Local Unitarity

A representation of differential cross-sections That is locally free of infrared divergences at any order

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Hep-ph:2010.01068

In coll. with:
Z. Capatti, A. Pelloni, B. Ruijl

Humboldt Theory Seminar

17th November 2020

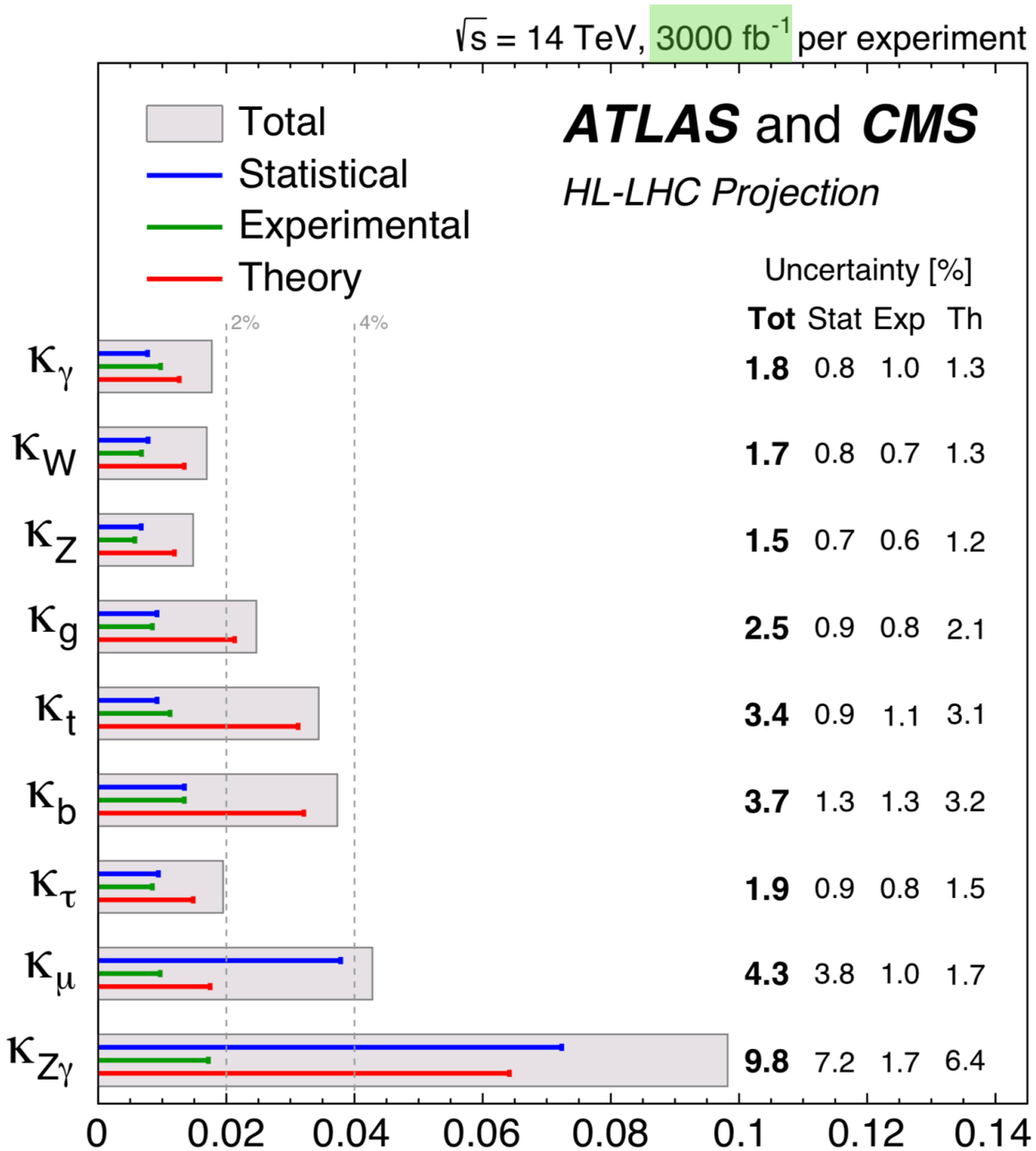
Plan

- Challenges in pushing fixed-order accuracy
- Local Unitarity : general concepts
- (Multi-)Loop Tree Duality : formulation
- (Multi-)Loop Tree Duality : singularities / deformation
- Local Unitarity : introduction by example
- Local Unitarity : numerical results
- Conclusion

Why precision ?

[CERN Yellow Report 2019: Report on the Physics at the HL-LHC]

The High-luminosity phase of the LHC reduces **statistical uncertainties** :



Higgs couplings

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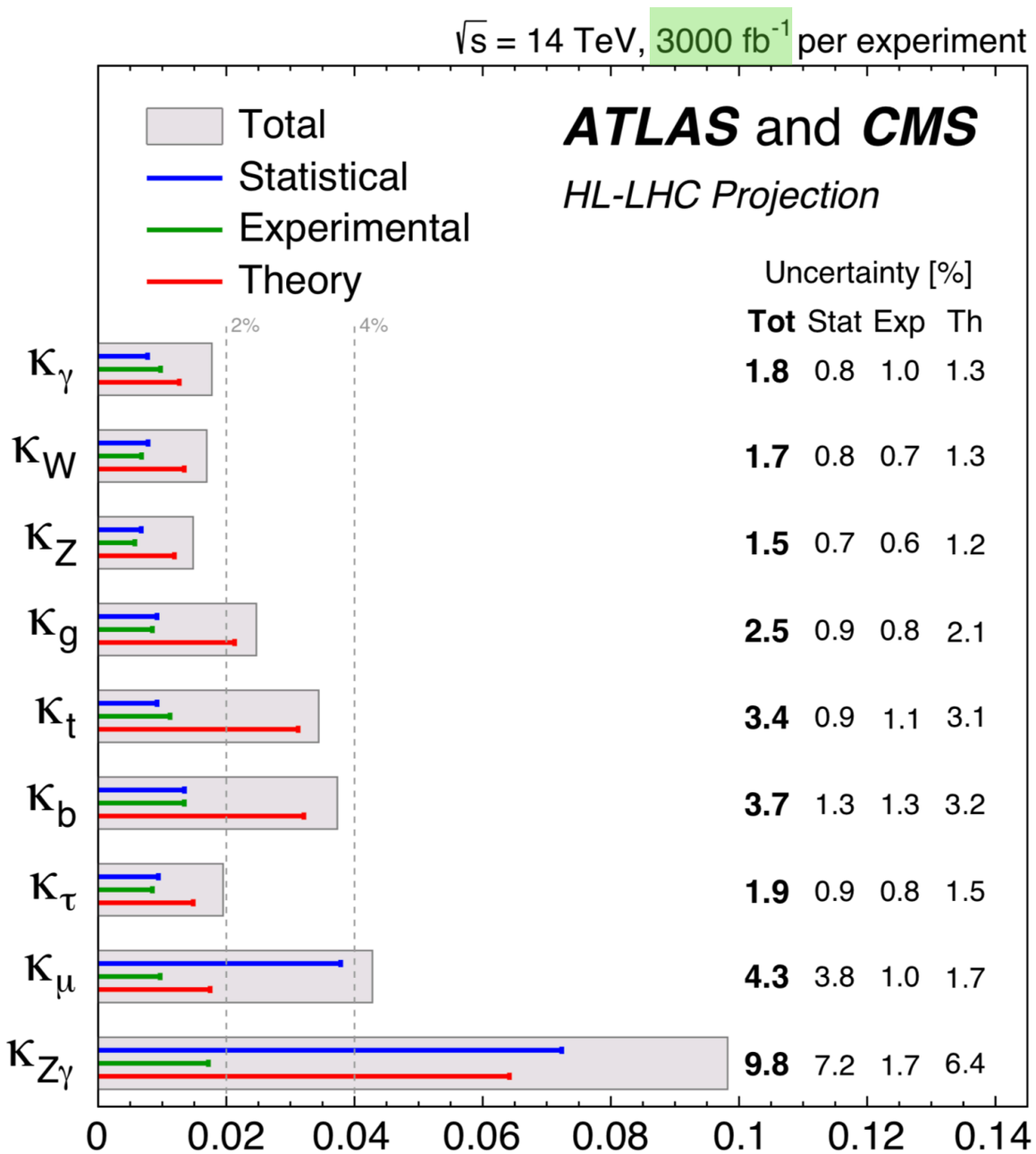

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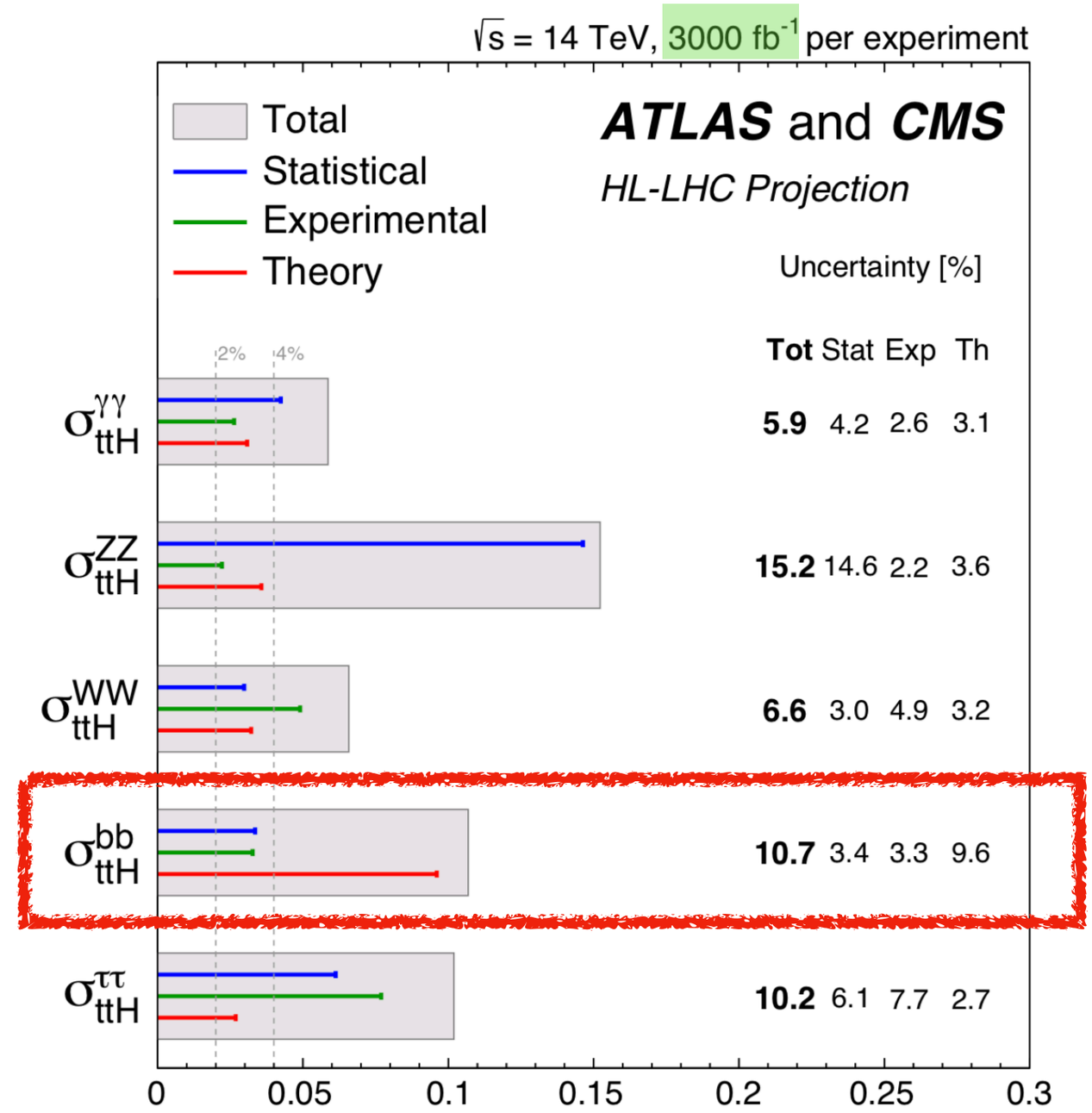

$pp \to t\bar{t}(H \to b\bar{b})$

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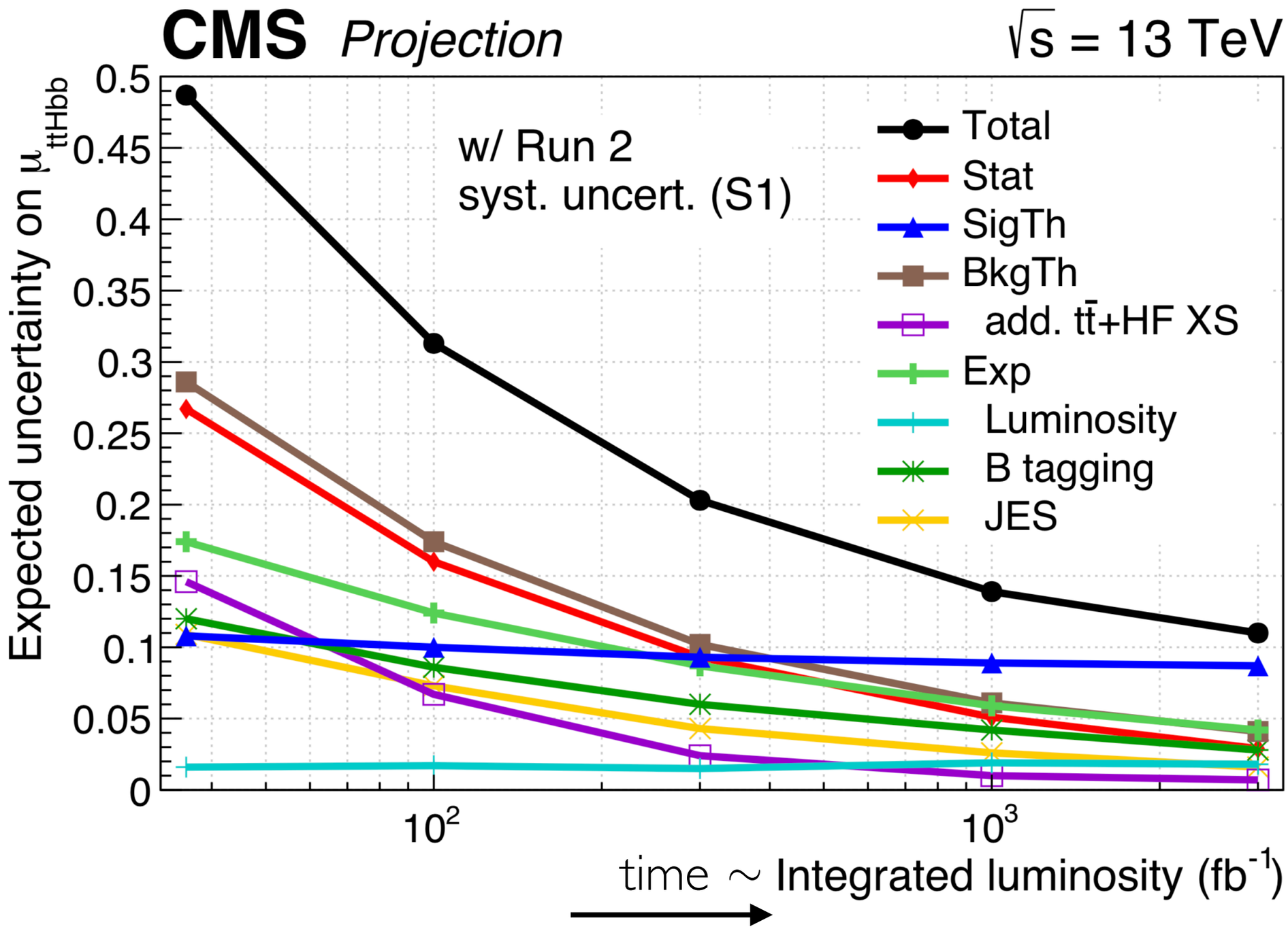

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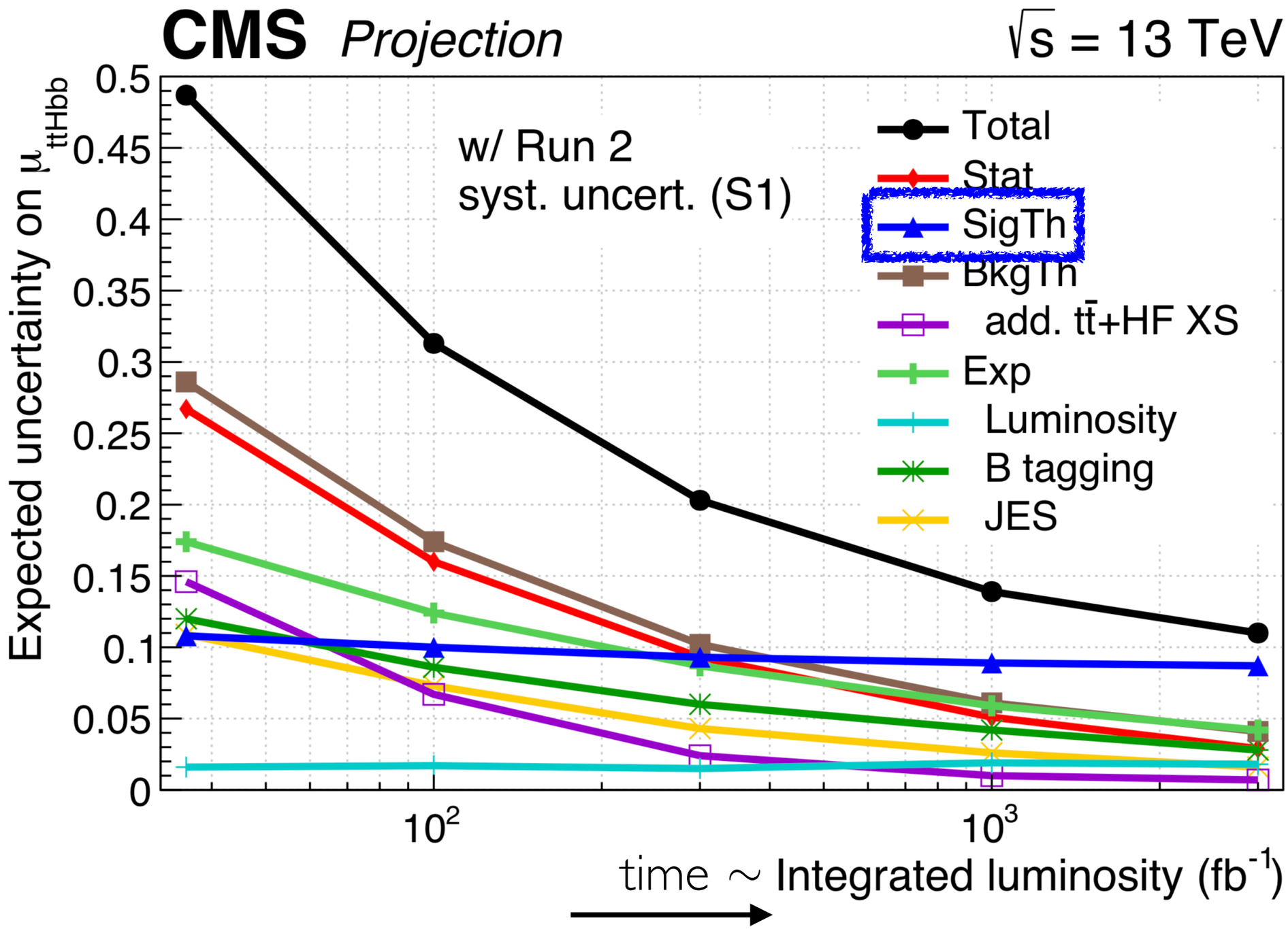

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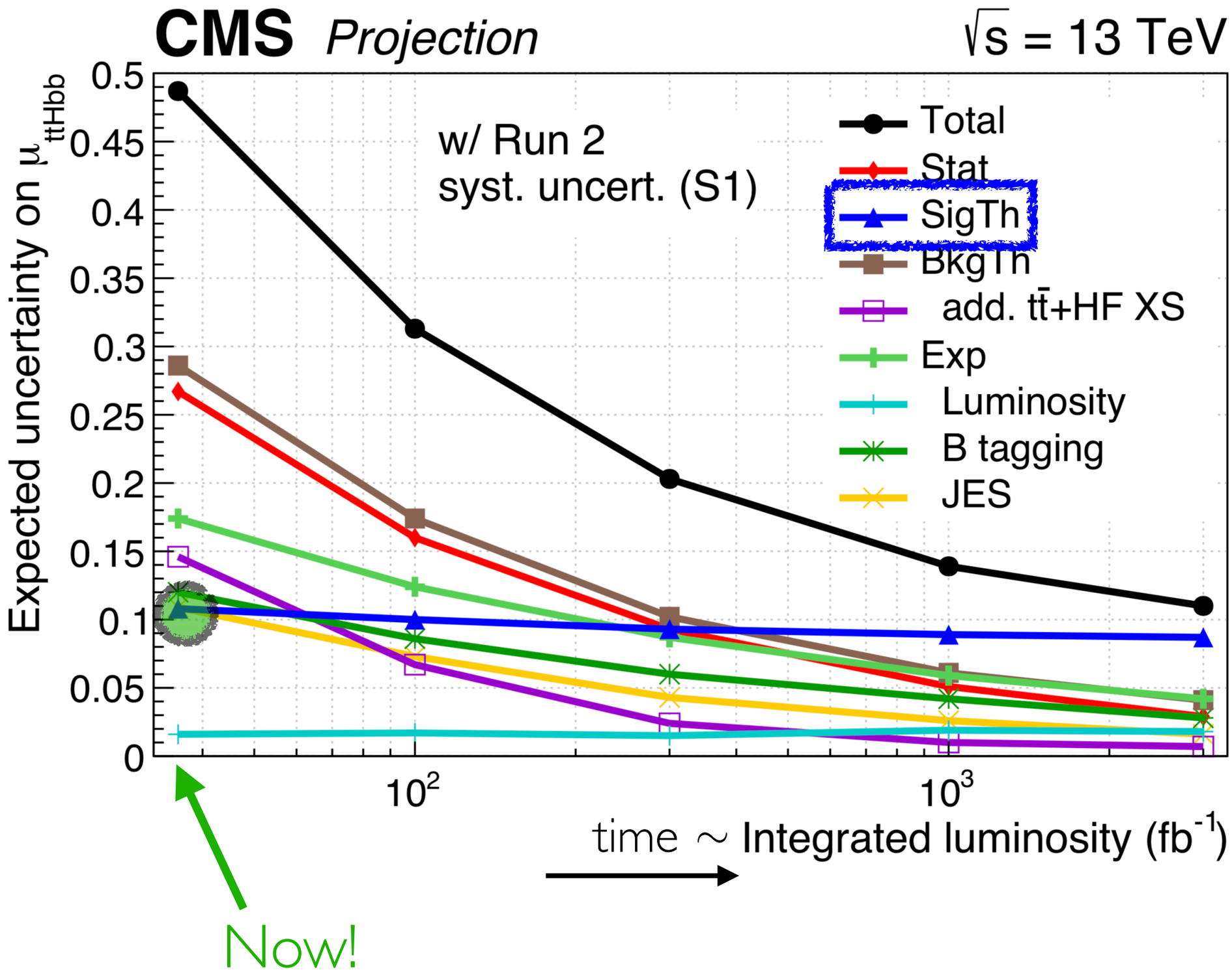

