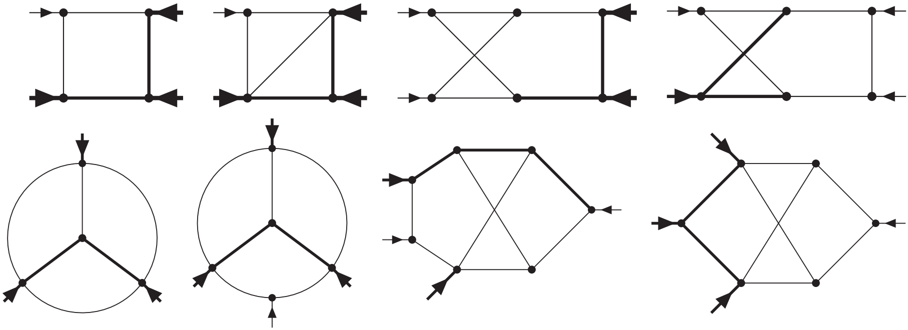
Non-trivial kinematics: up to 7 variables

Examples: massive integrals

Convention:

- thin lines: light-like momenta $p_e^2 = 0$, massless propagators $m_e = 0$
- thick lines: arbitrary masses m_e and external momenta p_e^2 (in particular the masses may be all different)

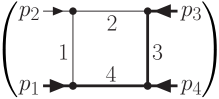
Then the following graphs are linearly reducible:



Non-trivial kinematics: up to 7 variables

Examples: box integral with two masses and three off-shell momenta

The one-loop box with four external momenta and $p_2^2 = m_1 = m_2 = 0$,

Φ 

$$= \Gamma(2+\varepsilon)\cdot m_3^{-4-2\varepsilon} \cdot \sum_{n=-1}^{\infty} f_n\left(m_3, m_4, p_1^2, p_3^2, p_4^2, s, u\right) \varepsilon^n,$$

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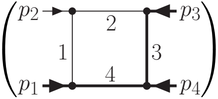
Φ 

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is linearly reducible. For space reasons we give an example only for the much simpler kinematics $p_3^2 = p_4^2 = 0$, $p = p_1^2$, $m = m_3^2 = m_4^2$ in $D = 4 - 2\varepsilon$ dimensions at $a_1 = a_2 = a_3 = a_4 = 1$. Then

$$f_n = \frac{m^2}{pm - us - um - ms}\widetilde{f_n} \quad \text{with} \quad \widetilde{f_{-1}} = \ln\frac{m(u+m)}{(p+m)(s+m)} \quad \text{and}$$

$$\begin{aligned}\widetilde{f_0} = & -2\,\mathrm{Li}_1\left(\tfrac{s}{p}\right)\mathrm{Li}_1\left(-\tfrac{p}{m}\right) + 2\,\mathrm{Li}_1\left(\tfrac{s(u+m)}{m(-u+p)}\right)\left[\mathrm{Li}_1\left(-\tfrac{p}{m}\right) - \mathrm{Li}_1\left(-\tfrac{u}{m}\right)\right] \\ & + \mathrm{Li}_1\left(-\tfrac{u}{m}\right) + 2\,\mathrm{Li}_2\left(-\tfrac{p}{m}\right) - 2\,\mathrm{Li}_{1,1}\left(1, -\tfrac{u}{m}\right) - 2\,\mathrm{Li}_{1,1}\left(1, -\tfrac{s}{m}\right) \\ & - 2\,\mathrm{Li}_2\left(-\tfrac{s}{m}\right) + 2\,\mathrm{Li}_{1,1}\left(-\tfrac{p}{m}, \tfrac{s}{p}\right) - 2\,\mathrm{Li}_{1,1}\left(\tfrac{u-p}{u+m}, \tfrac{s(u+m)}{m(p-u)}\right) \\ & + \mathrm{Li}_1\left(-\tfrac{s}{m}\right) - 2\,\mathrm{Li}_2\left(-\tfrac{u}{m}\right) - \mathrm{Li}_1\left(-\tfrac{p}{m}\right) + 2\,\mathrm{Li}_{1,1}\left(1, -\tfrac{p}{m}\right).\end{aligned}$$

Non-trivial kinematics: up to 7 variables

Examples: box integral with two masses and three off-shell momenta, term $\propto \varepsilon$

$$
\begin{aligned}
\widetilde{f_1} = & -2\,\mathrm{Li}_1\left(\tfrac{s}{p-u}\right)\mathrm{Li}_2\left(-\tfrac{p}{m}\right) + 2\,\mathrm{Li}_1\left(\tfrac{s}{p-u}\right)\mathrm{Li}_2\left(-\tfrac{u}{m}\right) - 2\,\mathrm{Li}_2\left(\tfrac{s(u+m)}{m(p-u)}\right)\mathrm{Li}_1\left(-\tfrac{u}{m}\right) + 2\,\mathrm{Li}_2\left(\tfrac{s(u+m)}{m(p-u)}\right)\mathrm{Li}_1\left(-\tfrac{p}{m}\right) \\
& - 2\,\mathrm{Li}_2\left(\tfrac{s}{p}\right)\mathrm{Li}_1\left(-\tfrac{p}{m}\right) - 2\,\mathrm{Li}_1\left(\tfrac{s}{p}\right)\mathrm{Li}_2\left(-\tfrac{p}{m}\right) + 4\,\mathrm{Li}_1\left(\tfrac{s(u+m)}{m(p-u)}\right)\mathrm{Li}_2\left(-\tfrac{p}{m}\right) - 4\,\mathrm{Li}_1\left(\tfrac{s(u+m)}{m(p-u)}\right)\mathrm{Li}_2\left(-\tfrac{u}{m}\right) \\
& + 2\,\mathrm{Li}_1\left(\tfrac{s(u+m)}{m(p-u)}\right)\mathrm{Li}_1\left(-\tfrac{u}{m}\right) + 2\,\mathrm{Li}_1\left(\tfrac{s}{p}\right)\mathrm{Li}_1\left(-\tfrac{p}{m}\right) - 2\,\mathrm{Li}_1\left(\tfrac{s(u+m)}{m(p-u)}\right)\mathrm{Li}_1\left(-\tfrac{p}{m}\right) + 4\,\mathrm{Li}_{1,2}\left(1, -\tfrac{p}{m}\right) \\
& + 2\,\mathrm{Li}_{1,1,1}\left(-\tfrac{p-u}{u+m}, \tfrac{u+m}{m}, \tfrac{s}{p-u}\right) - 2\,\mathrm{Li}_{1,1,1}\left(-\tfrac{p}{m}, \tfrac{p-u}{p}, \tfrac{s}{p-u}\right) + 2\,\mathrm{Li}_{2,1}\left(-\tfrac{p-u}{m}, \tfrac{s}{p-u}\right) + 2\,\mathrm{Li}_3\left(-\tfrac{p}{m}\right) \\
& - 2\,\mathrm{Li}_3\left(-\tfrac{u}{m}\right) + 4\,\mathrm{Li}_{1,1,1}\left(1, 1, -\tfrac{p}{m}\right) - 4\,\mathrm{Li}_{1,1,1}\left(1, -\tfrac{p-u}{u+m}, \tfrac{s(u+m)}{m(p-u)}\right) + 4\,\mathrm{Li}_{1,1,1}\left(1, -\tfrac{p}{m}, \tfrac{s}{p}\right) + 2\,\mathrm{Li}_{2,1}\left(-\tfrac{p}{m}, \tfrac{s}{p}\right) \\
& - 4\,\mathrm{Li}_{2,1}\left(-\tfrac{p-u}{u+m}, \tfrac{s(u+m)}{m(p-u)}\right) - 2\,\mathrm{Li}_{1,2}\left(-\tfrac{p-u}{u+m}, \tfrac{s(u+m)}{m(p-u)}\right) + 2\,\mathrm{Li}_{2,1}\left(1, -\tfrac{p}{m}\right) - 2\,\mathrm{Li}_3\left(-\tfrac{s}{m}\right) - 4\,\mathrm{Li}_{1,1,1}\left(1, 1, -\tfrac{u}{m}\right) \\
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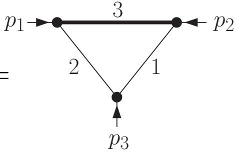
\end{aligned}
$$

Regularization in Schwinger parameters

Example: Vertex with one null momentum and one internal mass

Consider a vertex graph G with one internal mass and $p_3^2 = 0$:

$G =$ 

$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
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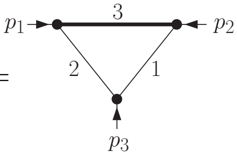
$G =$ 

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It is linearly reducible, but the parametric integral in the representation

$$\Phi(G) = \int \frac{\mathrm{d}^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)^{a_3}(k+p_2)^{2a_1}(k-p_1)^{2a_2}}$$
$$= \frac{\Gamma(\mathrm{sdd})}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int \frac{\alpha_1^{a_1-1}\alpha_2^{a_2-1}\alpha_3^{a_3-1}}{\psi^{D/2-\mathrm{sdd}}\varphi^{\mathrm{sdd}}}\Omega$$

diverges at $\varepsilon \to 0$ for $D = 4 - 2\varepsilon$ and $a_1 = a_2 = a_3 = 1$.