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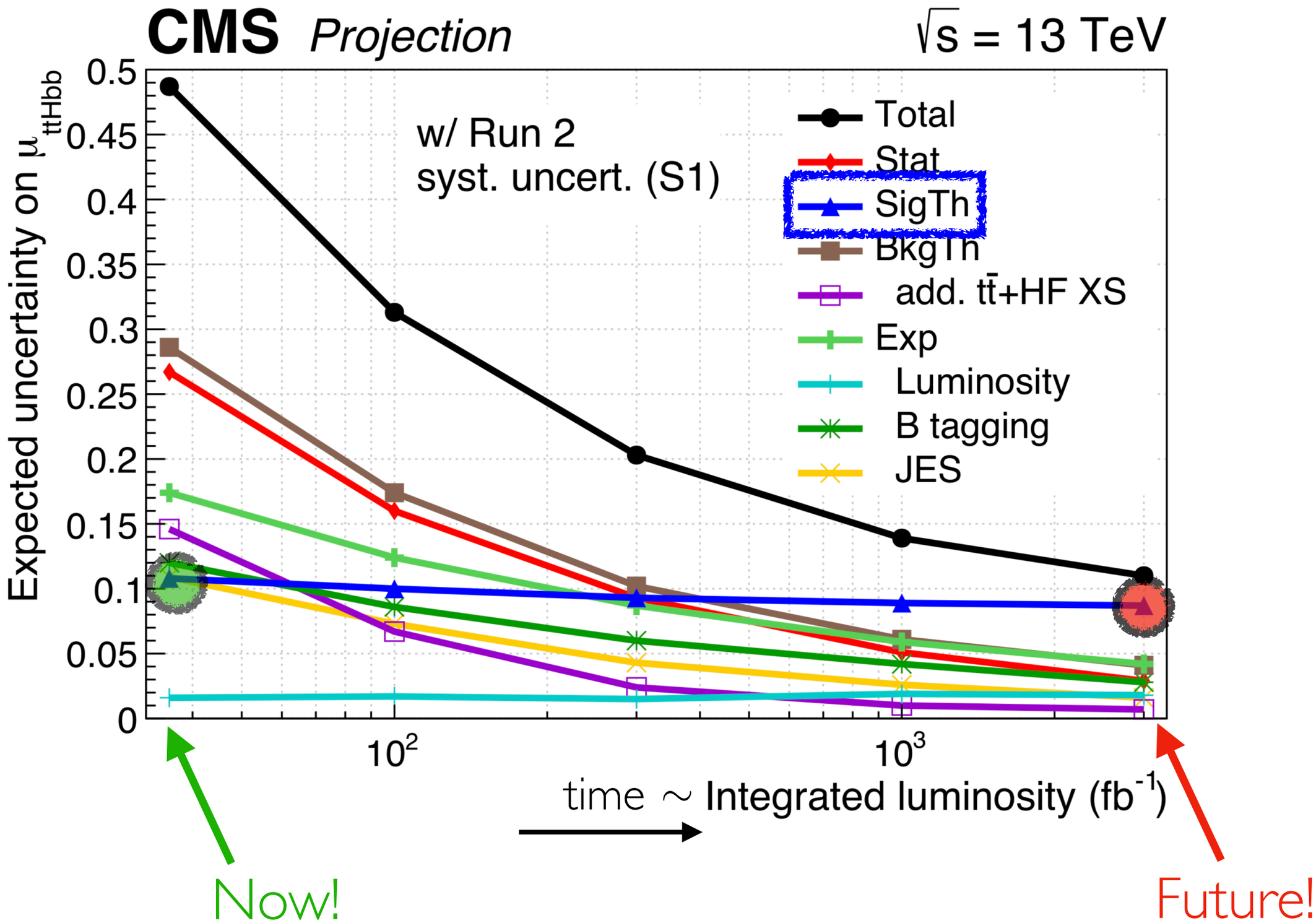
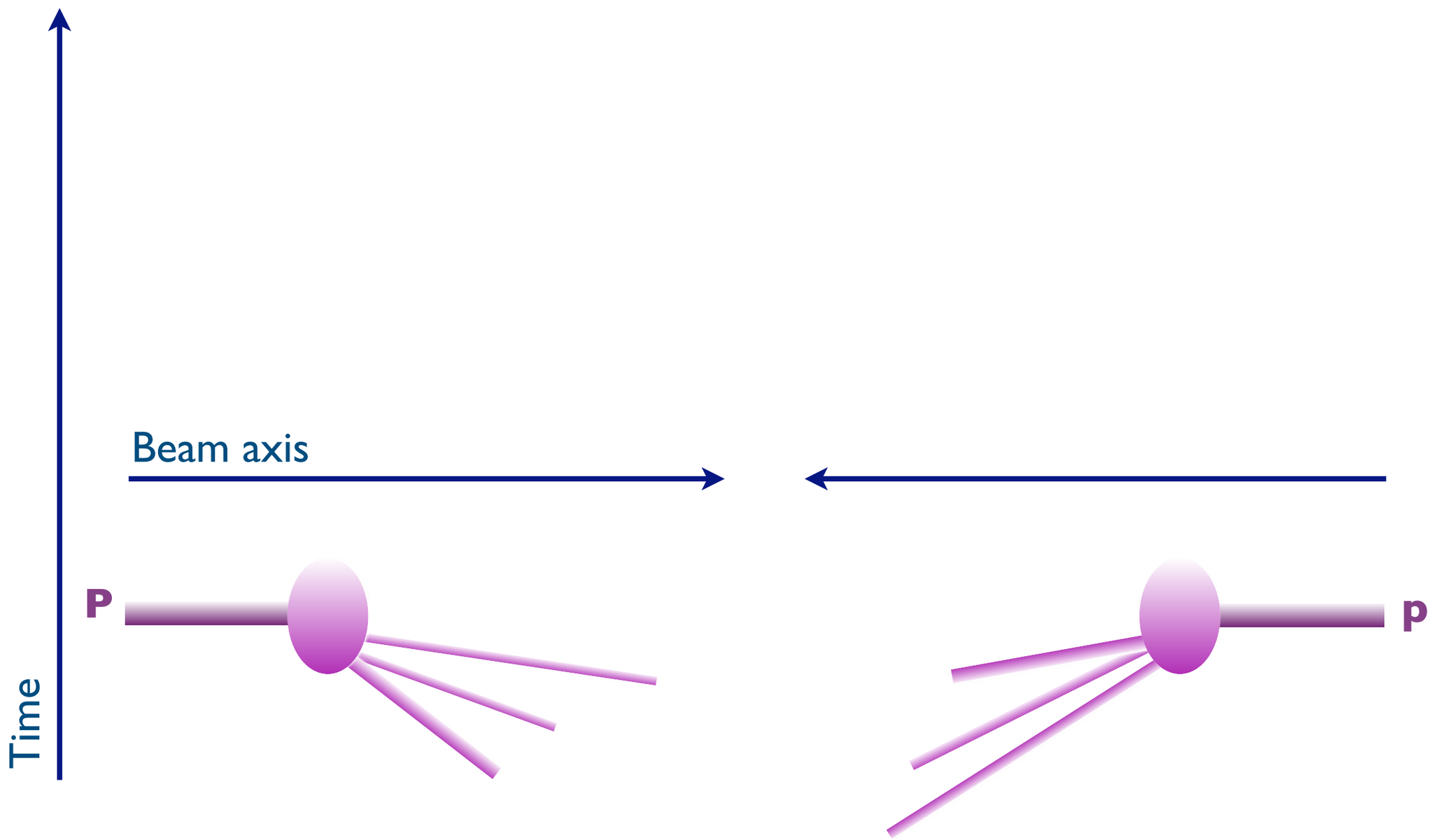


Broad context of my research



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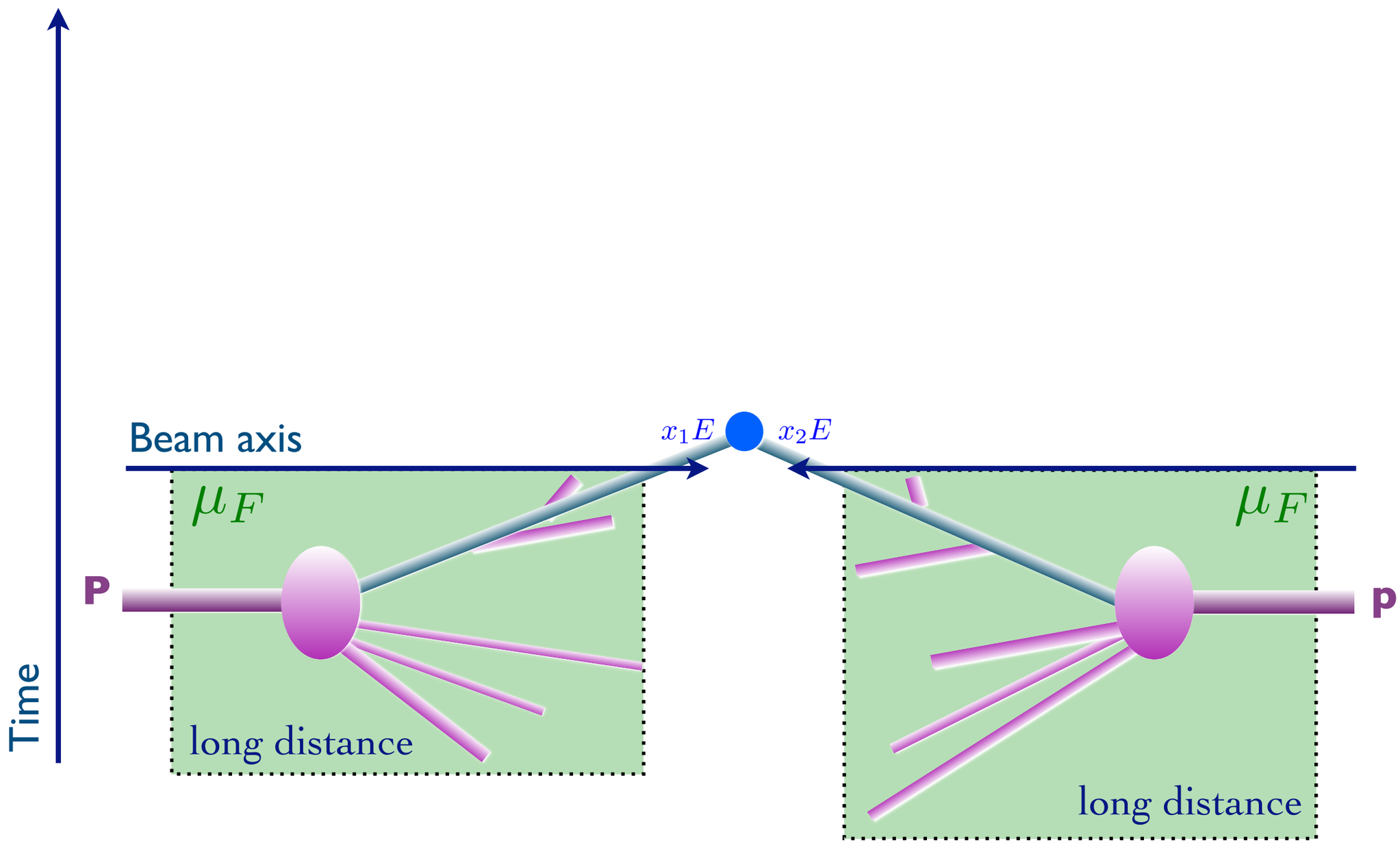

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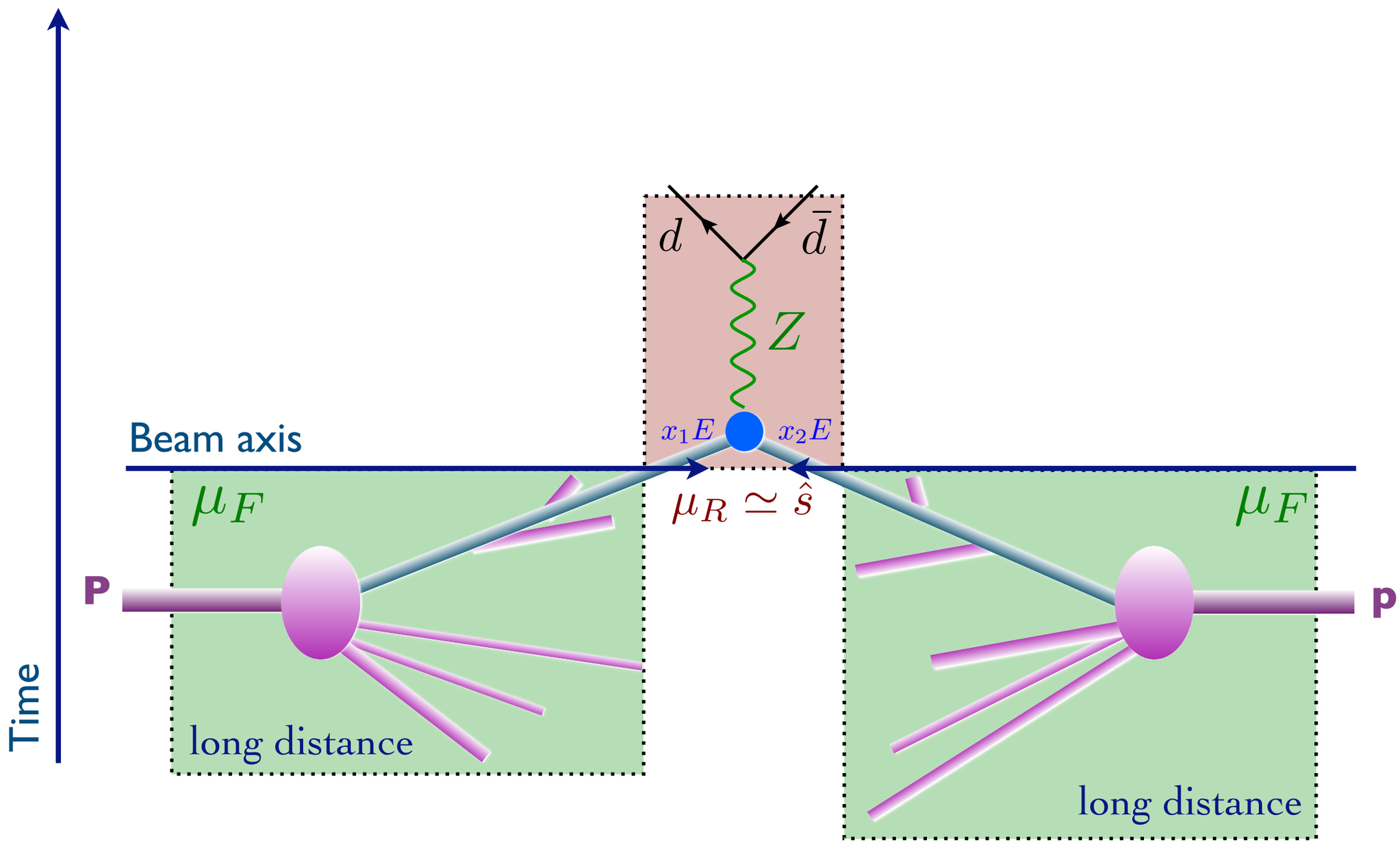

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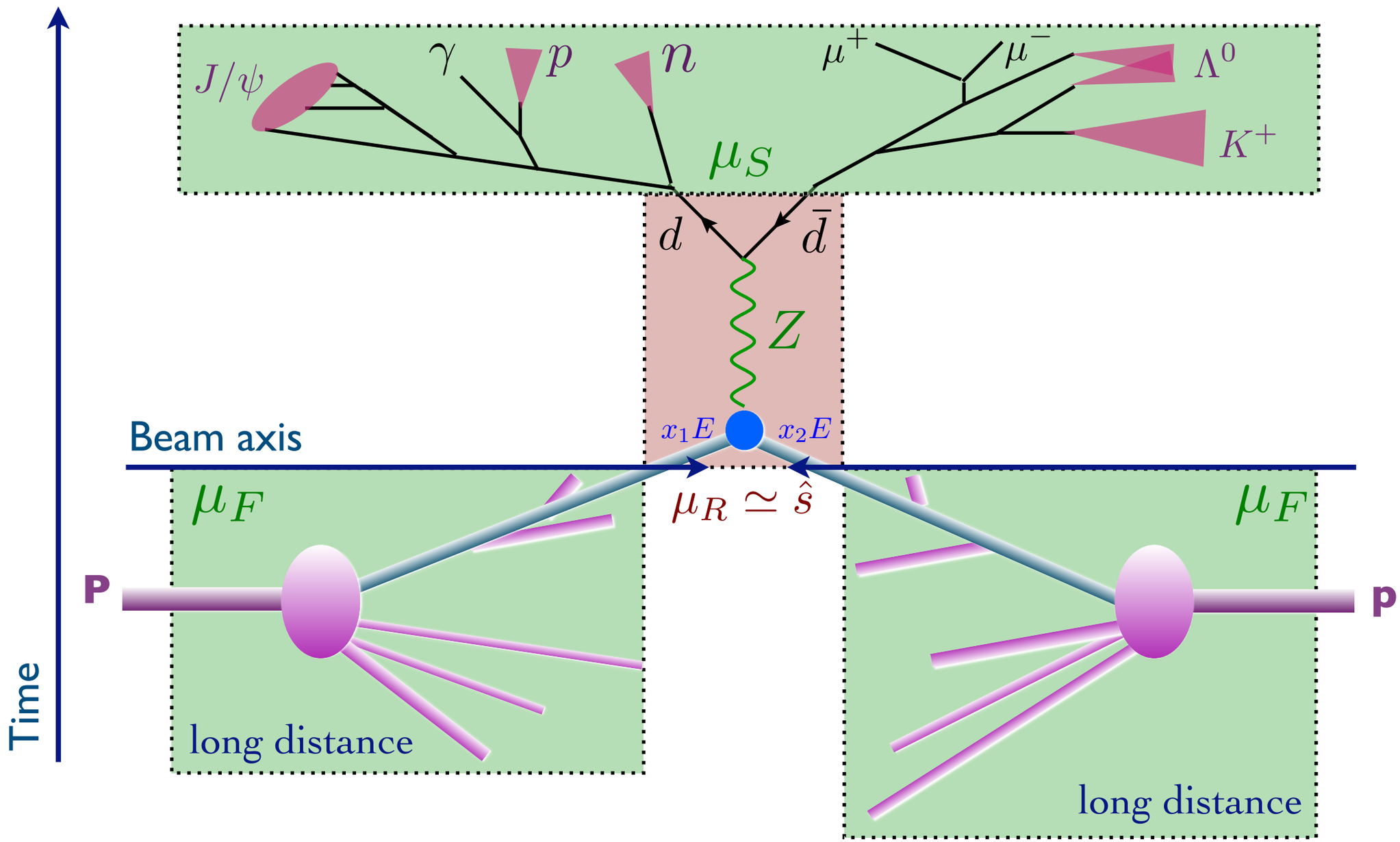
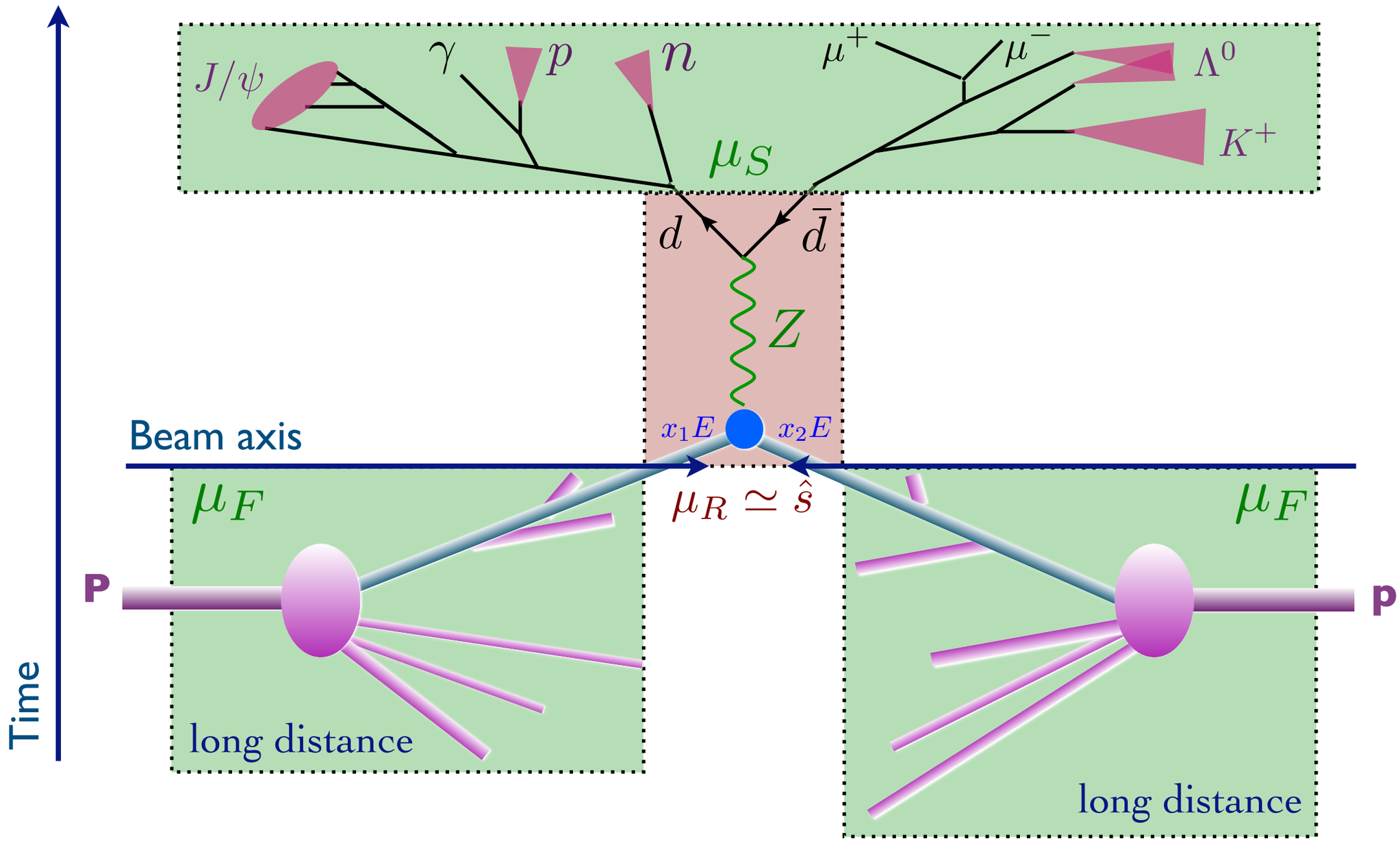


Broad context of my research

$$\sum_{a,b} \int dx_1 dx_2 d\Phi_{\rm FS}\, f_a(x_1,\mu_F) f_b(x_2,\mu_F)\, \hat{\sigma}_{pp\to\{P\}}(\mu_F,\mu_R,\mu_S)\, \mathcal{S}_{\{P\}\to\{H\}}(\mu_S)$$

Phase-space integral — Parton density functions — Parton-level differential cross section — Parton evolution operator



PERTURBATIVE EXPANSIONS

The differential cross section can be written as a perturbation series, using the coupling constant as an expansion parameter :

$$\hat{\sigma} = \sigma^{\text{Born}} \left(1 + \frac{\alpha_s}{2\pi} \sigma^{(1)} + \left(\frac{\alpha_s}{2\pi} \right)^2 \sigma^{(2)} + \left(\frac{\alpha_s}{2\pi} \right)^3 \sigma^{(3)} + \dots \right)$$

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LO predictions

Easy

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NLO corrections

Difficult but **automated**.

My past research

MadGraph5

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Case-by-case only

Automation at this order and beyond is the goal of my current research

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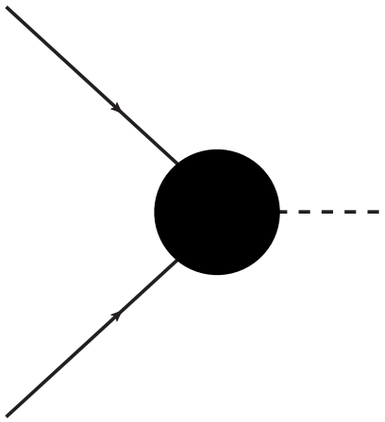
LO predictions	NLO corrections	NNLO corrections	NNNLO corrections
Easy	Difficult but **automated**. My past research MadGraph5	Case-by-case only Automation at this order and beyond is the goal of my current research	pp → V/H only

NLO Anatomy

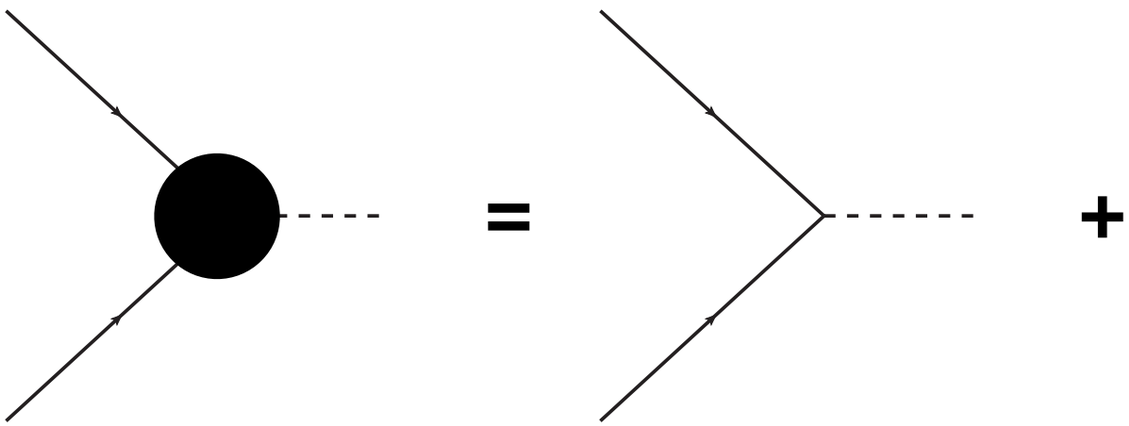


=

σ^{NLO} =

NLO inclusive

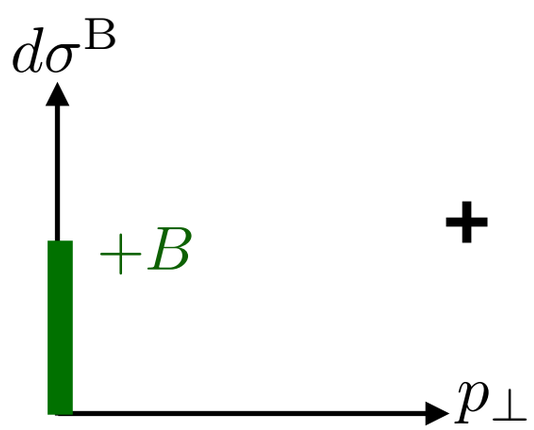
NLO Anatomy



$$\sigma^{\mathrm{NLO}} = \int_m d^{(4)}\sigma^B +$$

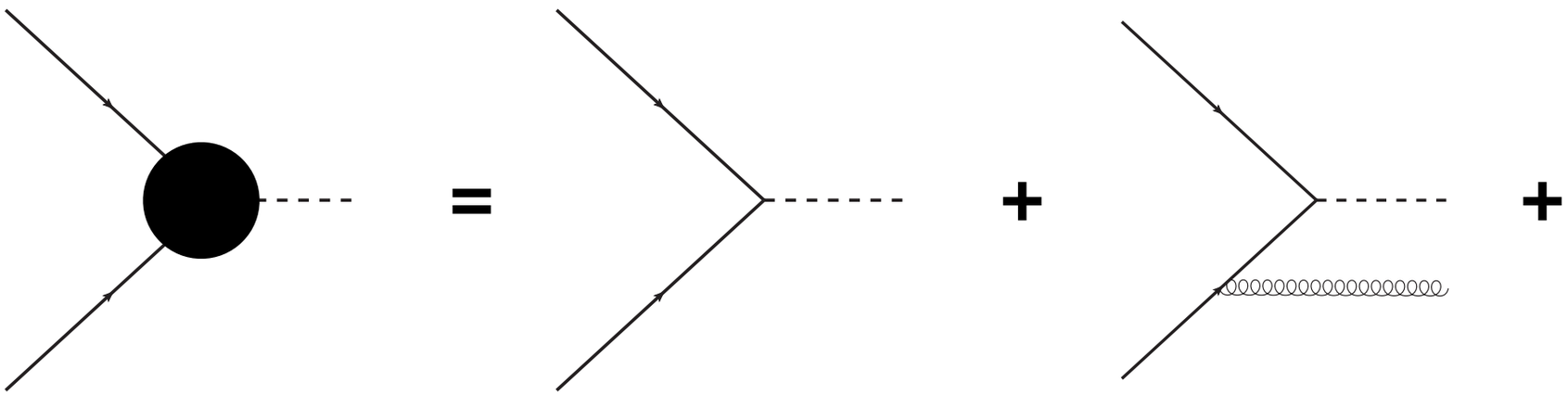
NLO inclusive

Born (B)



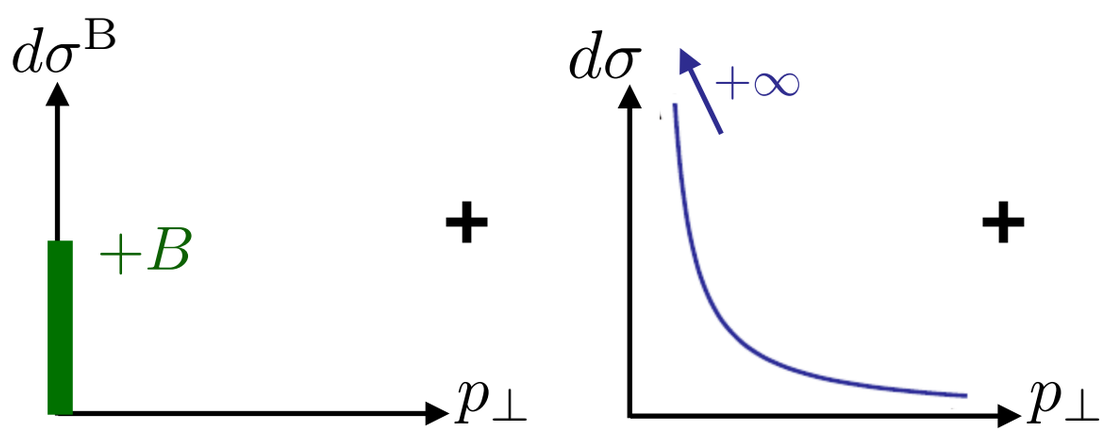
B

NLO Anatomy



$$\sigma^{\mathrm{NLO}} = \int_m d^{(4)}\sigma^B + \int_{m+1} d^{(d)}\sigma^R +$$

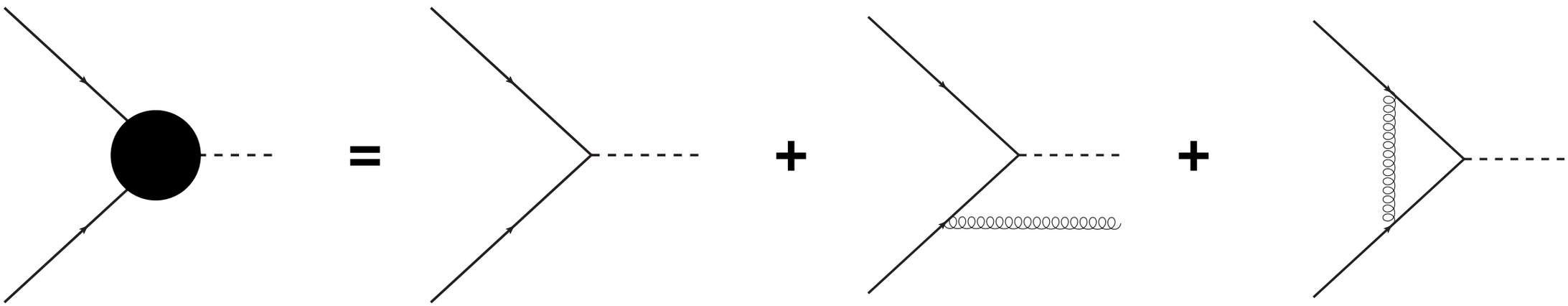
NLO inclusive — Born (B) — Real-emission (R)



B

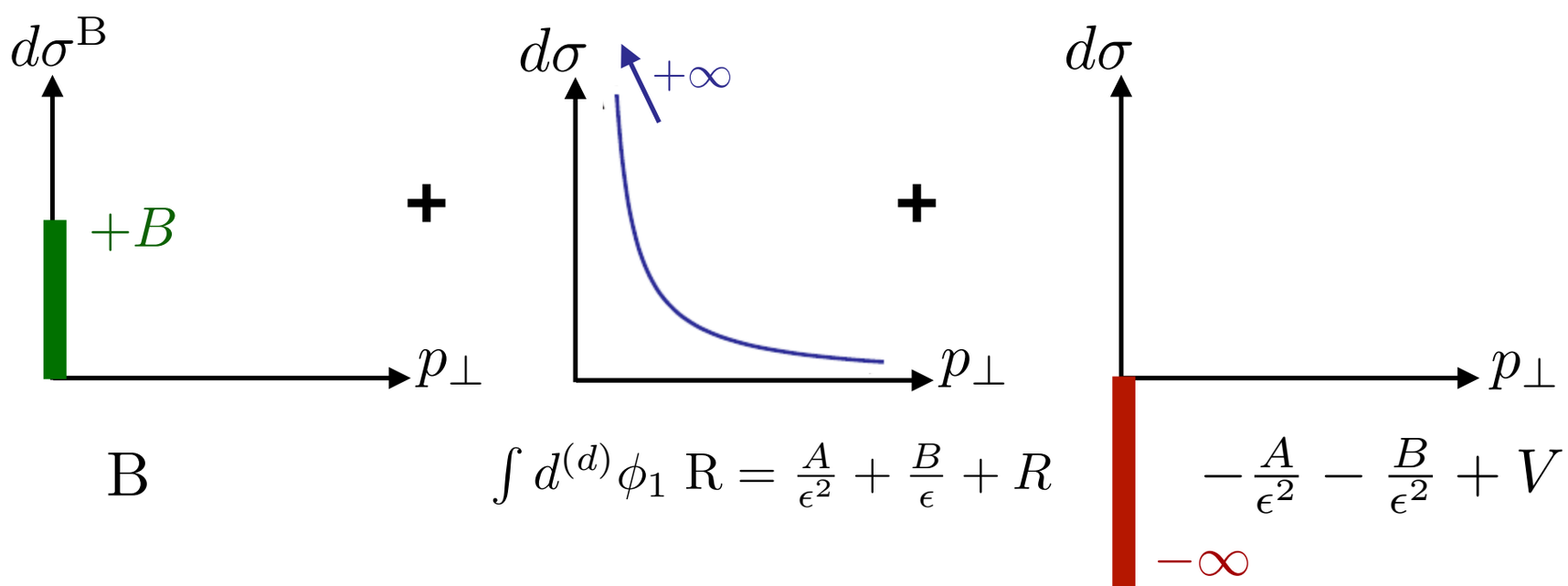
$$\int d^{(d)}\phi_1 \ \mathrm{R} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + R$$

NLO Anatomy



$$\sigma^{\mathrm{NLO}} = \int_m d^{(4)}\sigma^B + \int_{m+1} d^{(d)}\sigma^R + \int_m d^{(4)} \int_1 d^{(d)}\sigma^V$$

NLO inclusive — Born (B) — Real-emission (R) — Virtual (V)



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$\int d^{(d)}\phi_1 \, \mathrm{R} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + R$

$-\frac{A}{\epsilon^2} - \frac{B}{\epsilon^2} + V$

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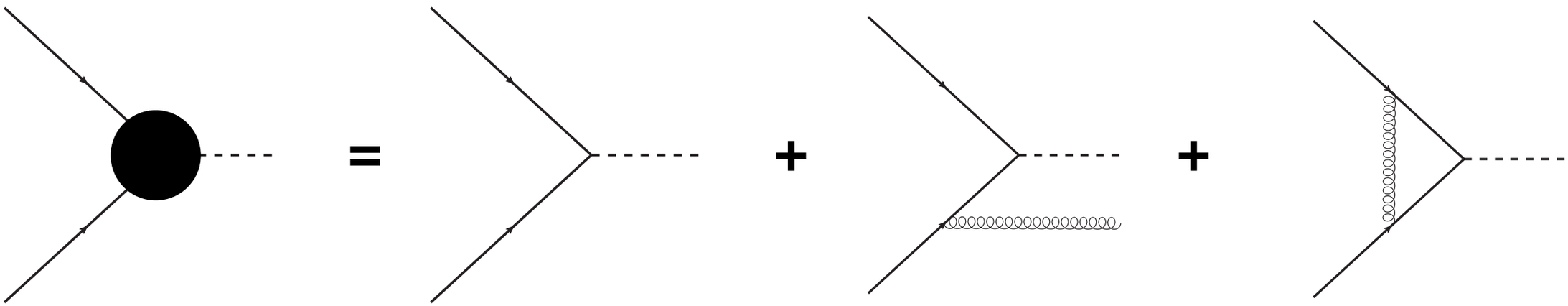

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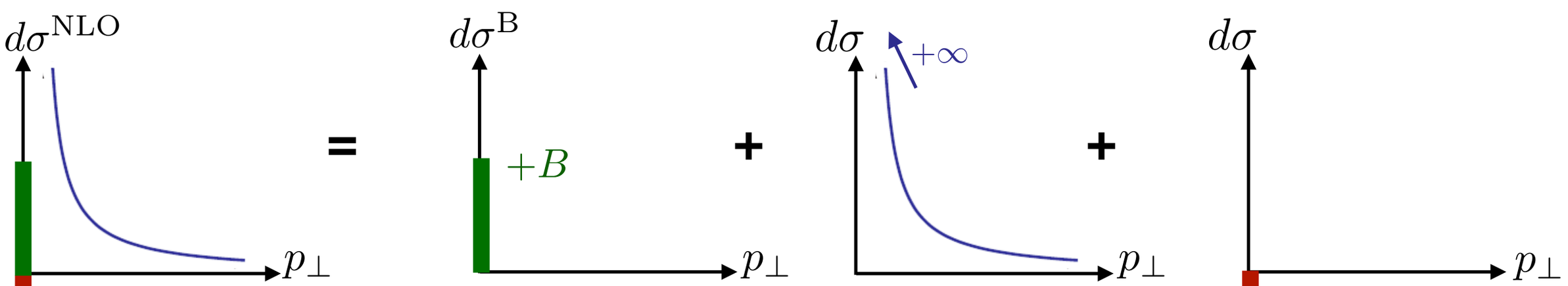


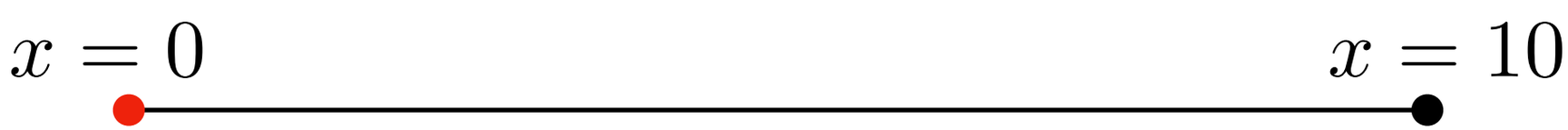
$\underbrace{B}_{\sigma^{\mathrm{LO}}} + \underbrace{R+V}_{\text{NLO correction}}$ $\quad$ B $\quad$ $\int d^{(d)}\phi_1\ \mathrm{R} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + R$ $\quad$ $-\frac{A}{\epsilon^2} - \frac{B}{\epsilon^2} + V$

$-\infty$

One-dimensional toy example

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \frac{\cos(x)}{x} \mathcal{J}(x) + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



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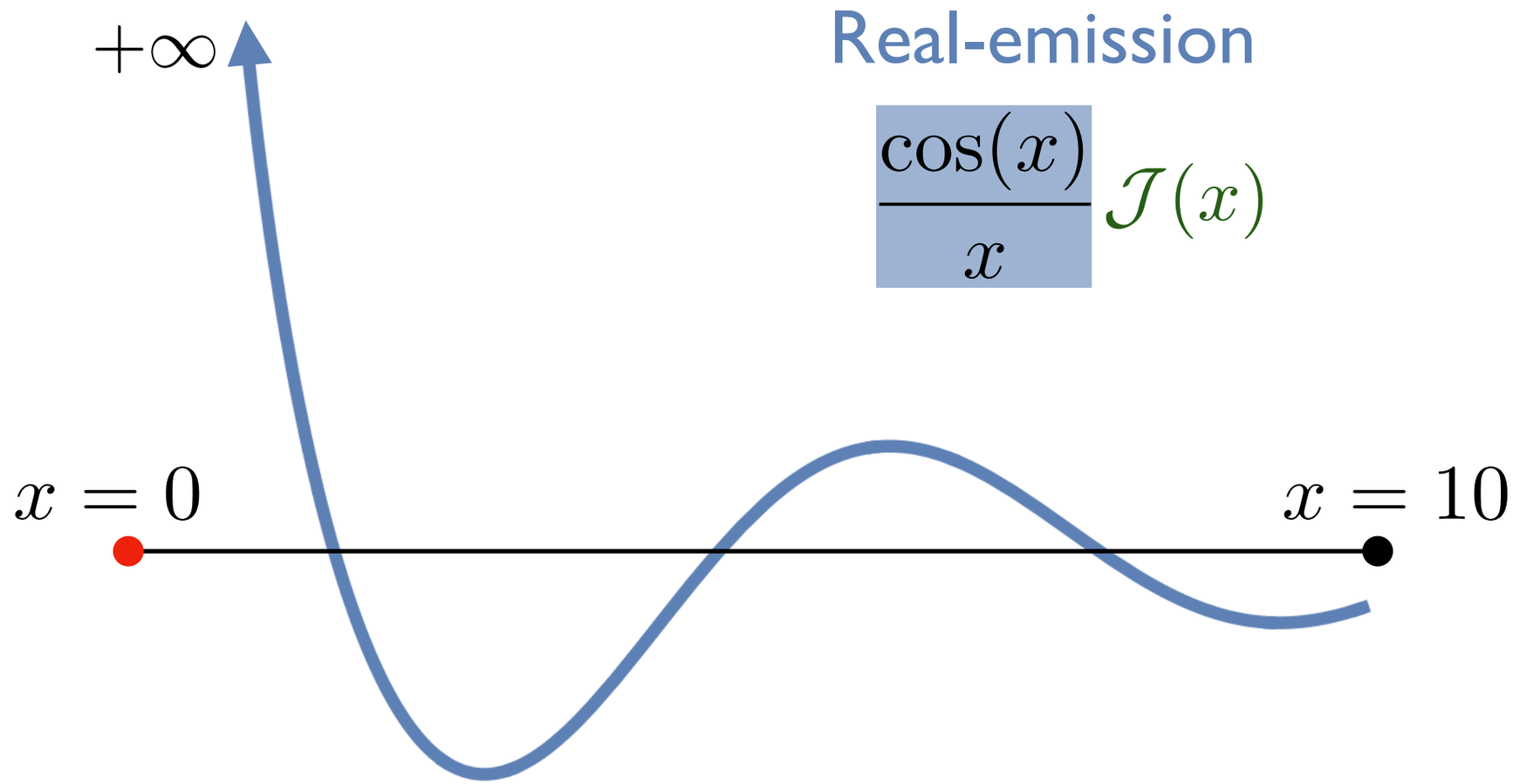

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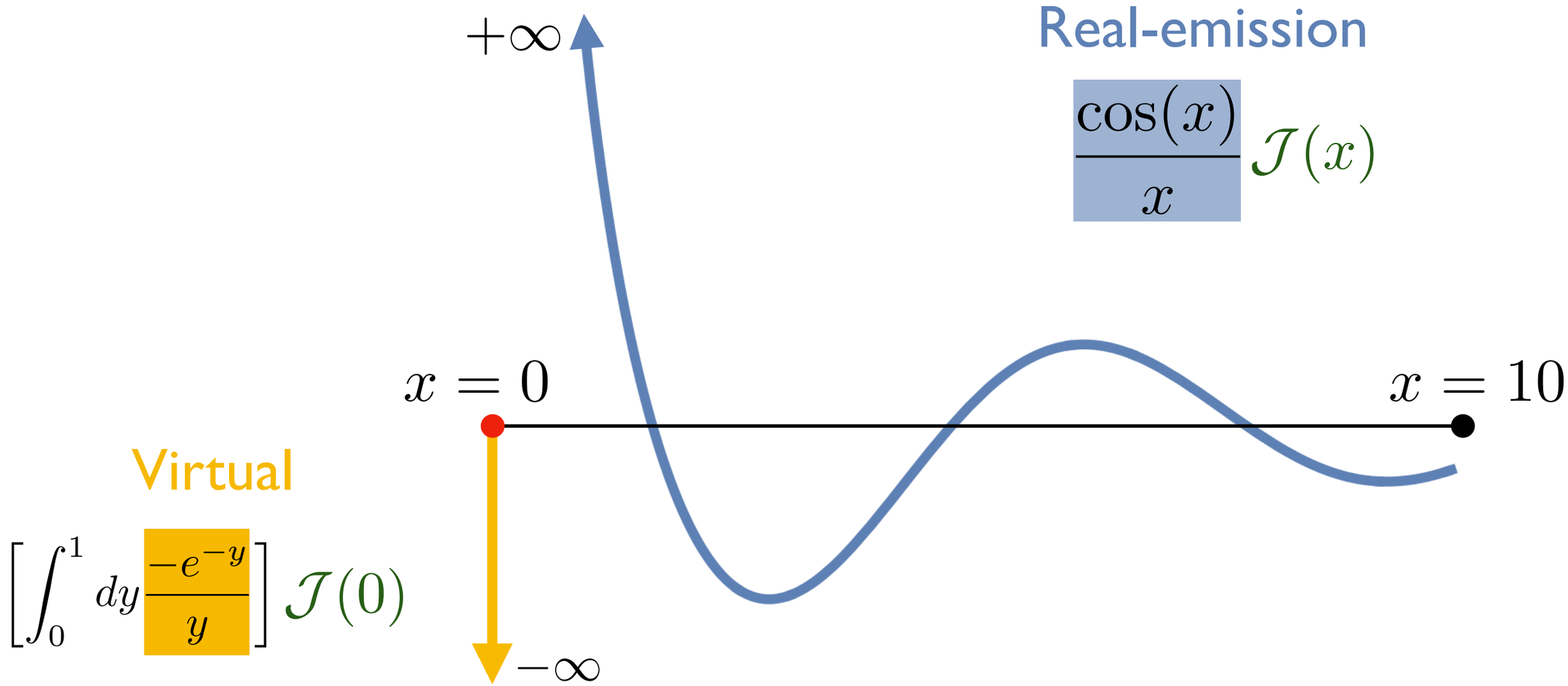

Subtraction vs Local Unitarity

- One can introduce the following local (in x) **counterterm**:

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NUMERICAL

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- My research introduces a new paradigm, **Local Unitarity**, whereby :

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) + \frac{-e^{-x}}{x}\mathcal{J}(0)\Theta(1-x)\right]$$

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \left|\mathcal{M}^{(L)}(\{p_i\})\right|^2 \delta p_1^2 \ldots \delta p_N^2 \delta^4\left(p_1 + \ldots p_N\right)$$

Numerical

LO

$d\bar{d} \to Z$

{}

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Numerical

LO	NLO
$d\bar{d} \to Z$	$d\bar{d} \to Zg$
{}	

```
| L0 : ((S(4),),)
| L1 : ((C(1,4),),)
| L2 : ((C(2,4),),)
| L3 : ((C(S(4),1),),)
| L4 : ((C(S(4),2),),)
```

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Numerical

LO	NLO	NNLO
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{}		

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```

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((C(4,5),),)
| L3 : ((C(1,4),),)
| L4 : ((C(2,4),),)
[ … 71 limits … ]
| L69 : ((C(C(S(4),2),S(5),),),)
| L70 : ((C(S(C(S(4),5),),2),),)
| L71 : ((C(S(C(S(5),4),),2),),)
```

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 \left(p_1 + \ldots p_N\right)$$

Numerical

LO	NLO	NNLO	N3LO	…
$d\bar{d} \to Z$	$d\bar{d} \to Zg$	$d\bar{d} \to Zgg$	$d\bar{d} \to Zggg$	
{}				

```
| L0 : ((S(4),),)
| L1 : ((C(1,4),),)
| L2 : ((C(2,4),),)
| L3 : ((C(S(4),1),),)
| L4 : ((C(S(4),2),),)
```

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((C(4,5),),)
| L3 : ((C(1,4),),)
| L4 : ((C(2,4),),)
[ … 71 limits … ]
| L69 : ((C(C(S(4),2),S(5),),),)
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| L71 : ((C(S(C(S(5),4),),2),),)
```

```
| L0 : ((S(4),),)
| L1 : ((S(5),),)
| L2 : ((S(6),),)
| L3 : ((C(4,5),),)
| L4 : ((C(4,6),),)
[ … 1396 limits … ]
| L1394 : ((C(S(C(S(C(S(6),4),),5),),2),),)
| L1395 : ((C(S(C(S(C(S(5),6),),4),),2),),)
| L1396 : ((C(S(C(S(C(S(6),5),),4),),2),),)
```

OBSTACLES: INFRARED LIMITS

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 \left(p_1 + \ldots p_N\right)$$

Numerical

LO	NLO	NNLO	N3LO	…
$d\bar{d} \to Z$	$d\bar{d} \to Zg$	$d\bar{d} \to Zgg$	$d\bar{d} \to Zggg$	
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| L0 : ((S(4),),)
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```

```
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| L4 : ((C(4,6),),)
[ … 1396 limits … ]
| L1394 : ((C(S(C(S(C(S(6),4),),5),),2),),)
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| L1396 : ((C(S(C(S(C(S(6),5),),4),),2),),)
```

On top of :

- R^kV^{k-n} proliferation of contributions
- Convolution with PDF counterterms
- Growth of number of partonic channels

} Very challenging at and beyond NNLO…

OBSTACLES: MULTI-LOOPS MEs

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 \left(p_1 + \ldots p_N\right)$$

Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \boxed{\int d^dk_1 \cdots \int d^dk_L} \, I(\{p_i\}, \{k_i\})|$$

Analytical ?

Timings	Tree-level MG5aMC [arXiv:1405.0301]
$d\bar{d} \to ZZ$	7 μs
$d\bar{d} \to ZZg$	35 μs
$d\bar{d} \to ZZgg$	220 μs

OBSTACLES: MULTI-LOOPS MEs

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 \left(p_1 + \ldots p_N\right)$$

Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \boxed{\int d^dk_1 \cdots \int d^dk_L} \, I(\{p_i\}, \{k_i\})|$$

Analytical ?

Timings	Tree-level MG5aMC [arXiv:1405.0301]		One-loop MadLoop [arXiv:1103.0621]
$d\bar{d} \to ZZ$	7 μs	x 10^2 →	0.6 ms
$d\bar{d} \to ZZg$	35 μs	x 10^3 →	38 ms
$d\bar{d} \to ZZgg$	220 μs	x 10^4 →	1200 ms

OBSTACLES: MULTI-LOOPS MEs

$$\sigma \sim \boxed{\int d^4p_1 \ldots d^4p_N} \, |\mathcal{M}^{(L)}(\{p_i\})|^2 \, \delta p_1^2 \ldots \delta p_N^2 \delta^4 \left(p_1 + \ldots p_N\right)$$

Numerical

$$|\mathcal{M}^{(L)}(\{p_i\})|^2 \sim \boxed{\int d^dk_1 \cdots \int d^dk_L} \, I(\{p_i\}, \{k_i\})|$$

Analytical ?