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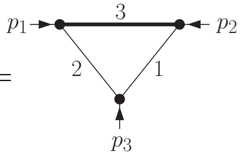
$G =$ 

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diverges at $\varepsilon \to 0$ for $D = 4 - 2\varepsilon$ and $a_1 = a_2 = a_3 = 1$. Then $\mathrm{sdd} = 1 + \varepsilon$ and the integral $\int_0^\infty \mathrm{d}\alpha_3$ divergces at the lower boundary:

$$\int \frac{\Omega}{(\alpha_1 + \alpha_2 + \alpha_3)^{1 - 2\varepsilon} \left[m^2(\alpha_1 + \alpha_2 + \alpha_3) + p_1^2 \alpha_2 + p_2^2 \alpha_1 \right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

Divergences in Schwinger parameters

Definition

Extend the degree of divergence $\operatorname{sdd} = \operatorname{sdd}(E)$ to all subsets $\emptyset \neq J \subseteq E$ as

$$\operatorname{sdd}(J) := \sum_{e \in J} a_e + (\operatorname{sdd} - D/2) \operatorname{ldeg}_J(\psi) - \operatorname{sdd} \cdot \operatorname{ldeg}_J(\varphi).$$

Here $\operatorname{ldeg}_J(P)$ counts the minimum degree w.r.t. edges in J among all non-zero monomials in P. In particular, the original integrand

$$F = \prod_{e \in E} \alpha_e^{a_e - 1} \cdot \psi^{\operatorname{sdd} - D/2} \cdot \varphi^{\operatorname{sdd}} \quad \text{scales under} \quad F_J := F|_{\alpha_e \mapsto \lambda \alpha_e \ \forall e \in J}$$

as $F_J = \lambda^{\operatorname{sdd}(J) - |J|} \cdot \widetilde{F_J}$ for small λ; $\widetilde{F_J}$ is finite at $\lambda \to 0$.

Finiteness

The integral $\int F\Omega$ is absolutely convergent if all coefficients of φ are positive (Euclidean domain) and $\operatorname{sdd}(J) > 0$ for all $\emptyset \neq J \subsetneq E$.

Analytic regularization in Schwinger parameters

A partial integration yields the relation

$$\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda} \lambda^{\mathsf{sdd}(J)} \widetilde{F_J}(\lambda) = \frac{\lambda^{\mathsf{sdd}(J)}}{\mathsf{sdd}(J)} \widetilde{F_J}(\lambda) \bigg|_{\lambda=0}^{\infty} - \frac{1}{\mathsf{sdd}(J)} \int_0^\infty \mathrm{d}\lambda \cdot \lambda^{\mathsf{sdd}(J)} \frac{\partial}{\partial \lambda} \widetilde{F_J}(\lambda)$$

with vanishing boundary contribution when $\mathsf{sdd}(J) > 0$ and $F_J(\lambda)$ falls off at $\lambda \to \infty$ faster than $\lambda^{-\mathsf{sdd}(J)}$.

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Example (Vertex with one mass and $p_3^2 = 0$ in $D = 4 - 2\varepsilon$)

We have $\mathsf{sdd}(\{3\}) = -\varepsilon$ with $\widetilde{F} = \psi^{2\varepsilon-1}\cdot\left[m^2\psi + p_2^2\alpha_1 + p_1^2\alpha_2\right]^{-1-\varepsilon}$:

$$\int \frac{\Omega}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}} = \frac{\alpha_3^{-\varepsilon}\widetilde{F}}{-\varepsilon}\Big|_{\alpha_3=0}^{\infty} + \frac{1}{\varepsilon}\cdot\int\frac{\Omega}{\alpha_3^{\varepsilon}}\frac{\partial}{\partial\alpha_3}\widetilde{F}$$

$$= \frac{1}{\varepsilon}\cdot\int\frac{\Omega\alpha_3^{1+\varepsilon}}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}}\left[\frac{2\varepsilon-1}{\psi} - \frac{(1+\varepsilon)\alpha_3 m^2}{\varphi}\right] \quad \text{when } \varepsilon > 0.$$

Regularization in Schwinger parameters

General construction

Proposition

For any subset $\emptyset \neq J \subsetneq E$*, acting with the differential operator*

$$\mathcal{D}_J := 1 - \frac{1}{\mathrm{sdd}(J)} \sum_{e \in J} \frac{\partial}{\partial \alpha_e} \alpha_e.$$

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Thus finitely many partial integrations suffice to make all $\operatorname{sdd}(J)$ positive, thereforeby yielding a convergent integral representation (suitable for hyperlogarithmic integration).

This procedure does not introduce new singularities, i.e. the polynomial reduction and linear reducibility is not affected.

Summary: Hyperlogarithmic integration

1. If a graph G (with specification of kinematics) is *linearly reducible*, all its periods are in principle computable with hyperlogarithms:
   - any order in ε, expansion near arbitrary even dimension $D|_{\varepsilon=0} \in 2\mathbb{N}$
   - any tensor structures
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2. A convergent representation is needed and can always be constructed in analytic (dimensional) regularization. But for highly divergent integrals, this generates huge expressions and a more economic way of subtracting divergences seems desireable.
3. Sometimes a graph can become linearly reducible by a change of variables (in particular for massive integrals). A systematic study of such transformations on Schwinger parameters is still lacking.
4. No combinatorial understanding of linear reducibility with kinematics is available, but tests on individual graphs are automated.
5. Still, many non-trivial integrals are accessible by this method.

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