Timings	Tree-level MG5aMC [arXiv:1405.0301]		One-loop MadLoop [arXiv:1103.0621]		Two-loop VVamp [arXiv:1503.08835]
$d\bar{d} \to ZZ$	7 μs	x 10^2 →	0.6 ms	x 10^3 →	$\mathcal{O}$(~1s)
$d\bar{d} \to ZZg$	35 μs	x 10^3 →	38 ms	? →	N/A
$d\bar{d} \to ZZgg$	220 μs	x 10^4 →	1200 ms		N/A

A claim :

The more N's you add …

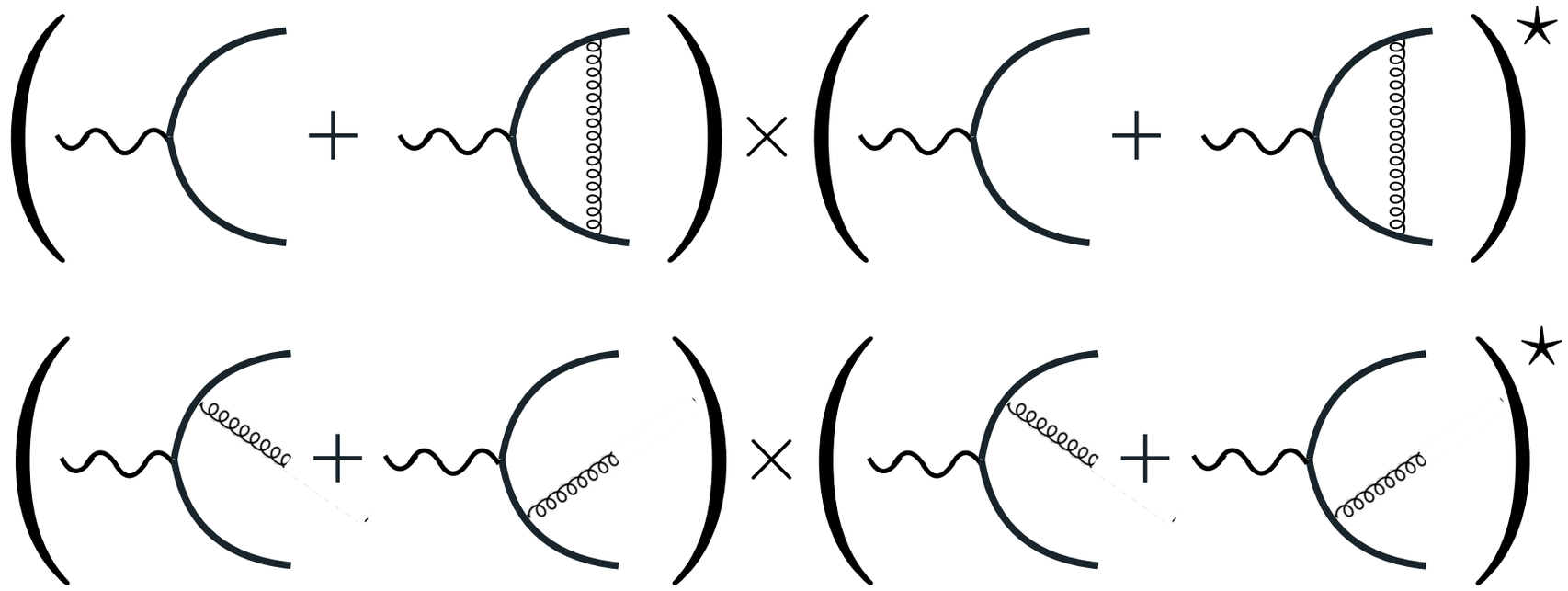
… the less natural it is to look at amplitudes.

Plan

- Challenges in pushing fixed-order accuracy
- Local Unitarity : general concepts
- (Multi-)Loop Tree Duality : formulation
- (Multi-)Loop Tree Duality : singularities / deformation
- Local Unitarity : introduction by example
- Local Unitarity : numerical results
- Conclusion

Remedy: forward-scattering graphs

$\sigma_{\gamma^\star \to d\bar{d}} =$



$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$

Remedy: forward-scattering graphs

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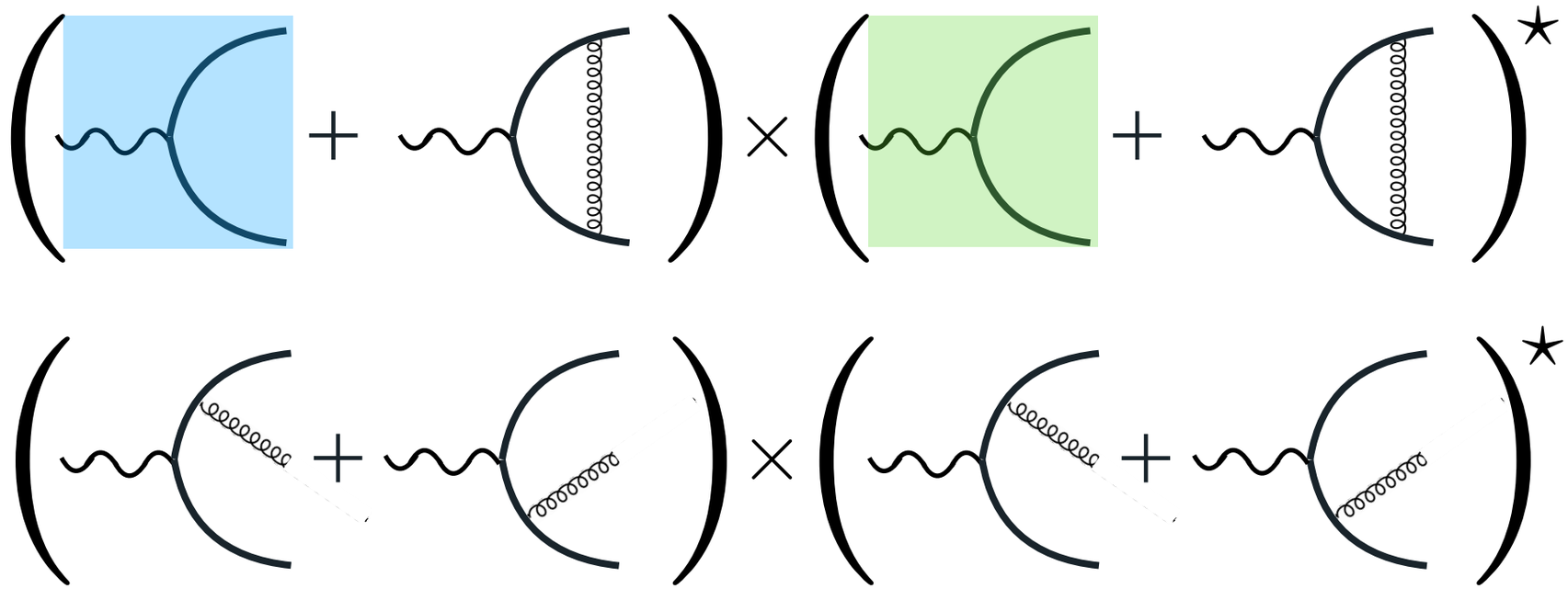

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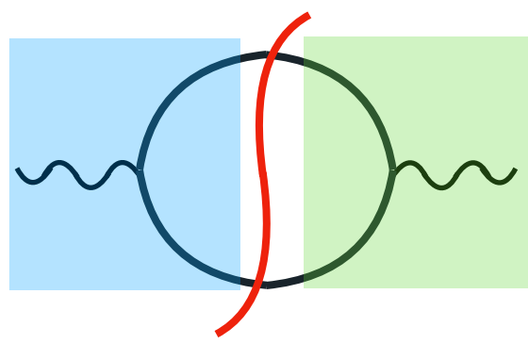

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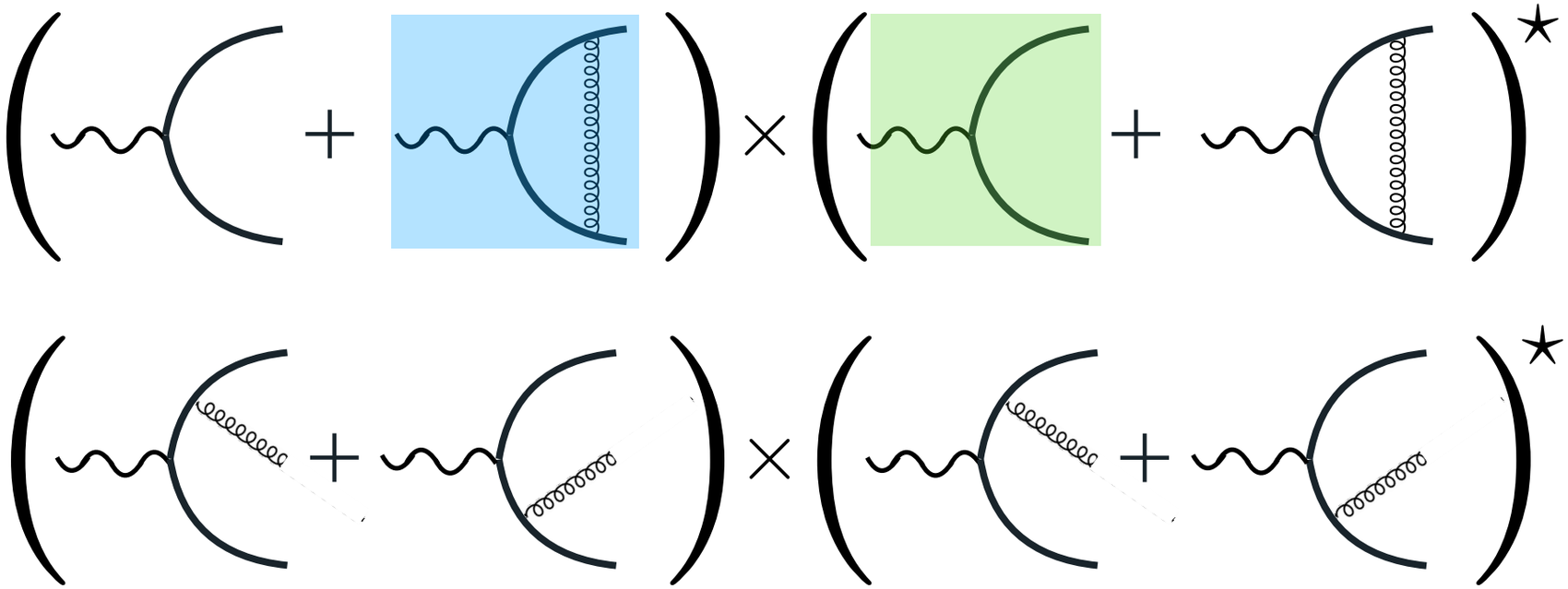

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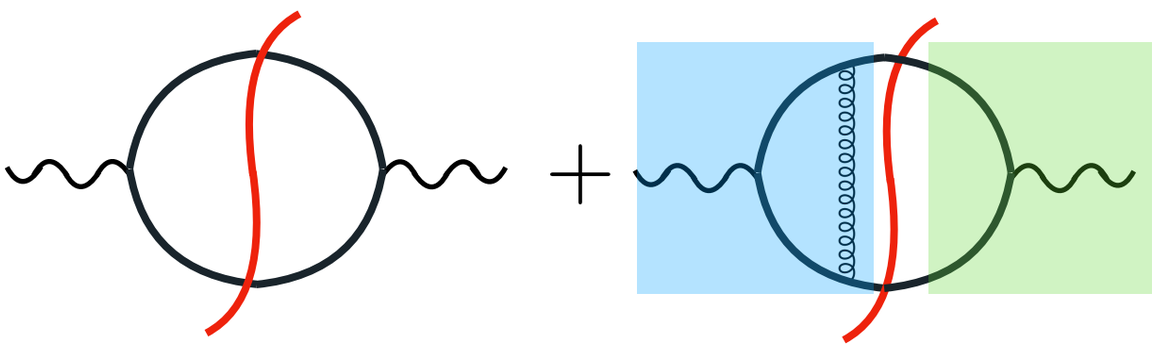

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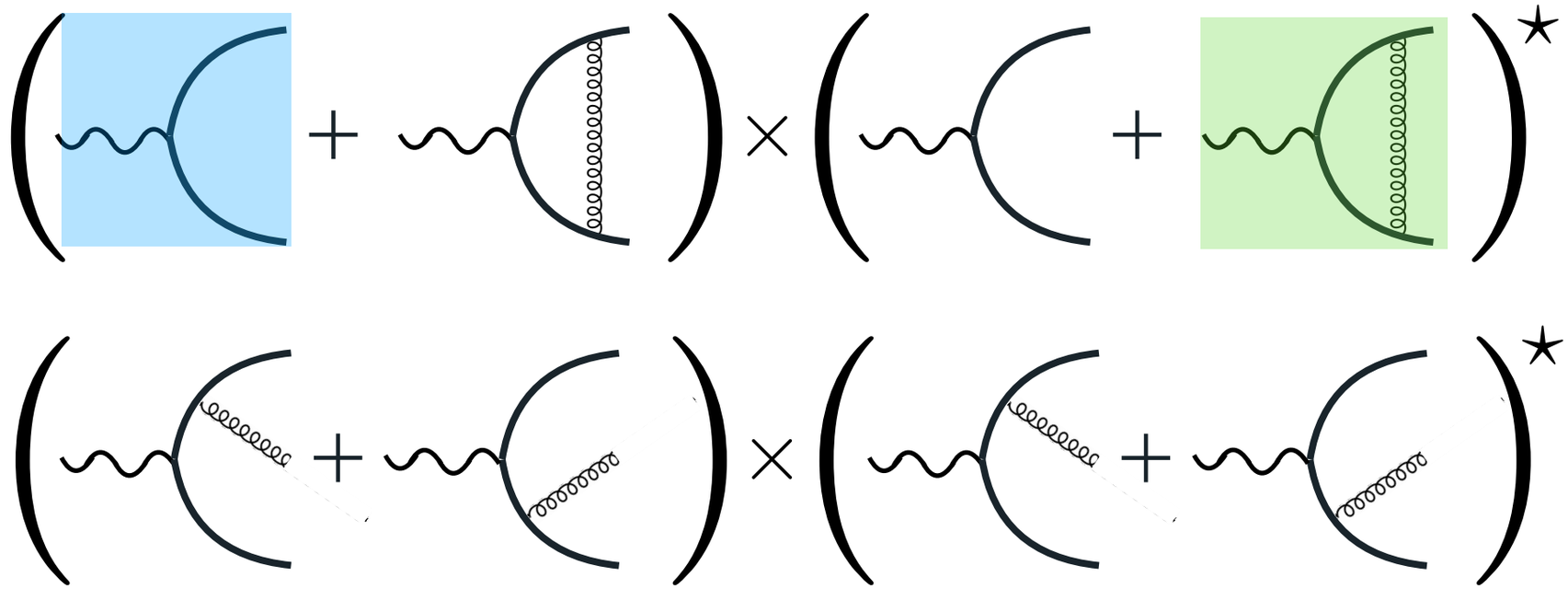

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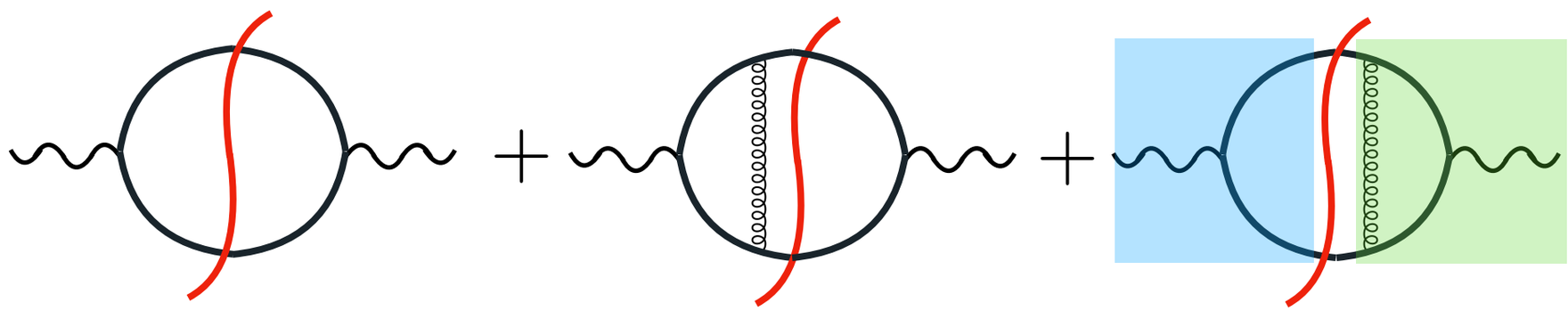

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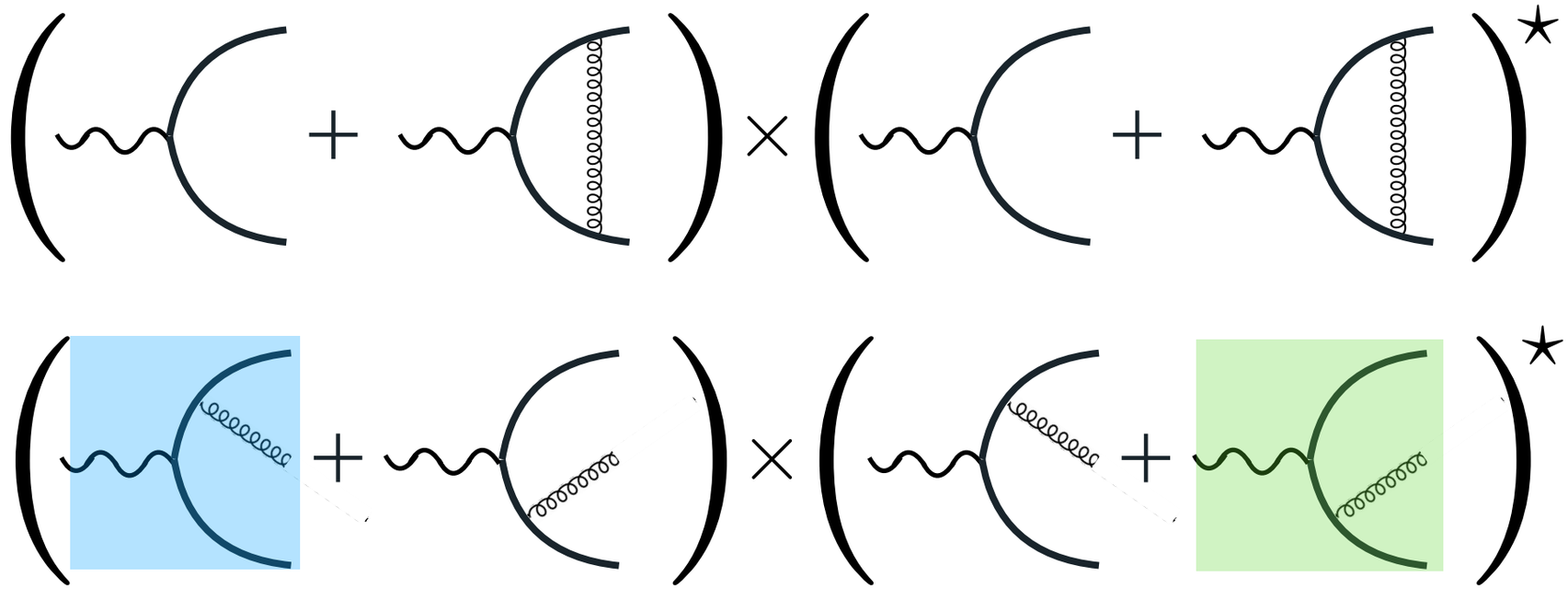

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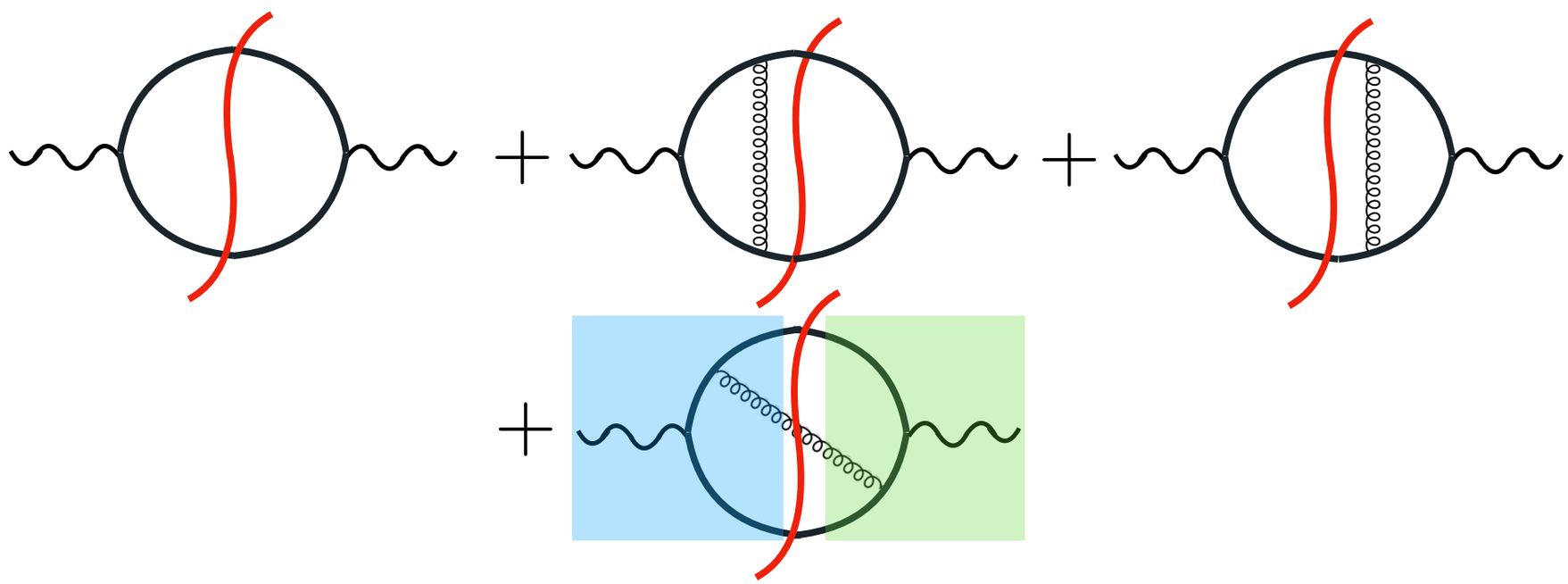

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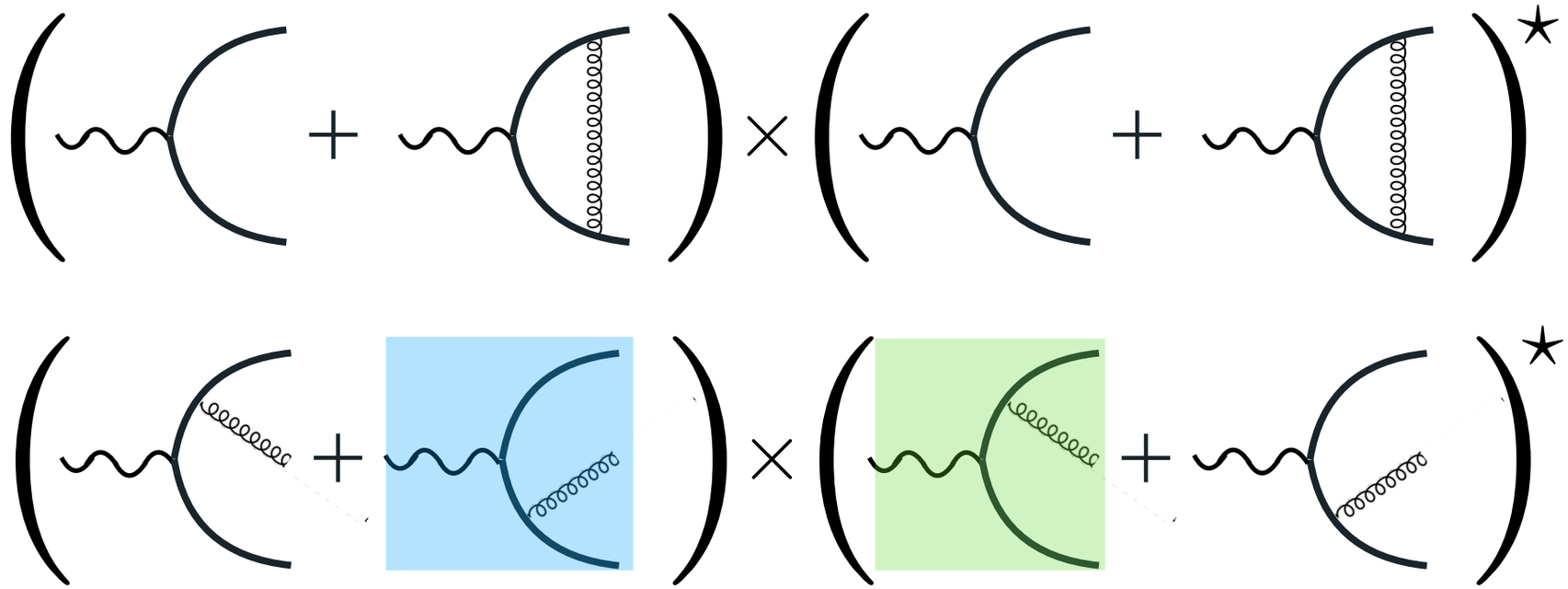

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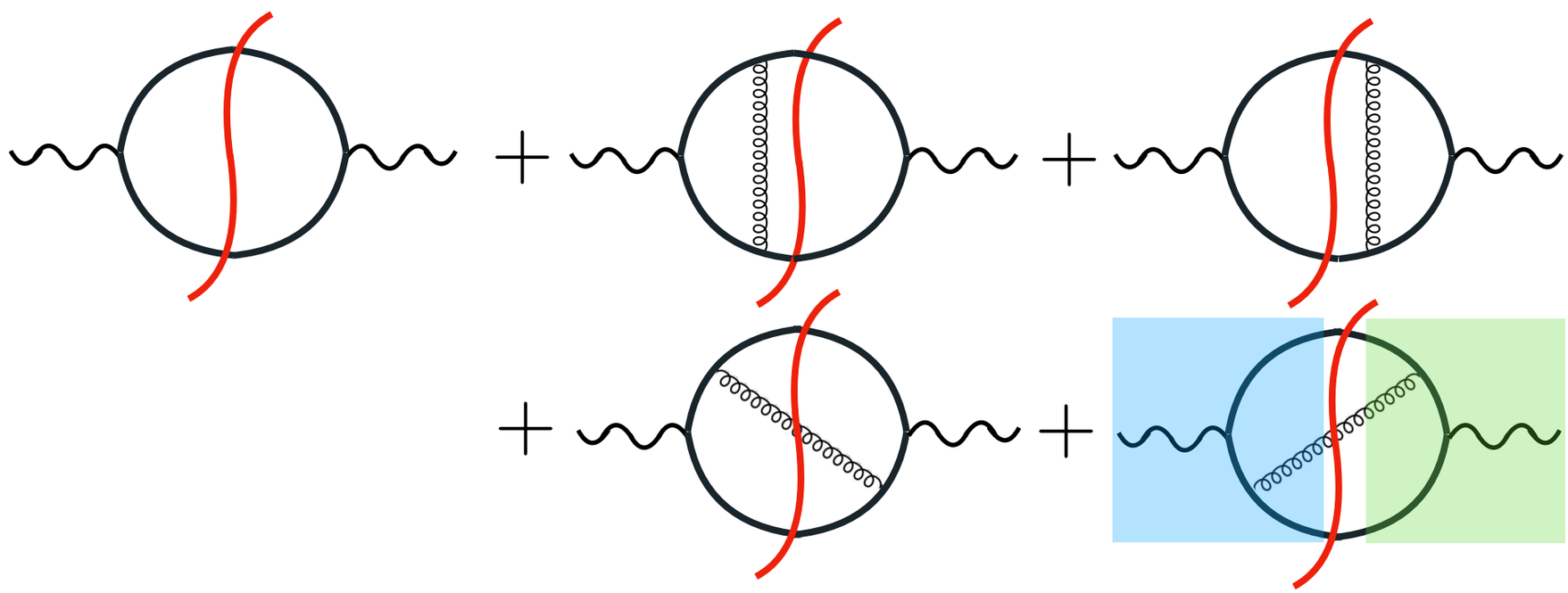

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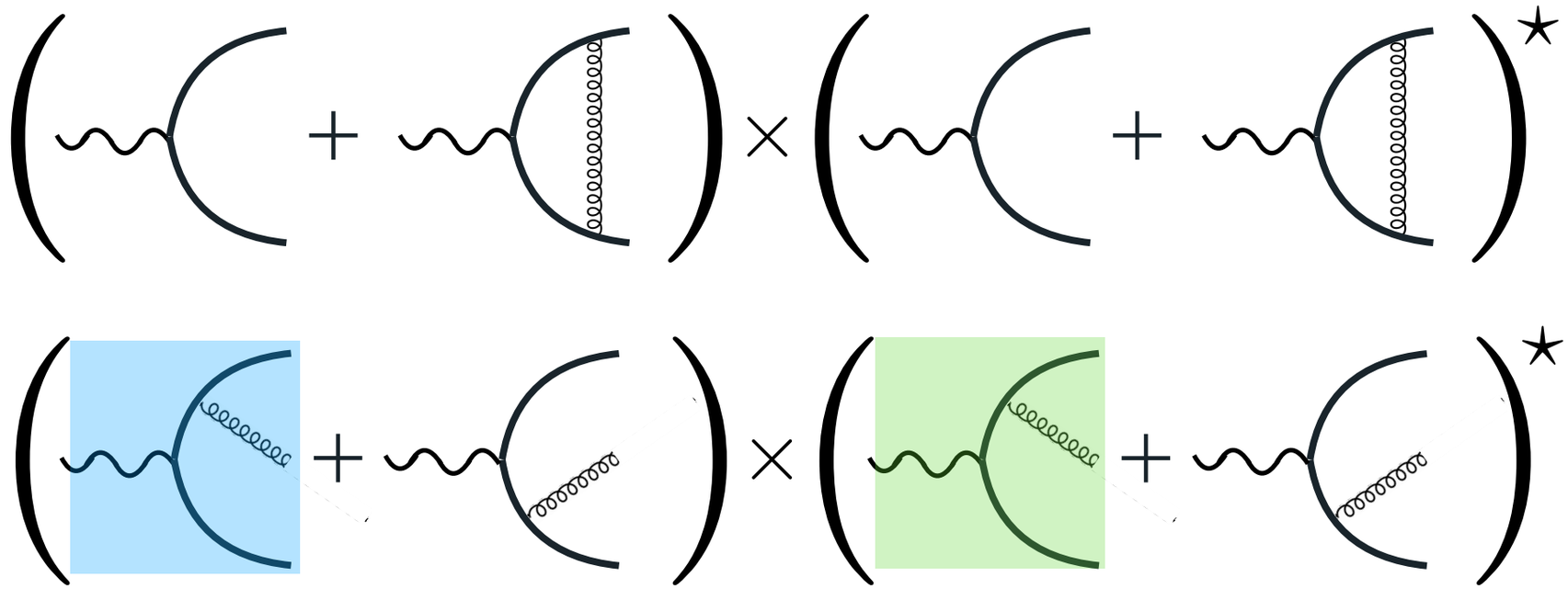

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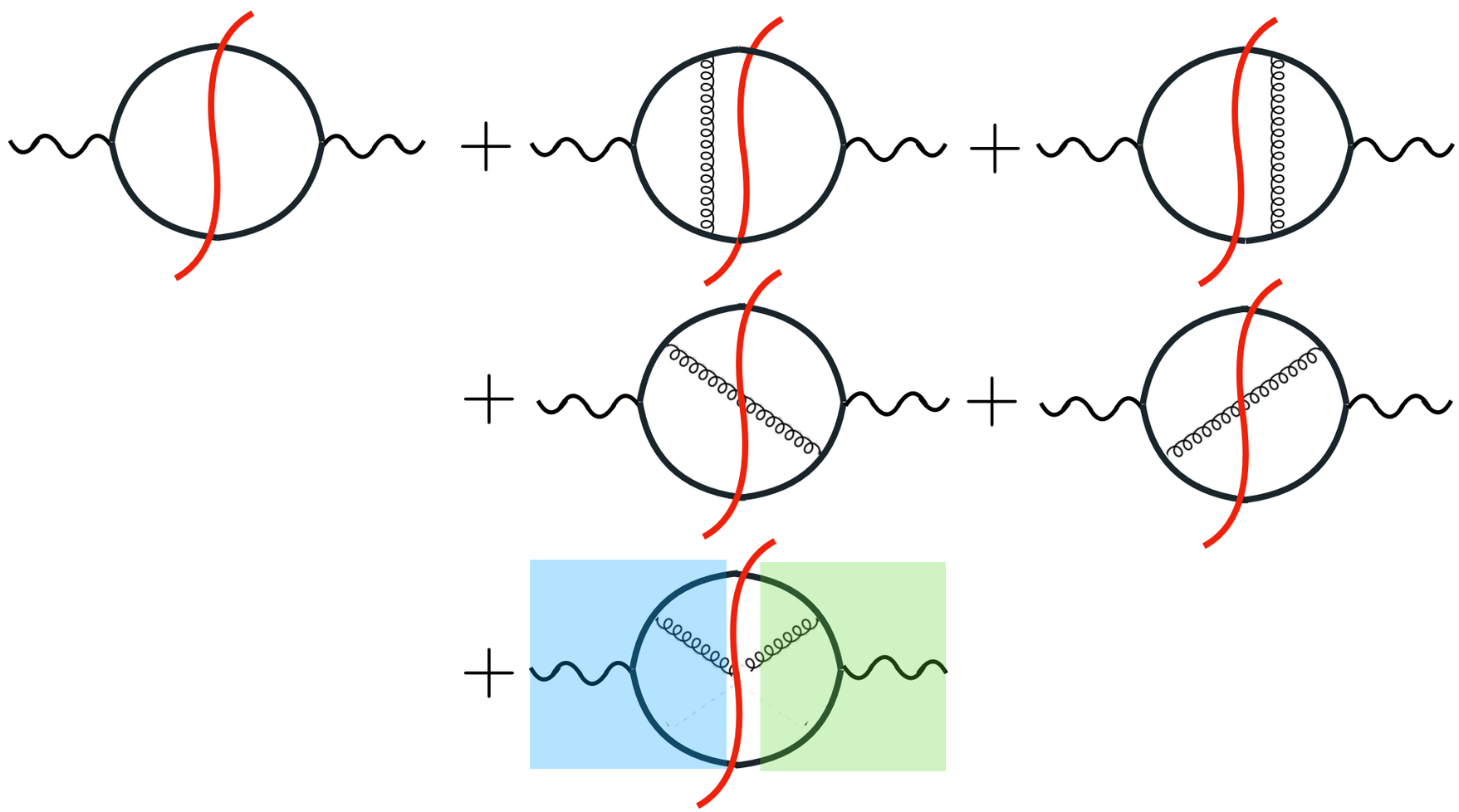

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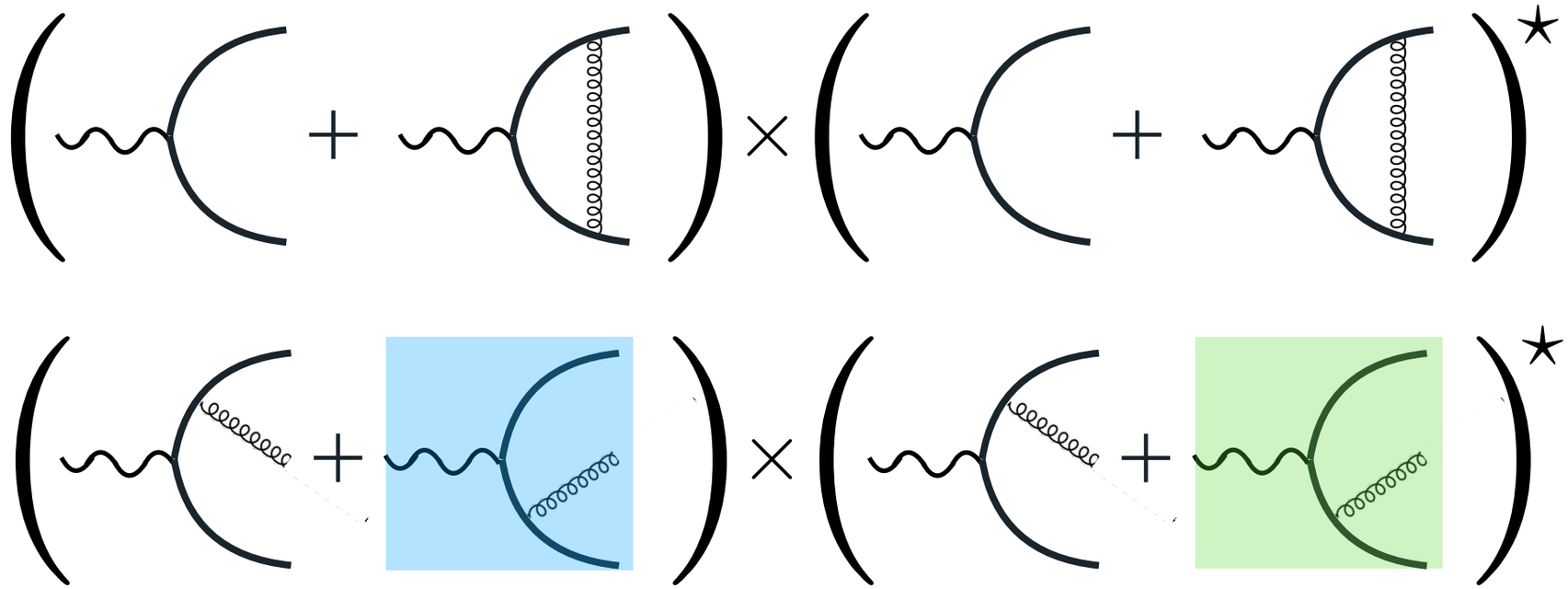

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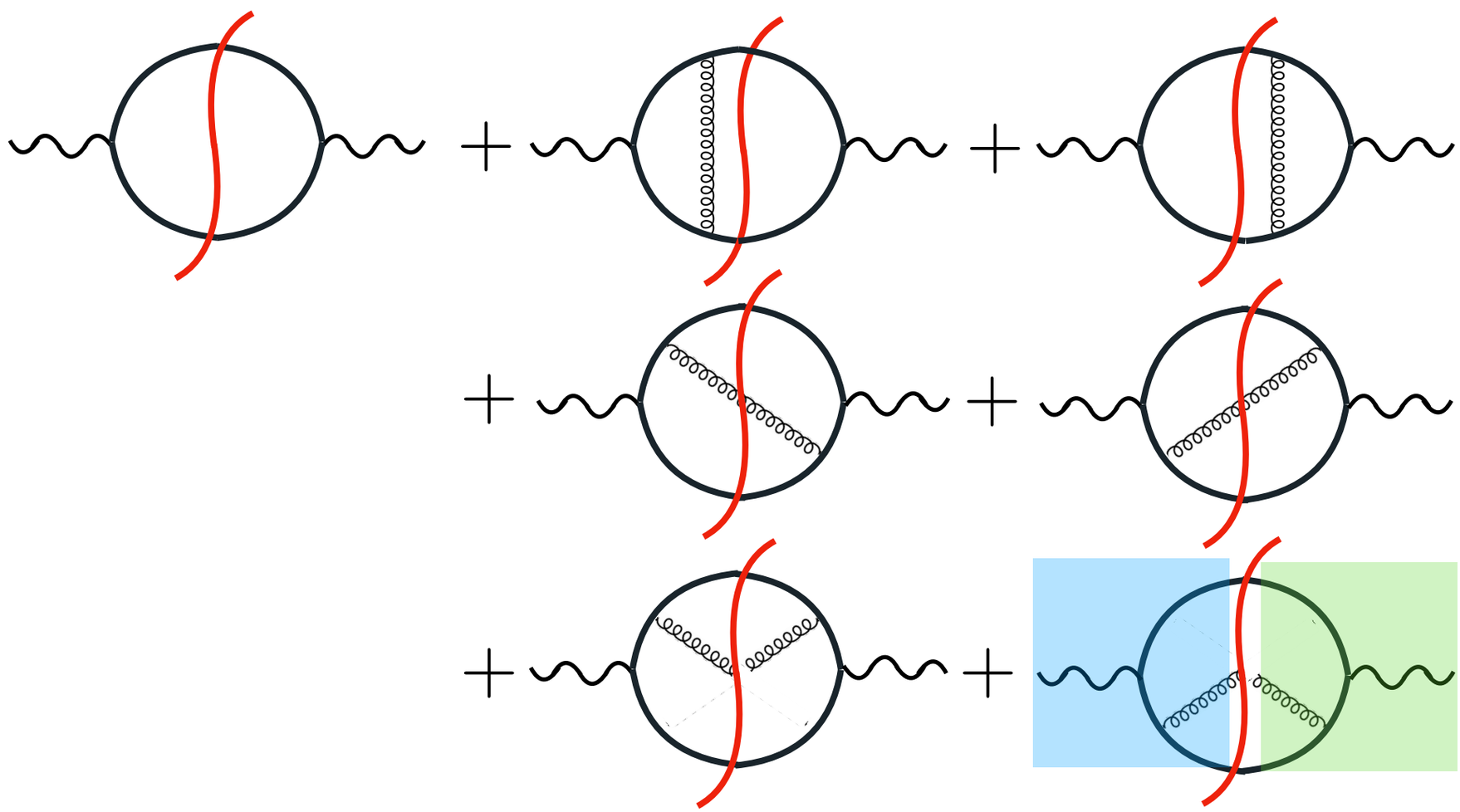

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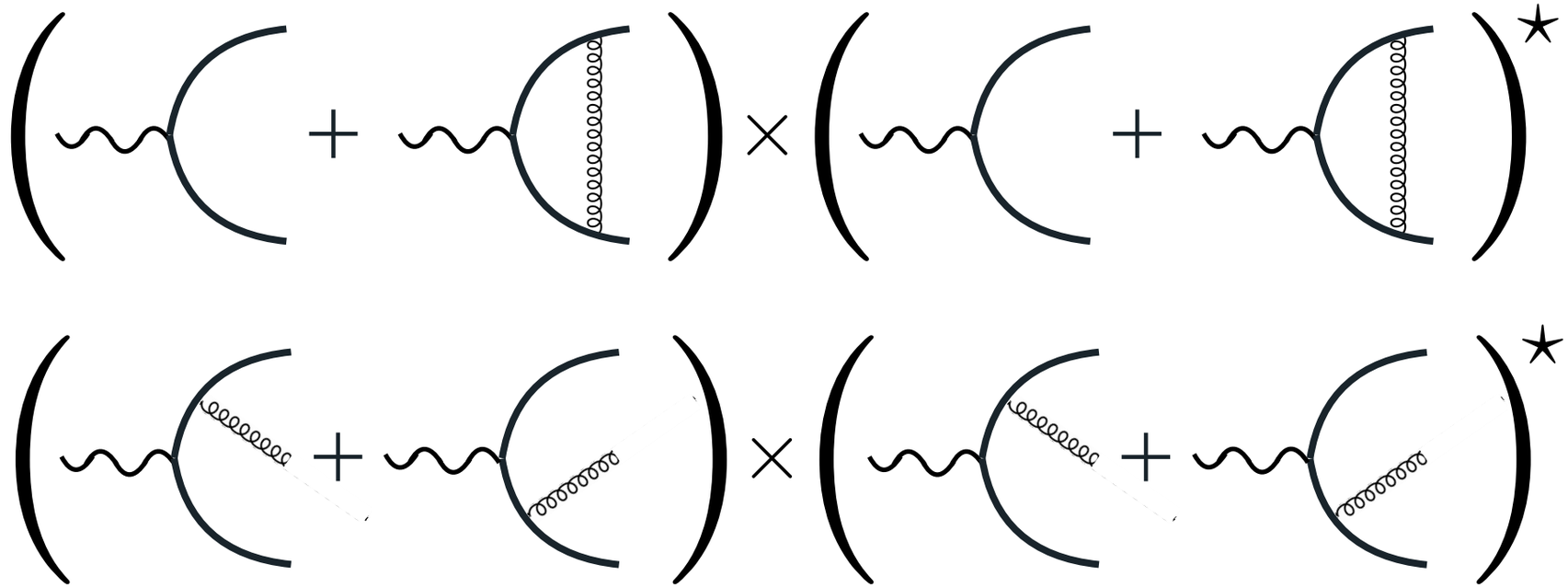

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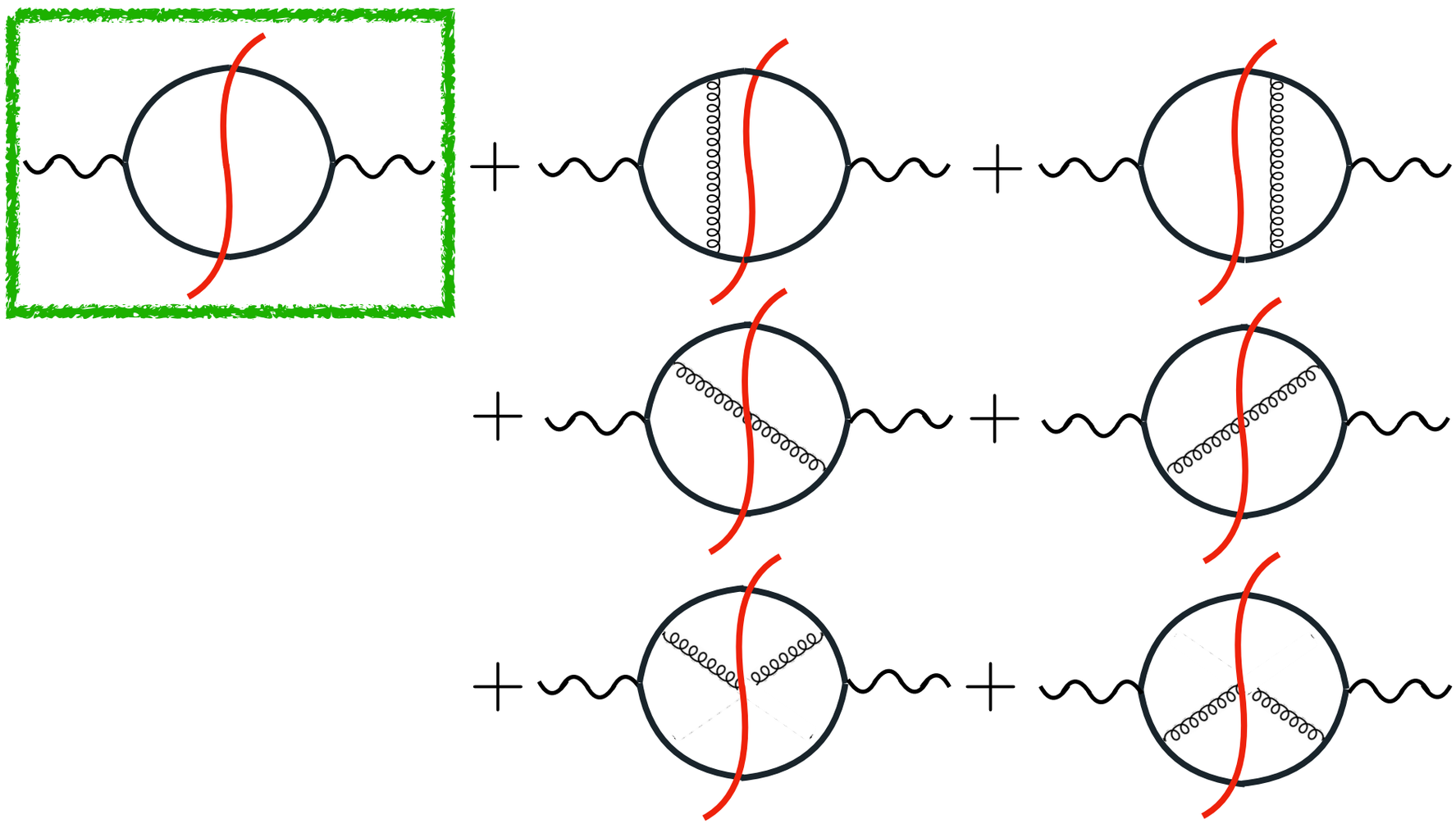
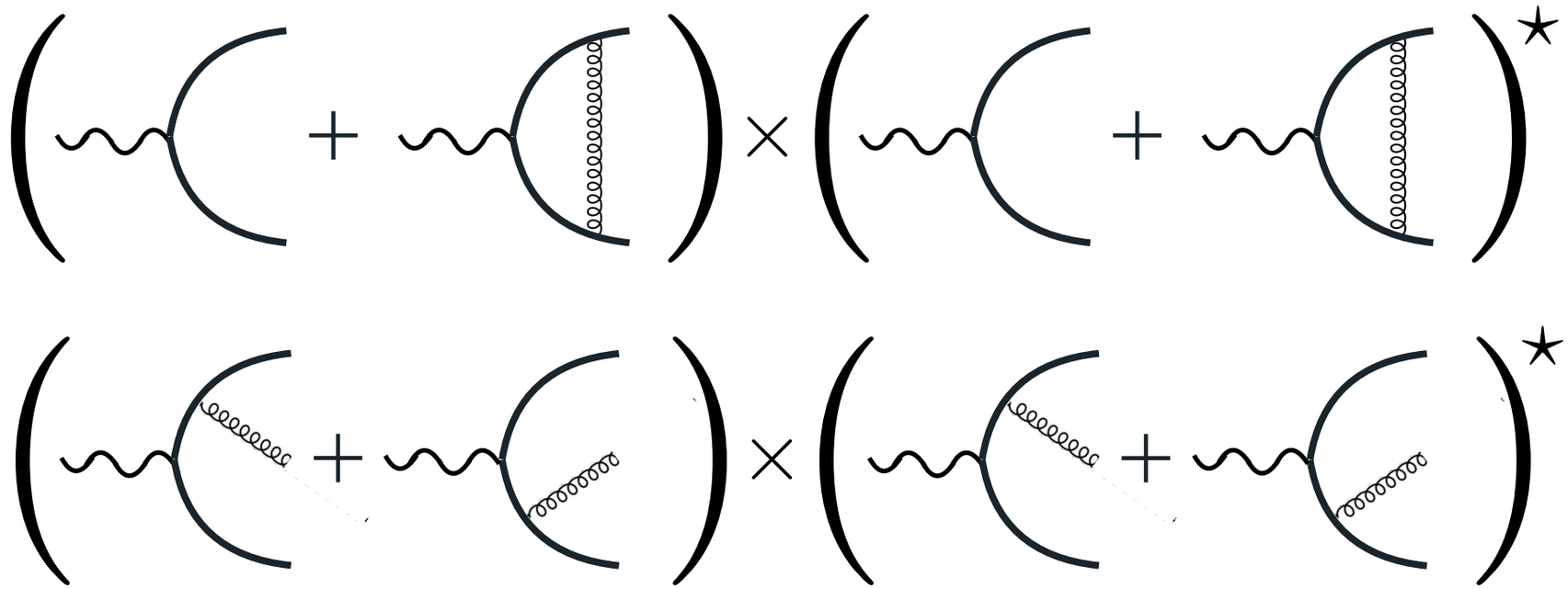


LO

Remedy: forward-scattering graphs

$$\sigma_{\gamma^\star \to d\bar{d}} =$$



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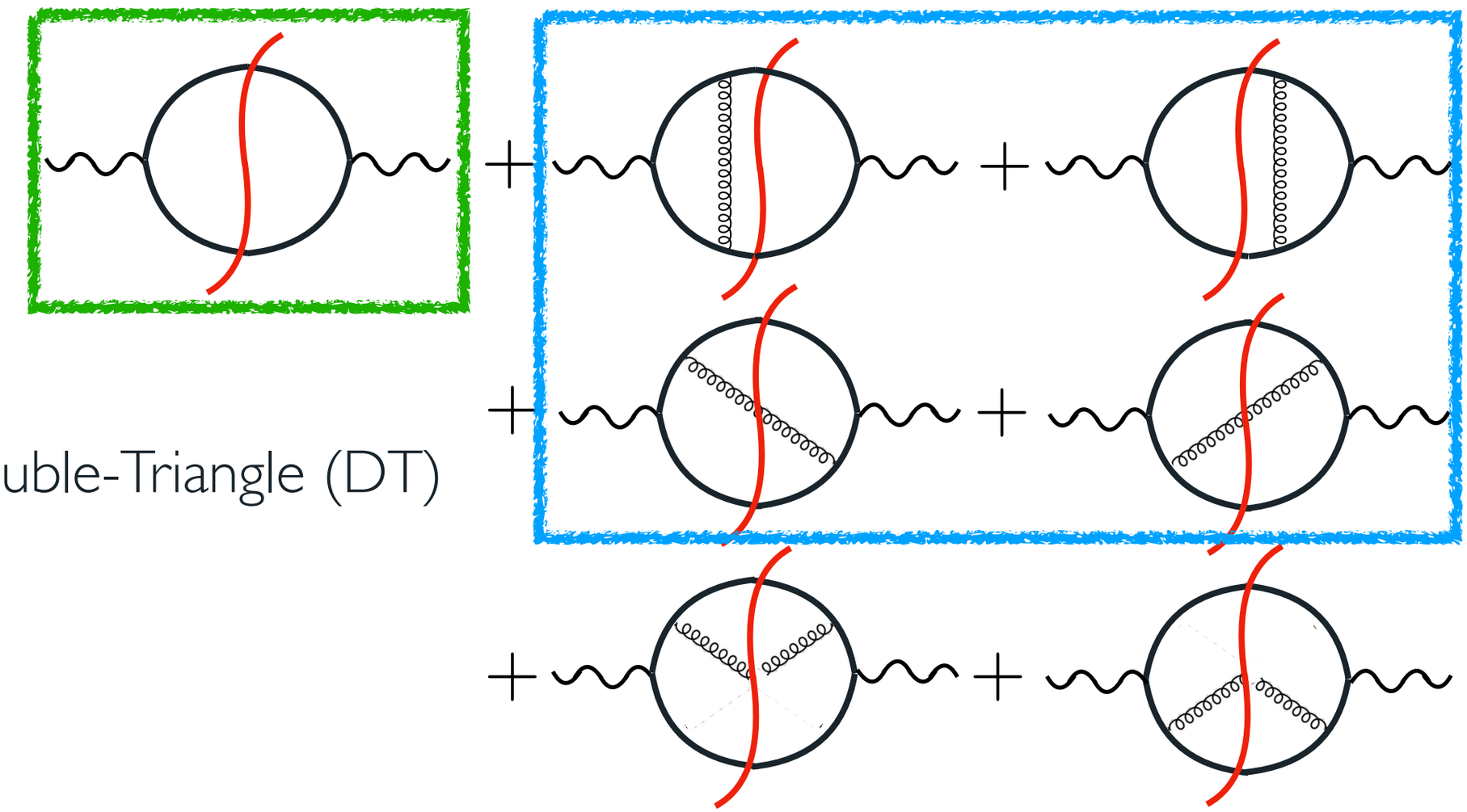
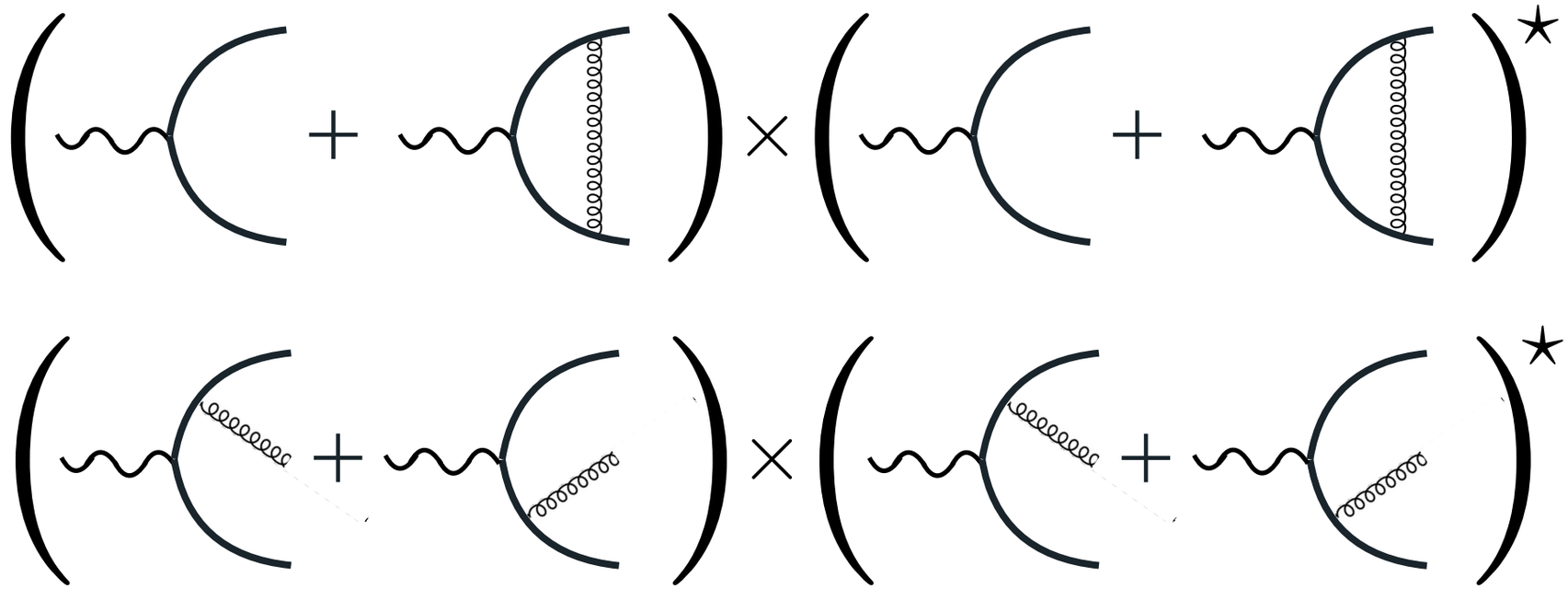


LO

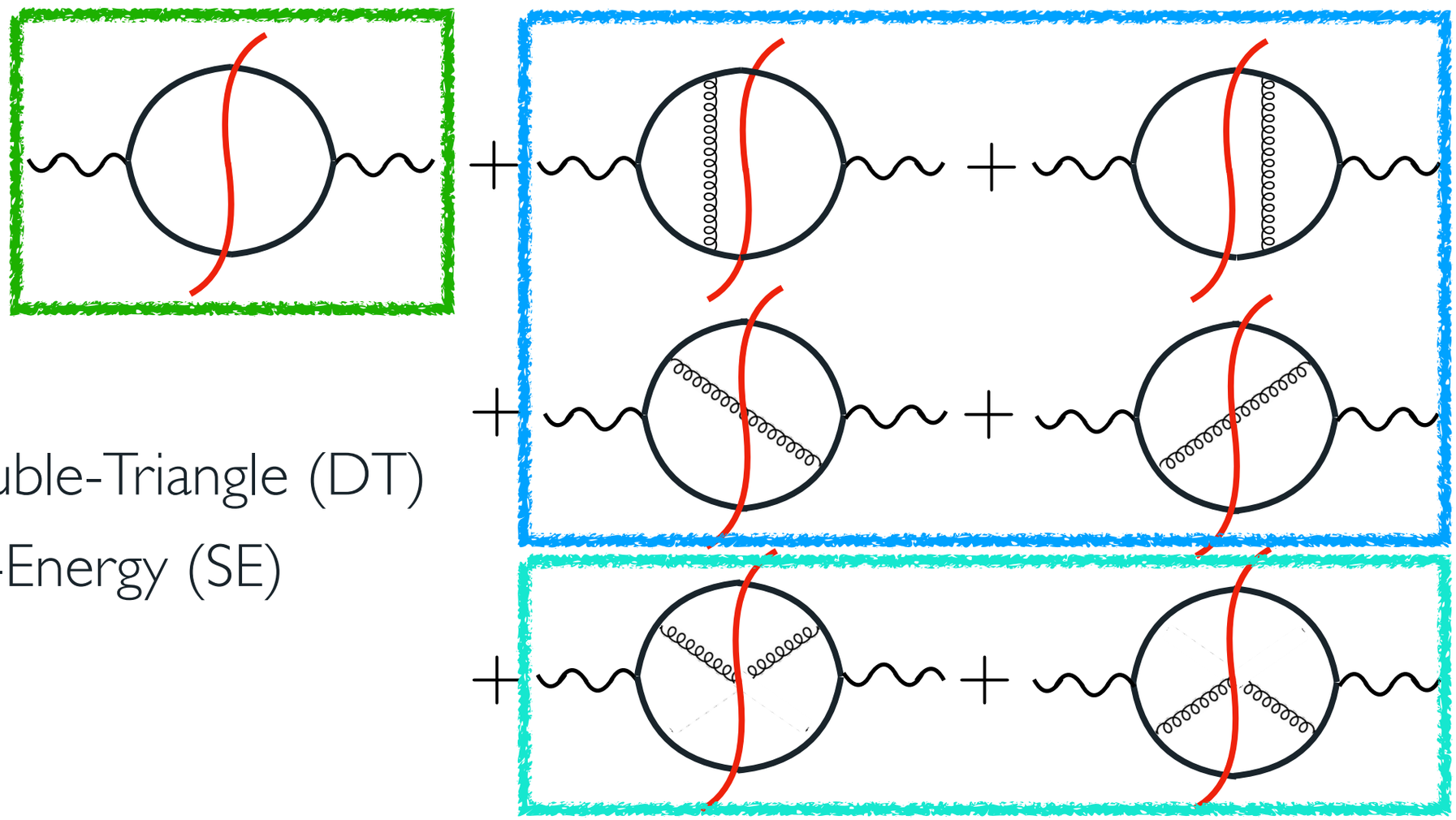
NLO, Double-Triangle (DT)

Remedy: forward-scattering graphs

$$\sigma_{\gamma^\star \to d\bar{d}} =$$



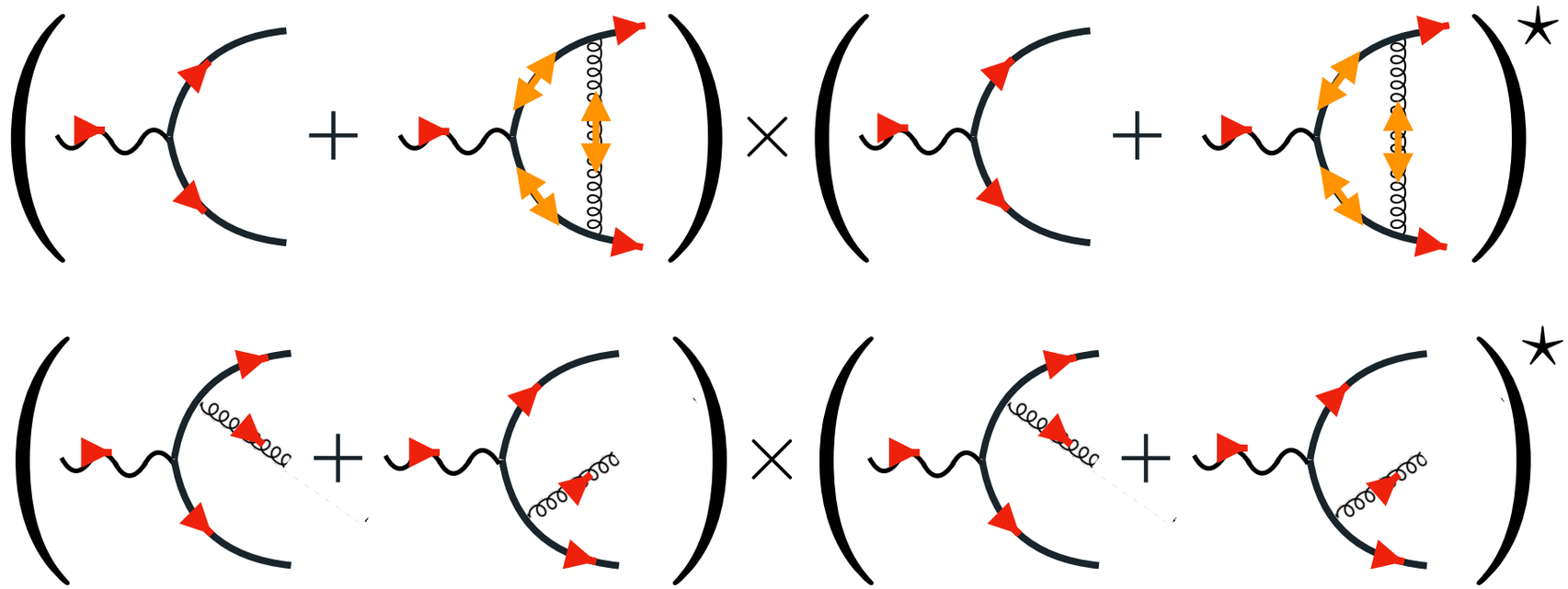
$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$$



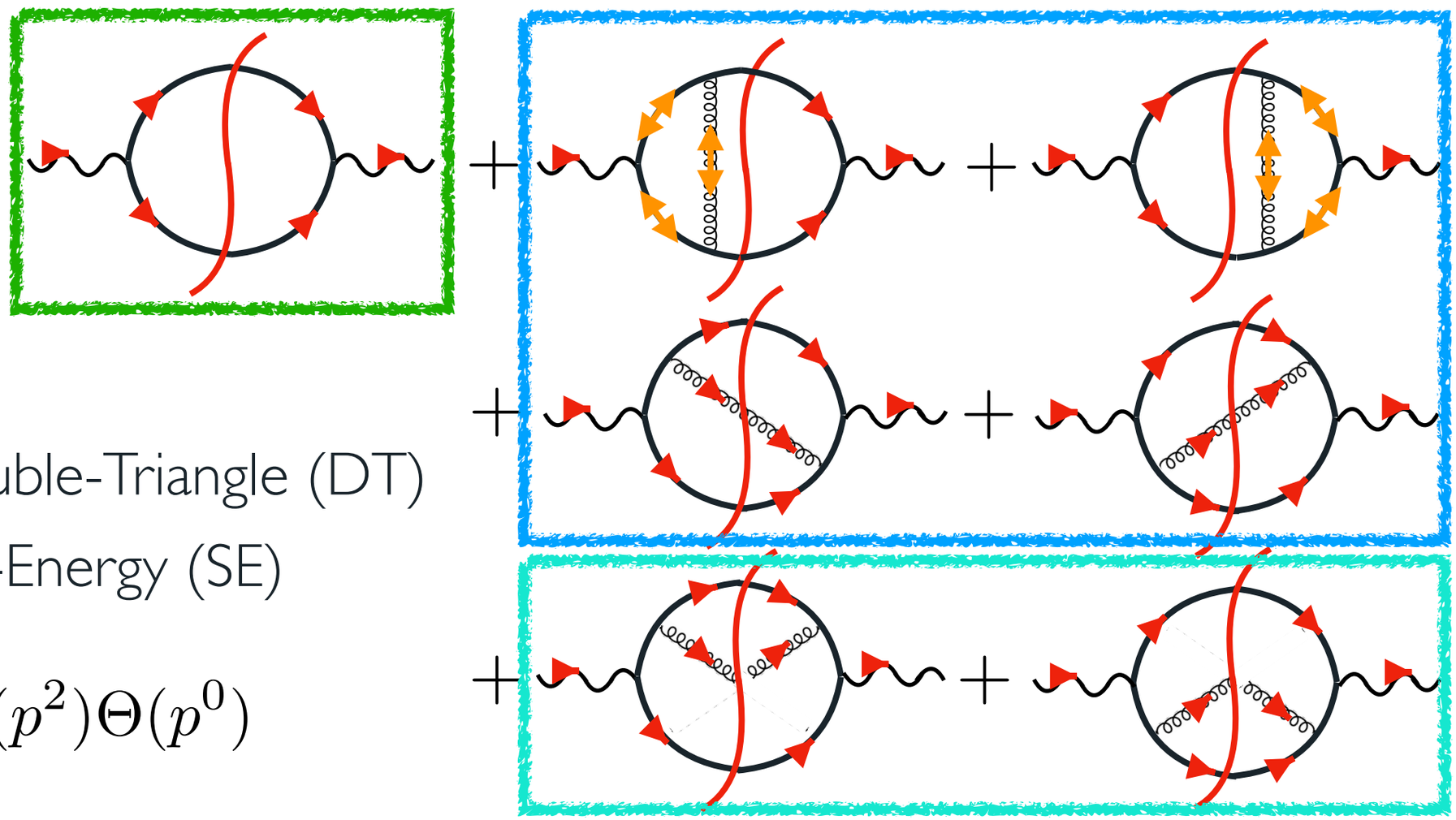
- LO
- **NLO**, Double-Triangle (DT)
- **NLO**, Self-Energy (SE)

Remedy: forward-scattering graphs

$\sigma_{\gamma^\star \to d\bar{d}} =$



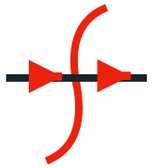
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$



LO

NLO, Double-Triangle (DT)

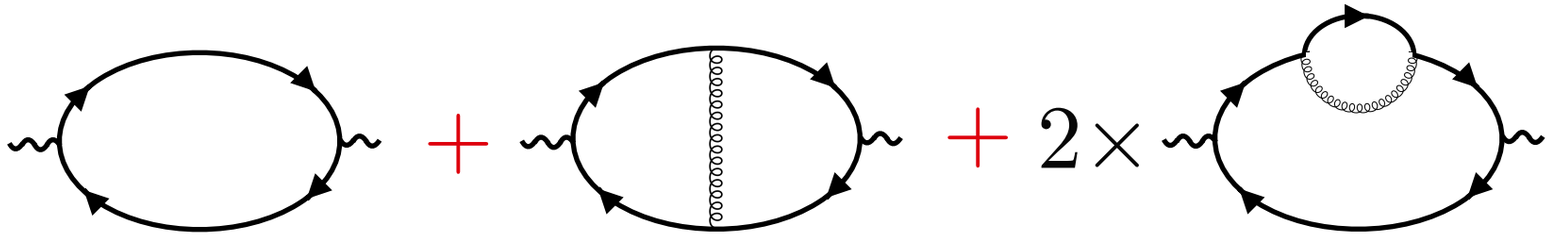
NLO, Self-Energy (SE)



$$\equiv \frac{p^2}{2p^0}\delta(p^2)\Theta(p^0)$$

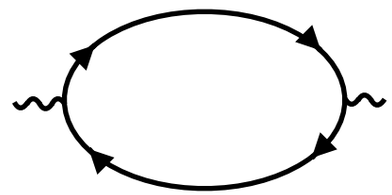
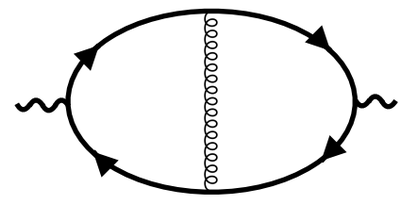
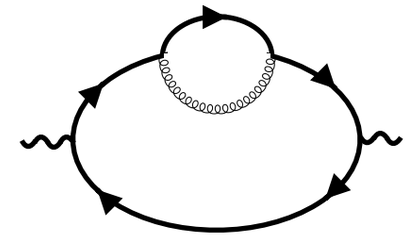
So what? It's just the optical theorem no?

$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$



$$\Big]$$

So what? It's just the optical theorem no?

$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$  $+$  $+\, 2\times$  $\Big]$

At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$

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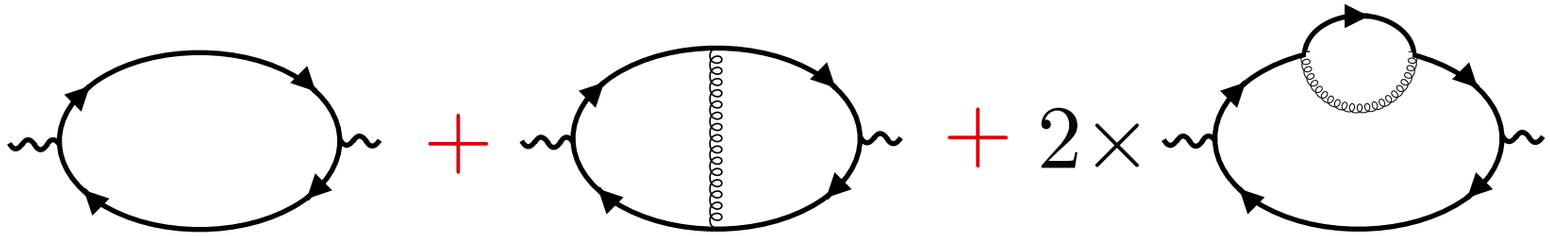
$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$  $$\Big]$$

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The **differential** version of LU is our new paradigm: **Local Unitarity**

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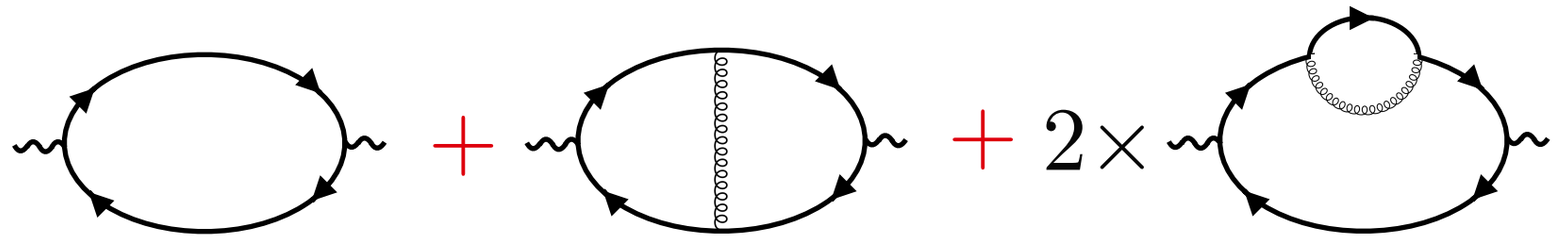
$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$  $$\Big]$$

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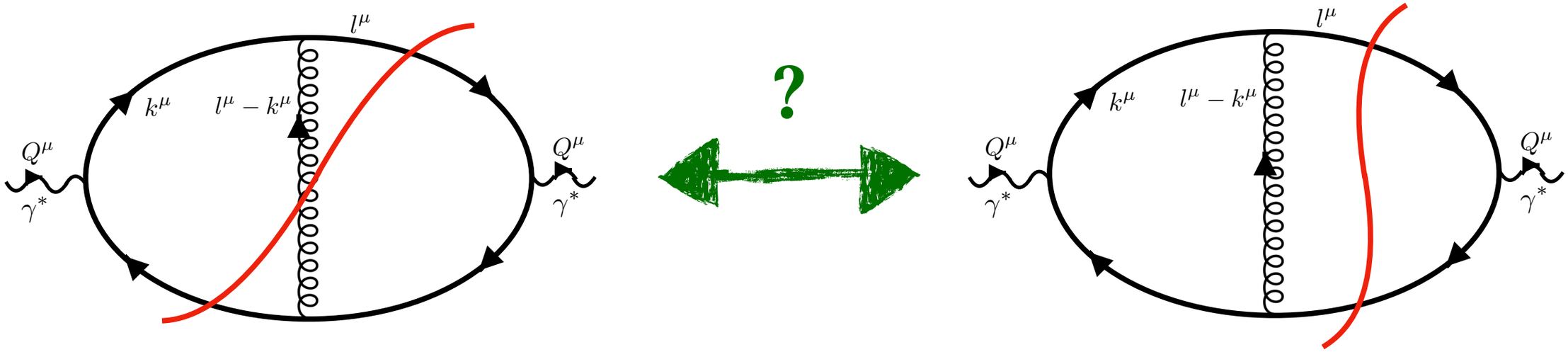
The **differential** version of LU is our new paradigm: **Local Unitarity**

Each supergraph is **individually locally finite**, how you ask?



$$\propto \delta^+(l^2)\delta^-((l-k)^2)\delta^-((k-Q)^2)$$

$$\propto \delta^+(l^2)\,\delta^-((l-Q)^2)$$

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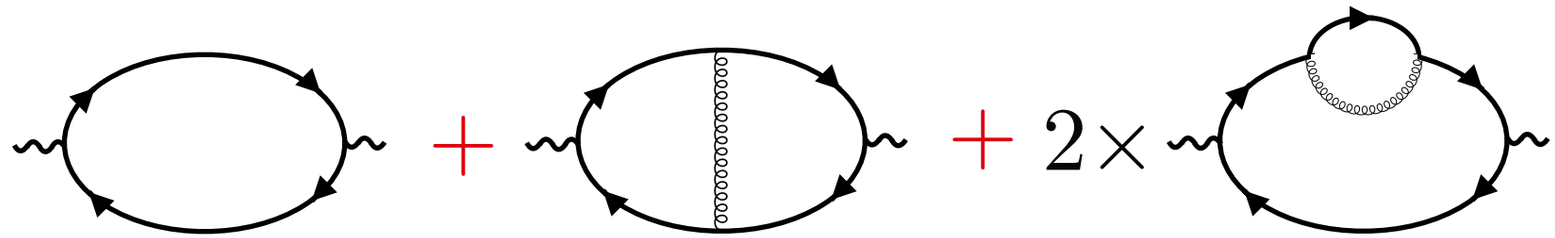
$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$  $$\Big]$$

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- (Multi-)Loop Tree Duality : singularities / deformation
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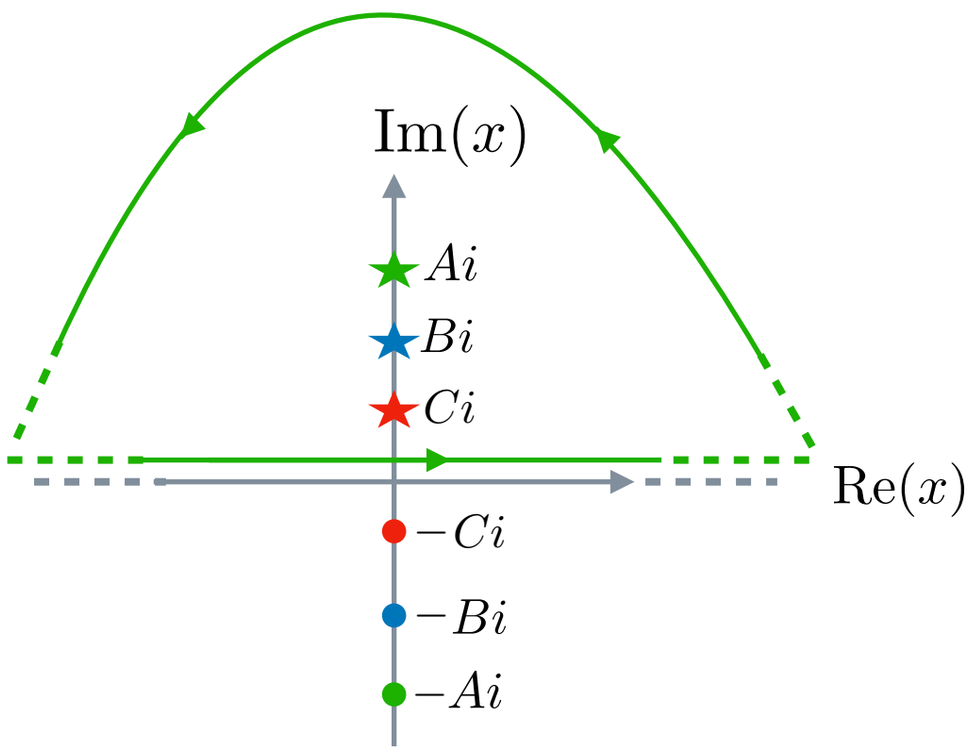
- Local Unitarity : numerical results
- Conclusion

(ONE)-LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx F(x) \qquad F(x) = \frac{1}{x^2 + A^2} \frac{1}{x^2 + B^2} \frac{1}{x^2 + C^2}$$

$$F(x) = \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(x - Bi)(x + Bi)} \frac{1}{(x - Ci)(x + Ci)}$$

$$\big(\text{Assumptions} \rightarrow \{A > 0, B > 0, C > 0\}\big)$$



Cauchy: $\left(R(x^\star) \equiv \mathrm{Res}(F, x = x^\star) \right)$

$$I = (-2\pi i)[R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(one-)Loop Tree duality

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta}$$

$$= \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

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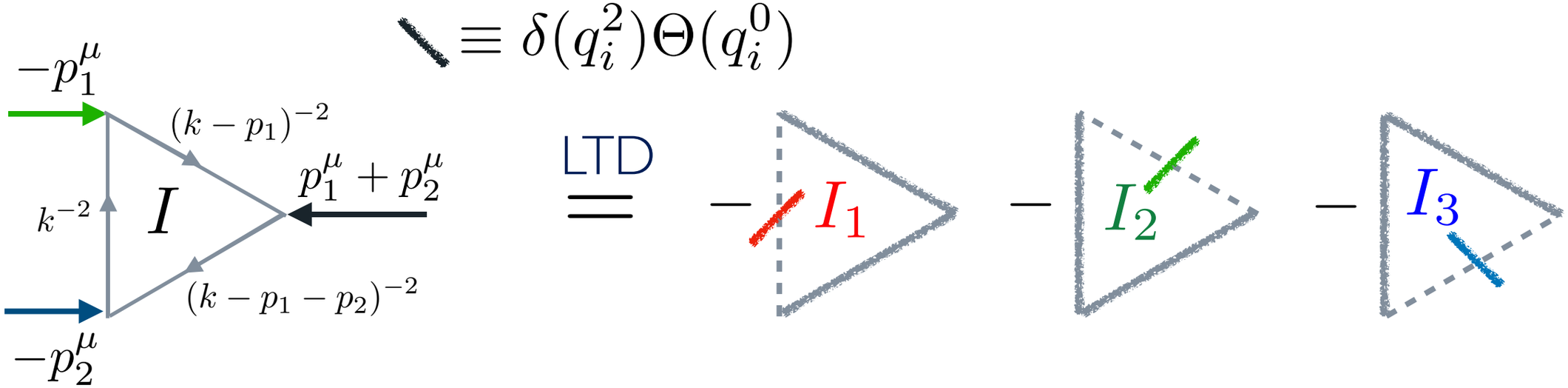
Analogous 1-loop triangle case:



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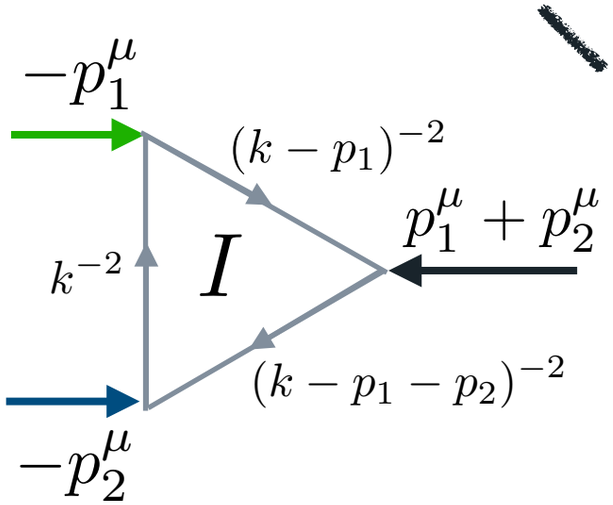
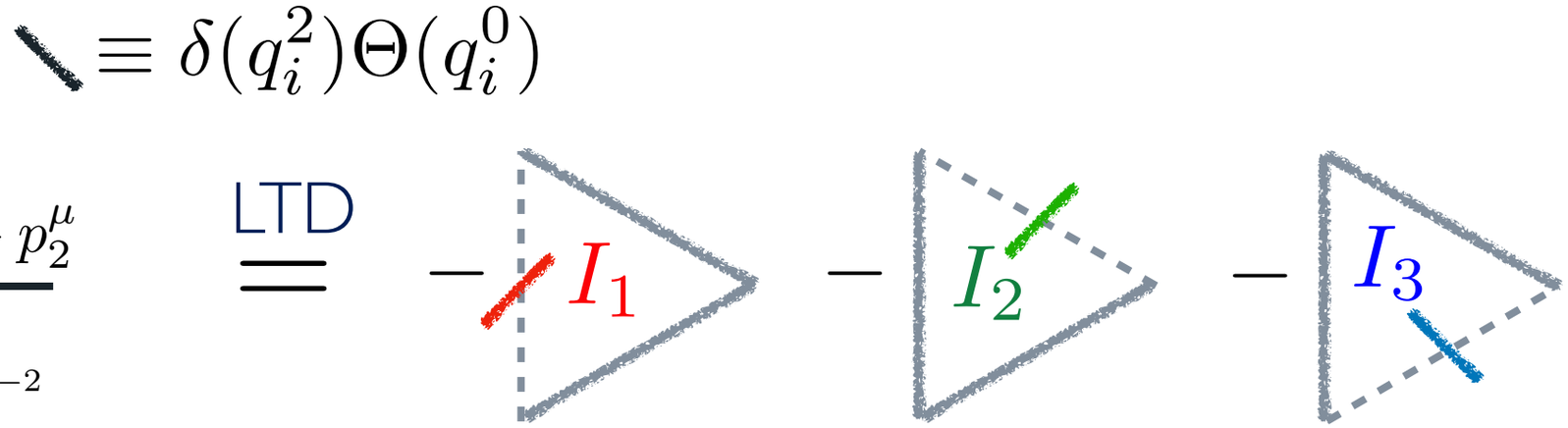




General 1-loop case:

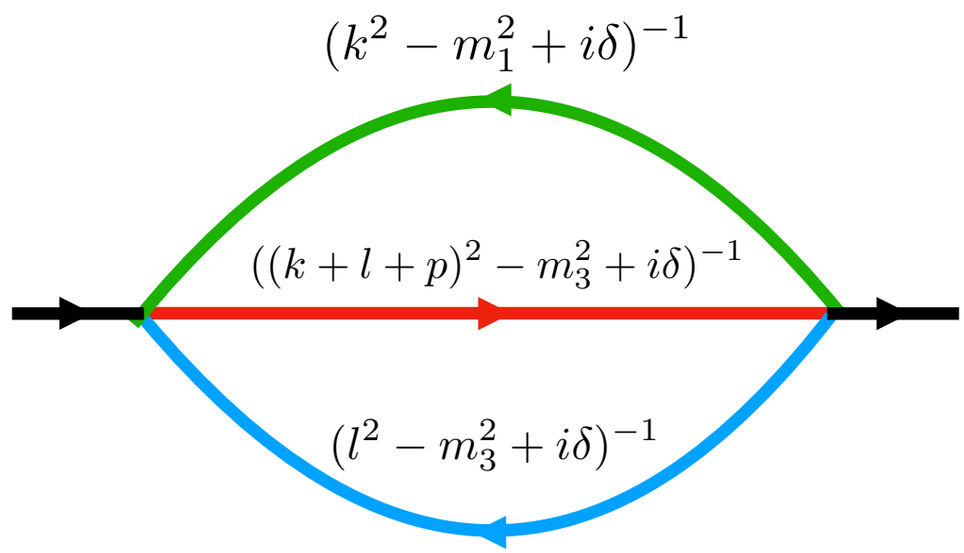
[Catani & al., arxiv:0804.3170]

$$I = -\int_q \sum_{i=1}^{N} \widetilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)}$$

Multi-Loop Tree Duality

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]

Analogous sunrise case:



$$I = (2\pi i)^2 \Big[\quad \mathtt{R(+1,+1,\ 0)} + \mathtt{R(-1,\ 0,+1)} + \mathtt{R(\ 0,+1,+1)} \Big]$$

Now, what about beyond two-loop and arbitrary topologies ?

- Distributional identities: [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- Averaging procedure: [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- Iterative procedure: [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]

Multi-Loop Tree Duality

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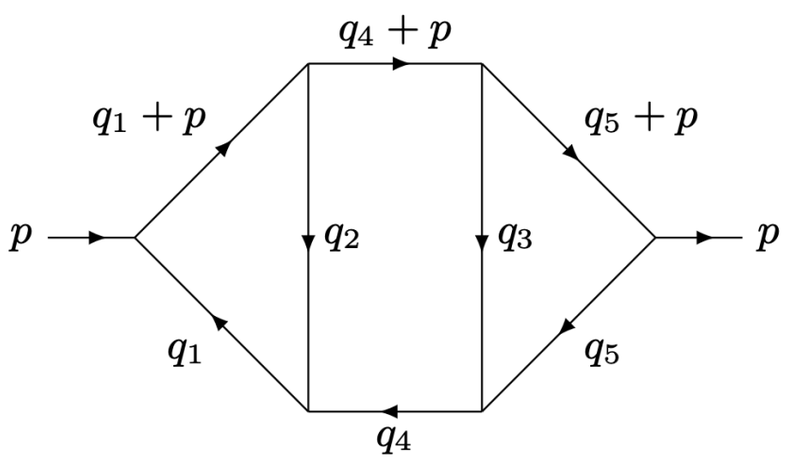
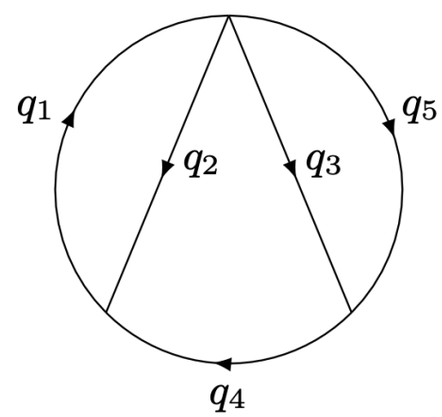
Three-loop example (showcasing the irrelevance of external momenta).



(a) Original diagram.



(b) Reduced diagram.

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\begin{aligned}
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(c) Momenta and signatures.

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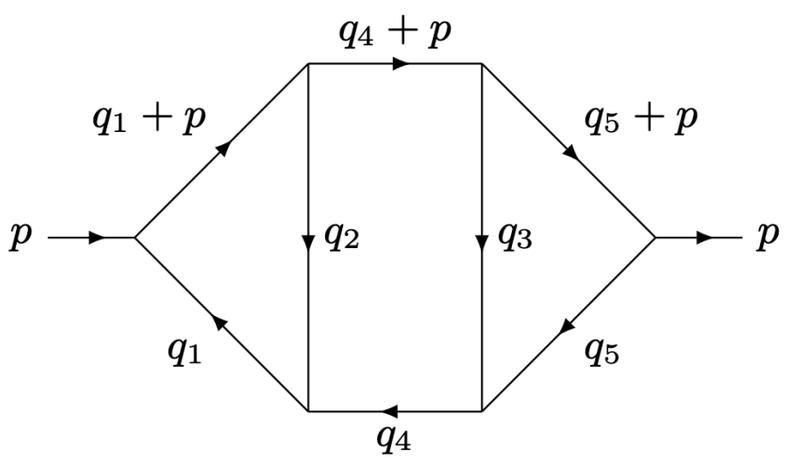

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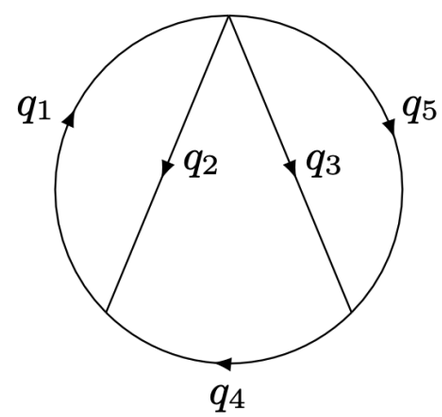

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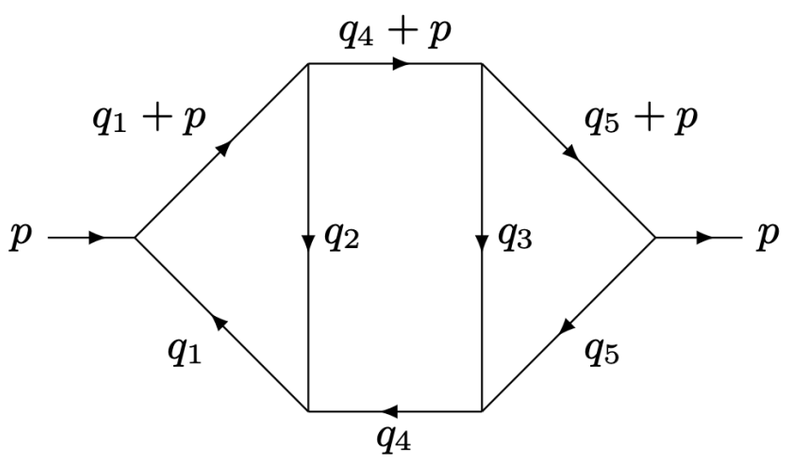

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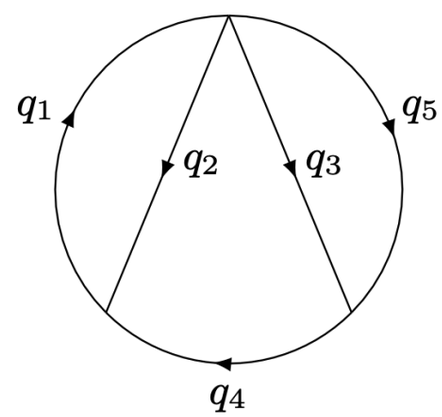

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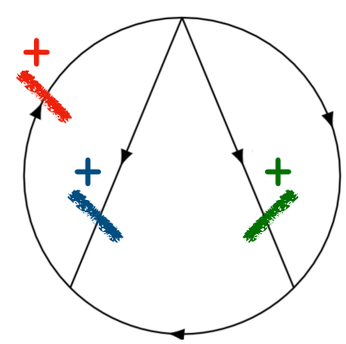

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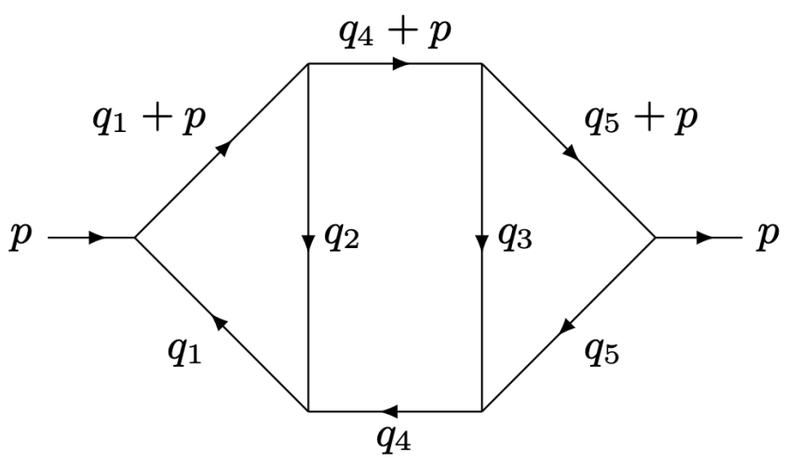

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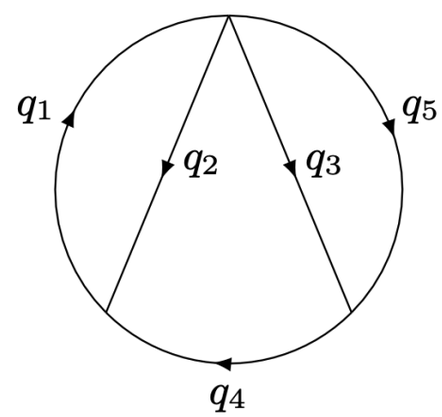

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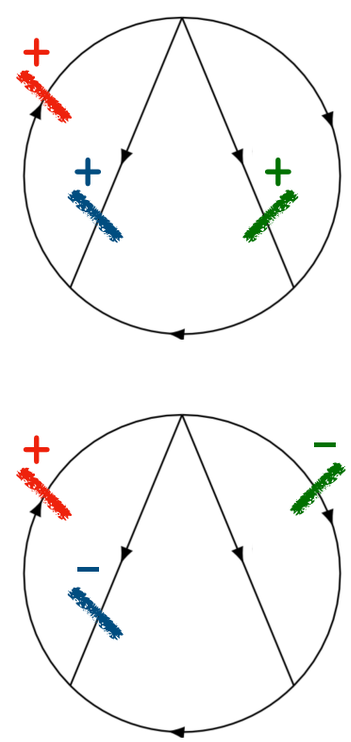

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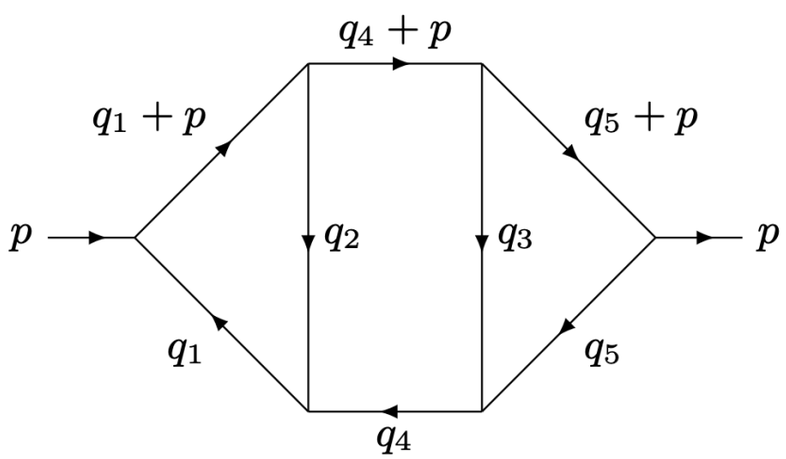

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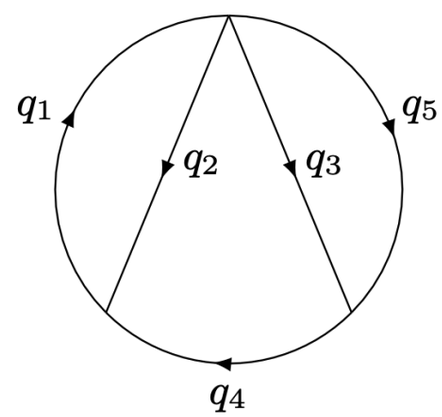

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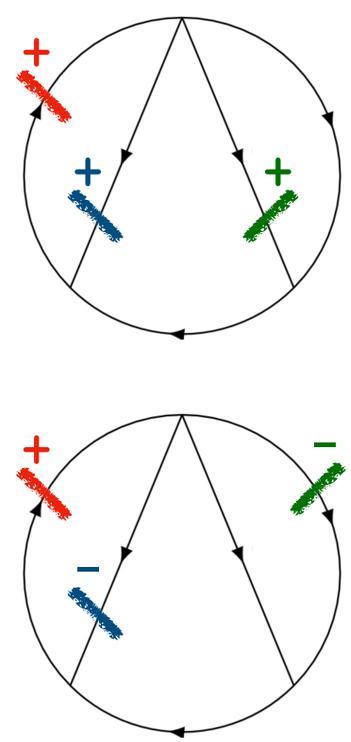
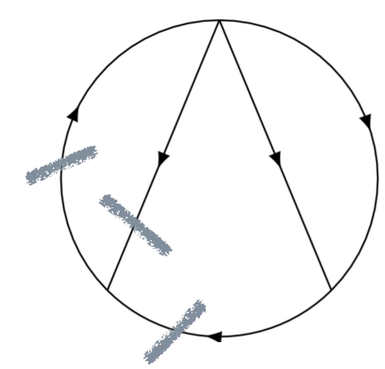
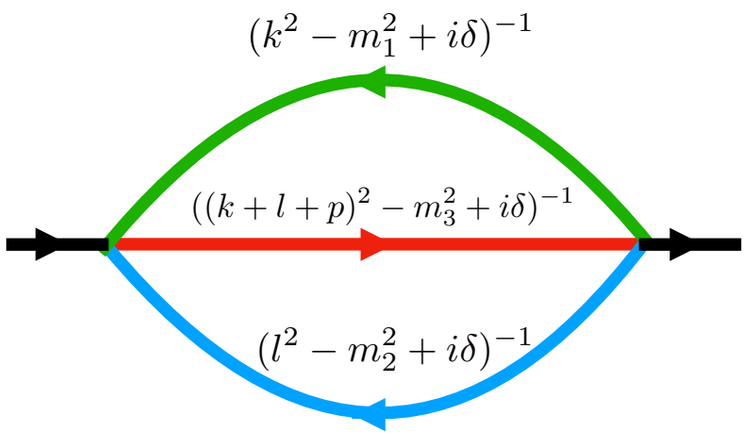


Ex. of an invalid spanning tree:



Multi-Loop Tree Duality: Singularities

The sunrise again, with more details:



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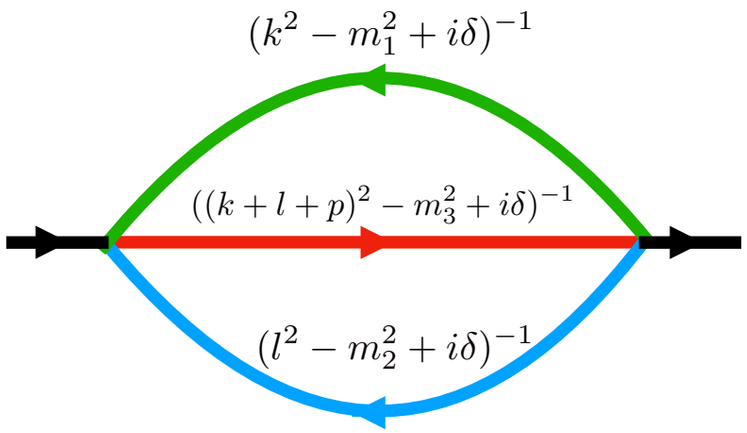

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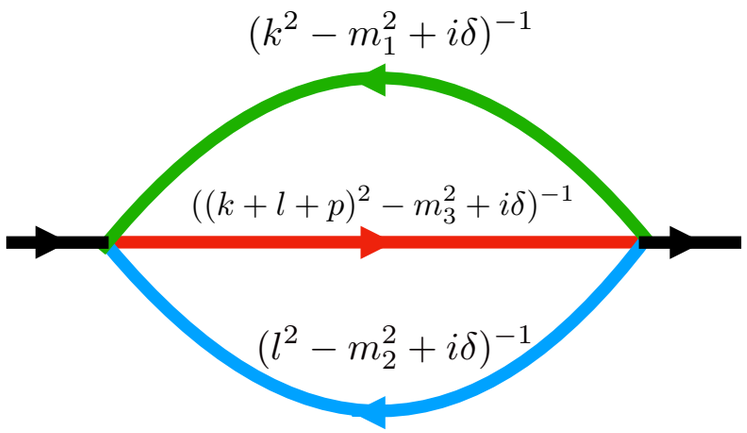

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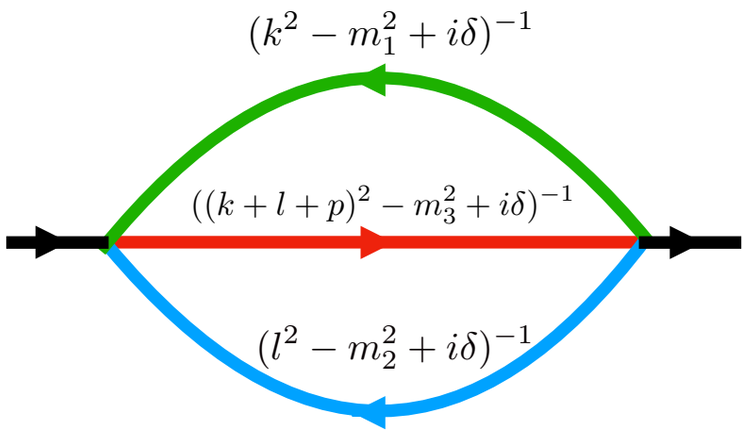

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identical signs : bounded ellipsoid-like surface requiring deformation: E-surfaces

otherwise : unbounded hyperboloid-like surface dual-cancelling: H-surfaces

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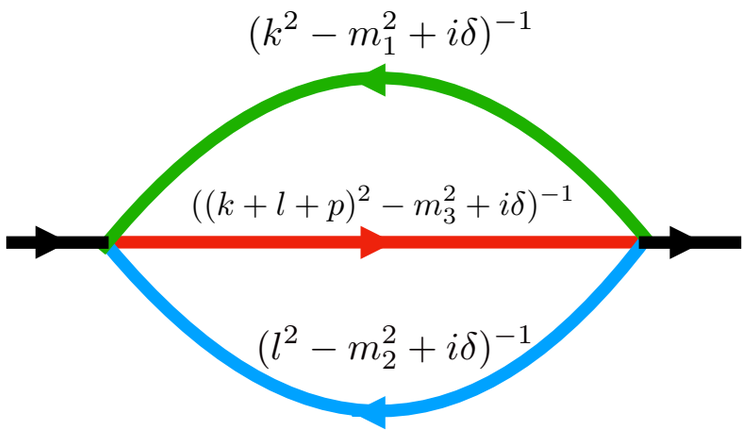

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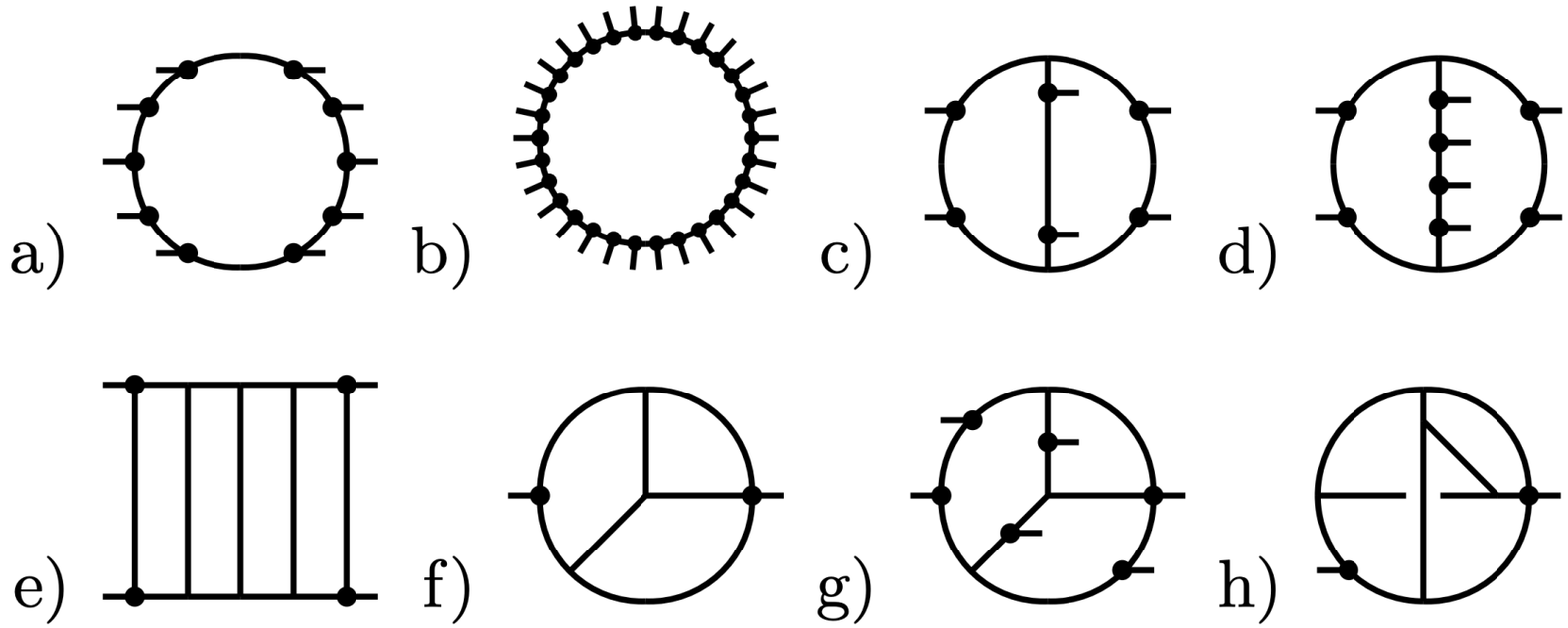
For Euclidean kinematics ($p_i^2 < 0$) : (on-shell energies: $E_i(\vec{k}_i) = \sqrt{\vec{k}_i^2 + m_i^2 - i\delta}$)

E-surface has no solution ➔ no non-integrable singularity ➔ no deformation needed

Numerical application - Euclidean case

[Capatti, VH, Kermanschah, Ruijl, arxiv:1906.06138]



G	Reference		Numerical LTD		N $[10^6]$	$[\mu s]$
a)*	[33]	$\mathrm{i}\,4.31638 \cdot 10^{-7}$	$\mathrm{i}\,4.31637(19) \cdot 10^{-7}$		110	1.1
b)	[33]	$\mathrm{i}\,0.358640$	$\mathrm{i}\,0.358646(29)$		210	5.9
c)	[7]	$1.1339(5) \cdot 10^{-4}$	$1.133719(58) \cdot 10^{-4}$		5500	2.5
c)*	[7]	$4.398(1) \cdot 10^{-8}$	$4.39825(17) \cdot 10^{-8}$		5500	2.5
d)*	[7]	$2.409(1) \cdot 10^{-8}$	$2.40869(27) \cdot 10^{-8}$		5500	3.5
e)	[34]	$-1.433521 \cdot 10^{-6}$	$-1.4338(18) \cdot 10^{-6}$		1500	27.4
f)	[35]	$\mathrm{i}\,5.26647 \cdot 10^{-6}$	$\mathrm{i}\,5.236(38) \cdot 10^{-6}$		7000	3.3
g)*	[7]	$\mathrm{i}\,1.7790(6) \cdot 10^{-10}$	$\mathrm{i}\,1.77648(48) \cdot 10^{-10}$		22000	11
h)	[35]	$-8.36515 \cdot 10^{-8}$	$-8.309(31) \cdot 10^{-8}$		7000	15.8

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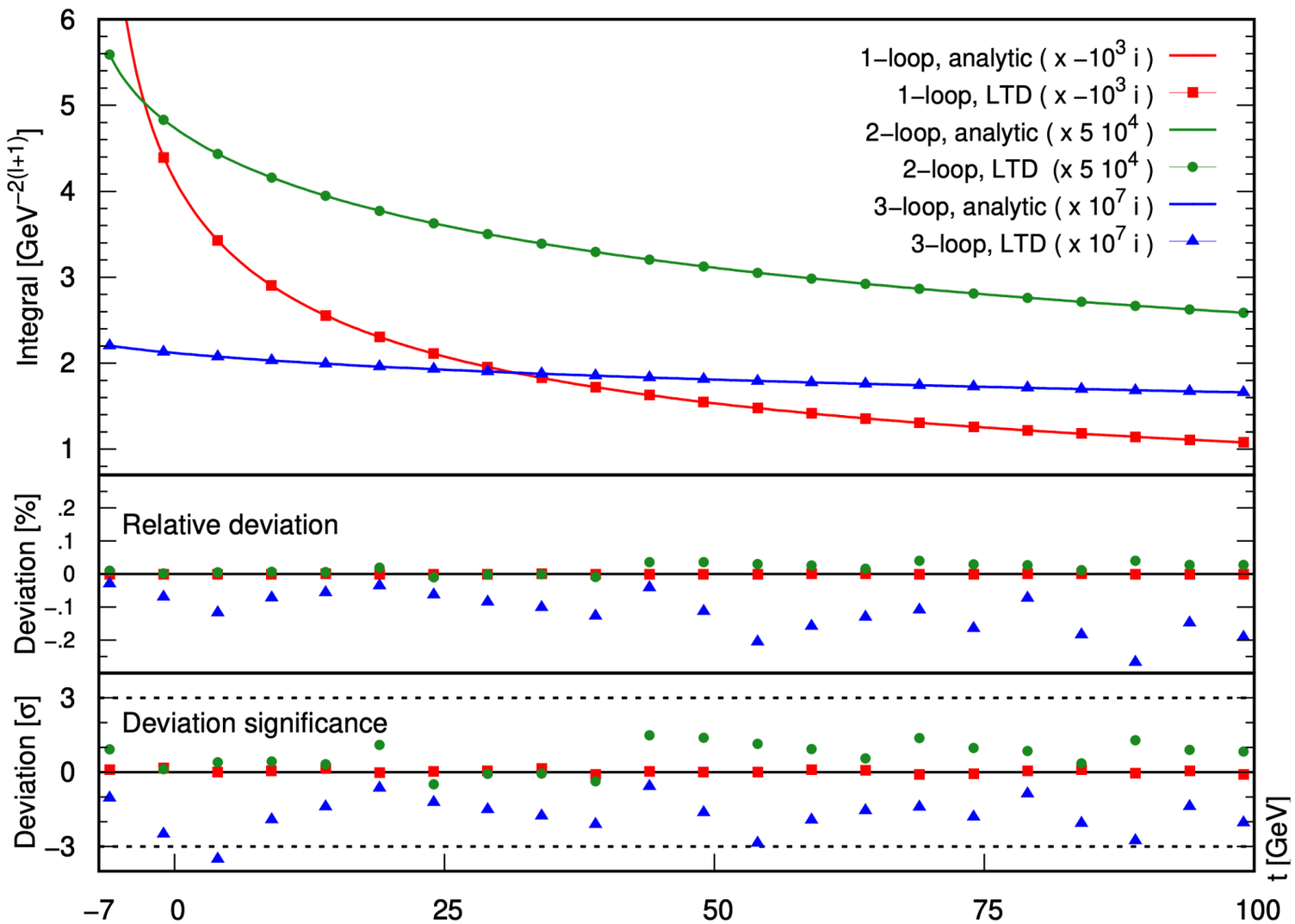


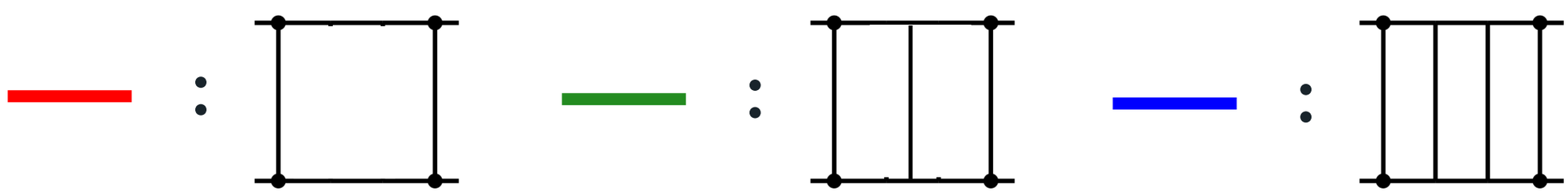
4-point scalar multi-loop scan:

$$p_1^2 = -5 \qquad p_{i=2,3,4}^2 = s = -1$$

$$t \in [-7, 100]$$

Analytic results from:

[Usyukina,Davydychev, PLB, 1993]



Plan

- Challenges in pushing fixed-order accuracy
- Local Unitarity : general concepts
- (Multi-)Loop Tree Duality : formulation
- (Multi-)Loop Tree Duality : singularities / deformation
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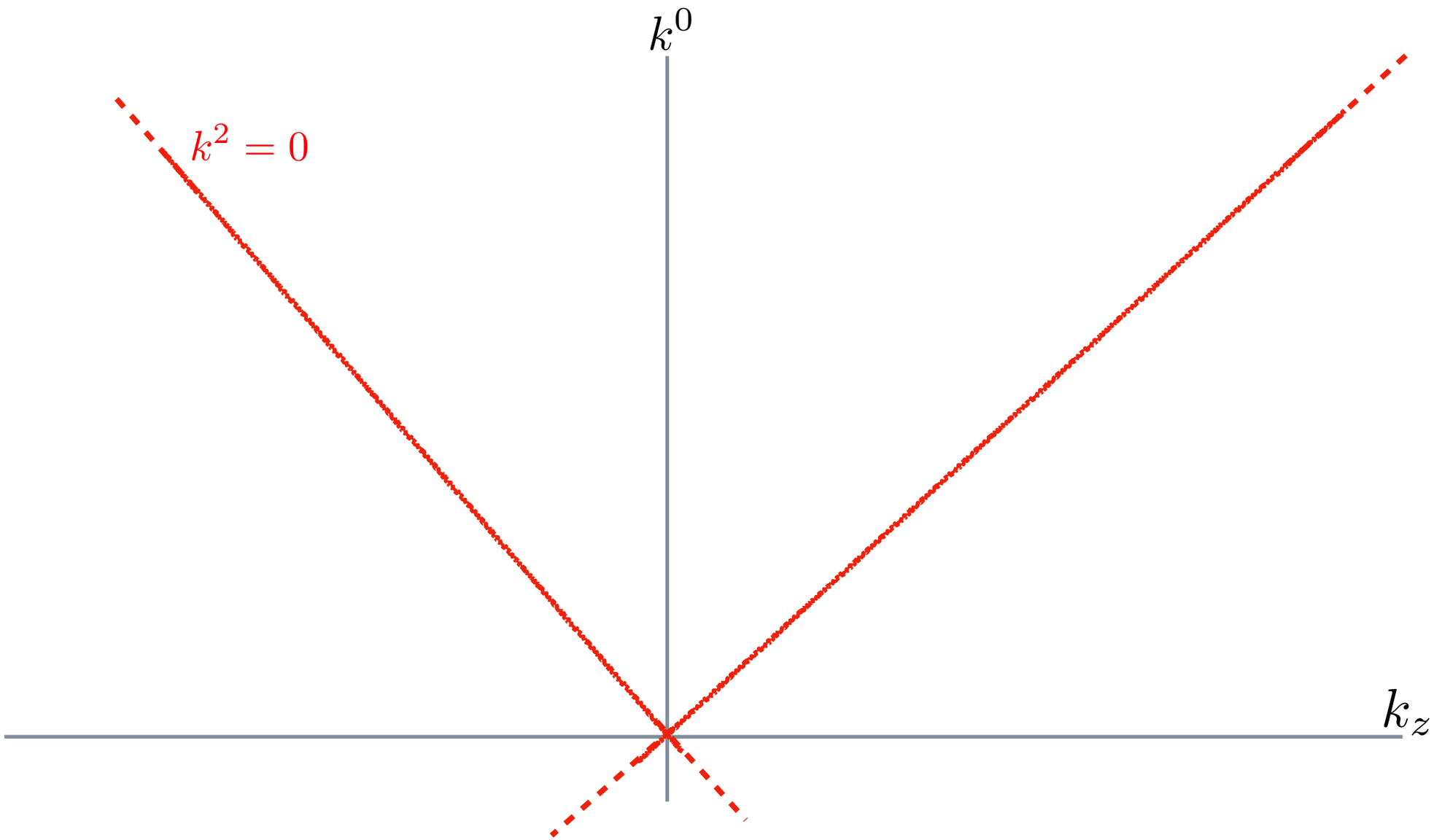
- Local Unitarity : numerical results
- Conclusion

Singular surfaces in Minkowski space

$$\int d^4k \frac{1}{k^2(k-p_1)^2(k-p_1-p_2)^2}$$



The integrand is singular along each of the coloured surface

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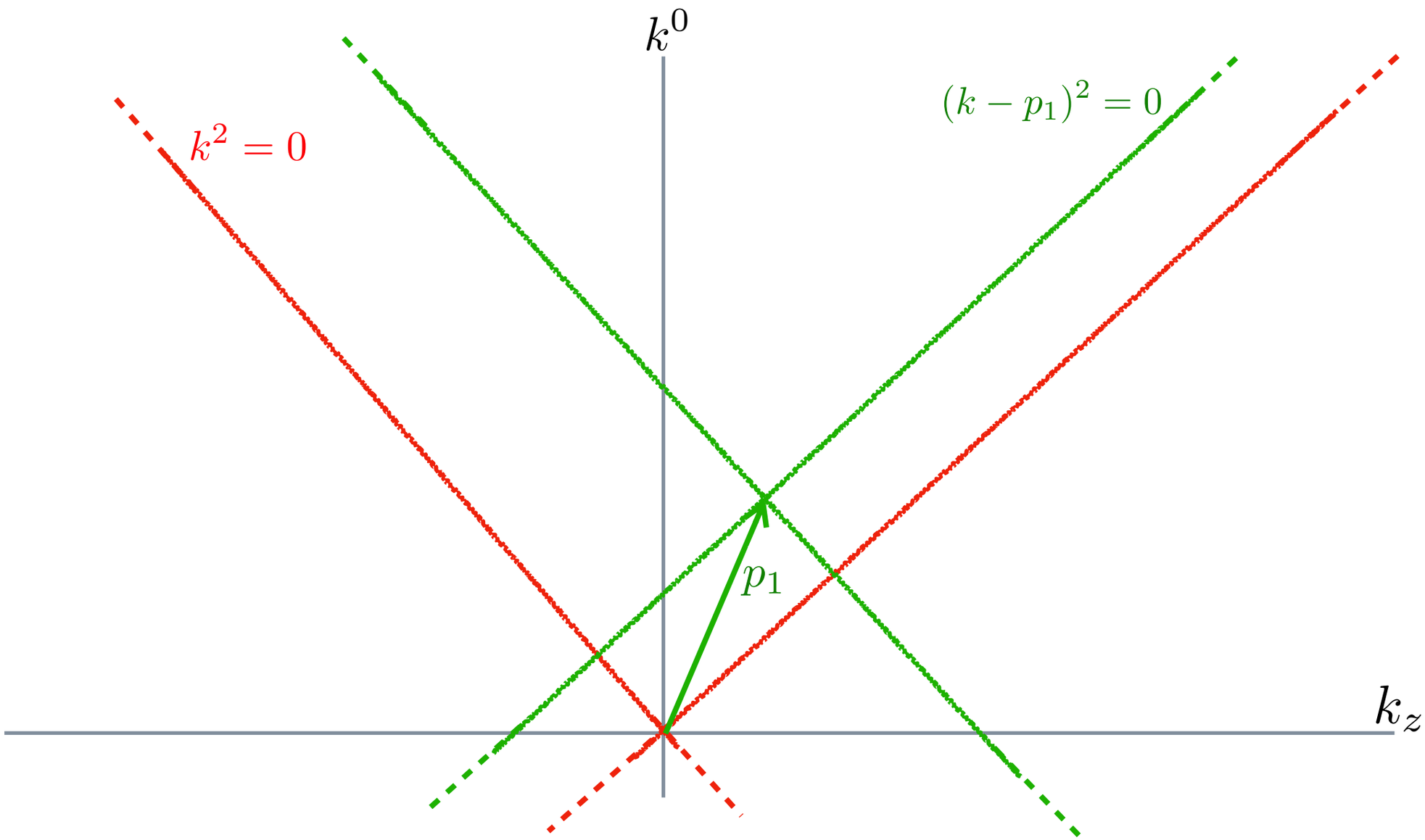

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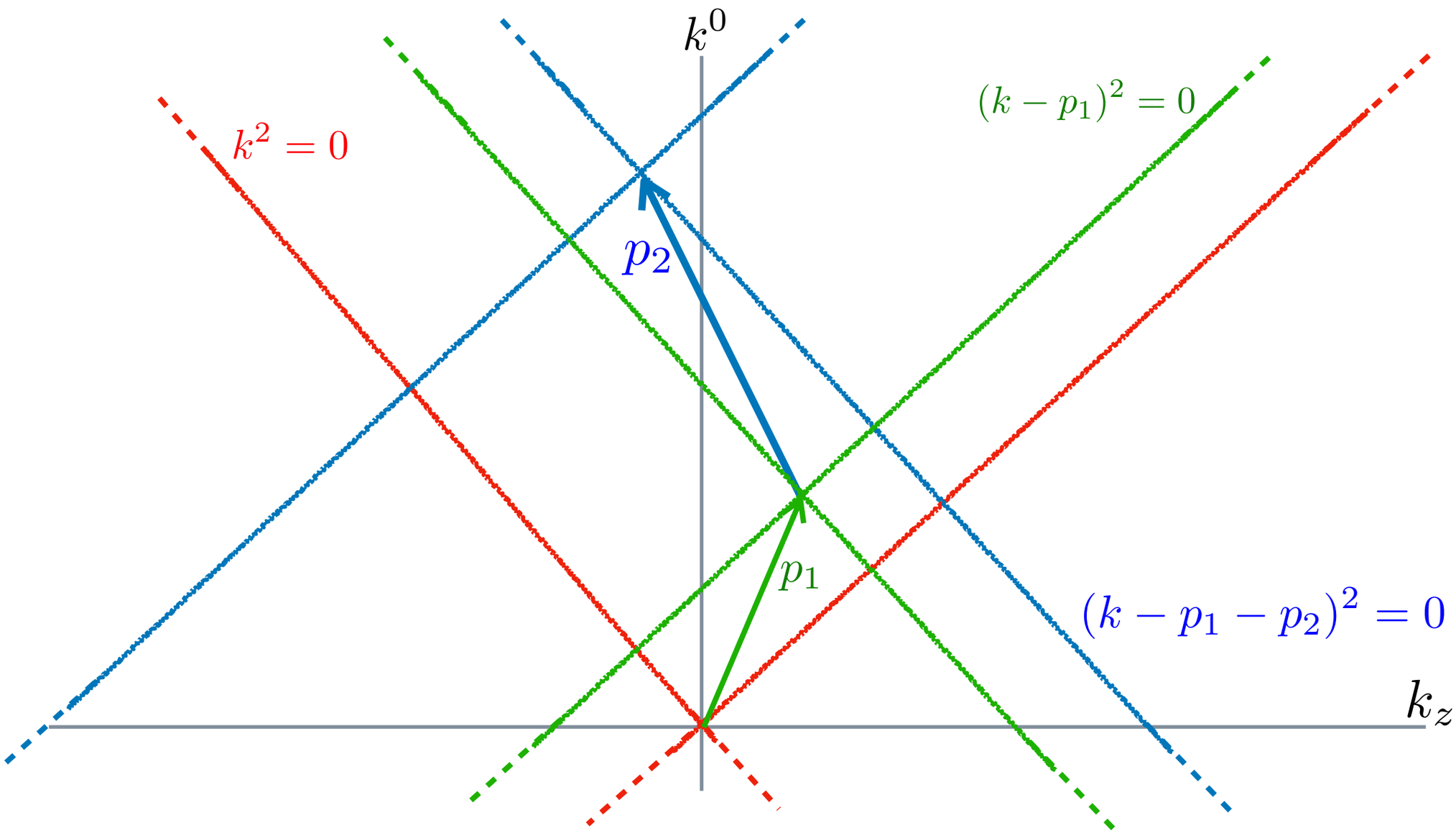

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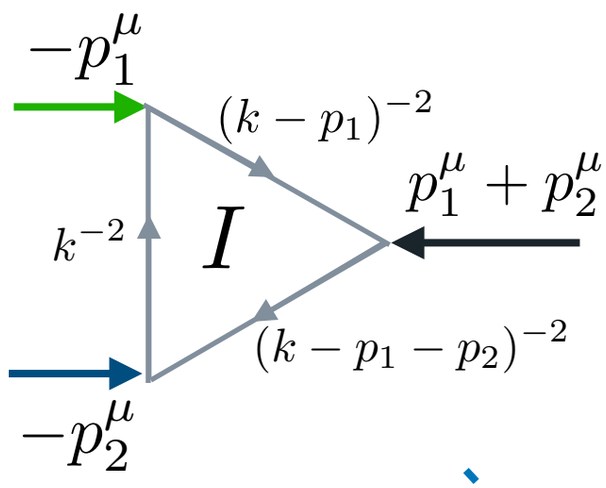
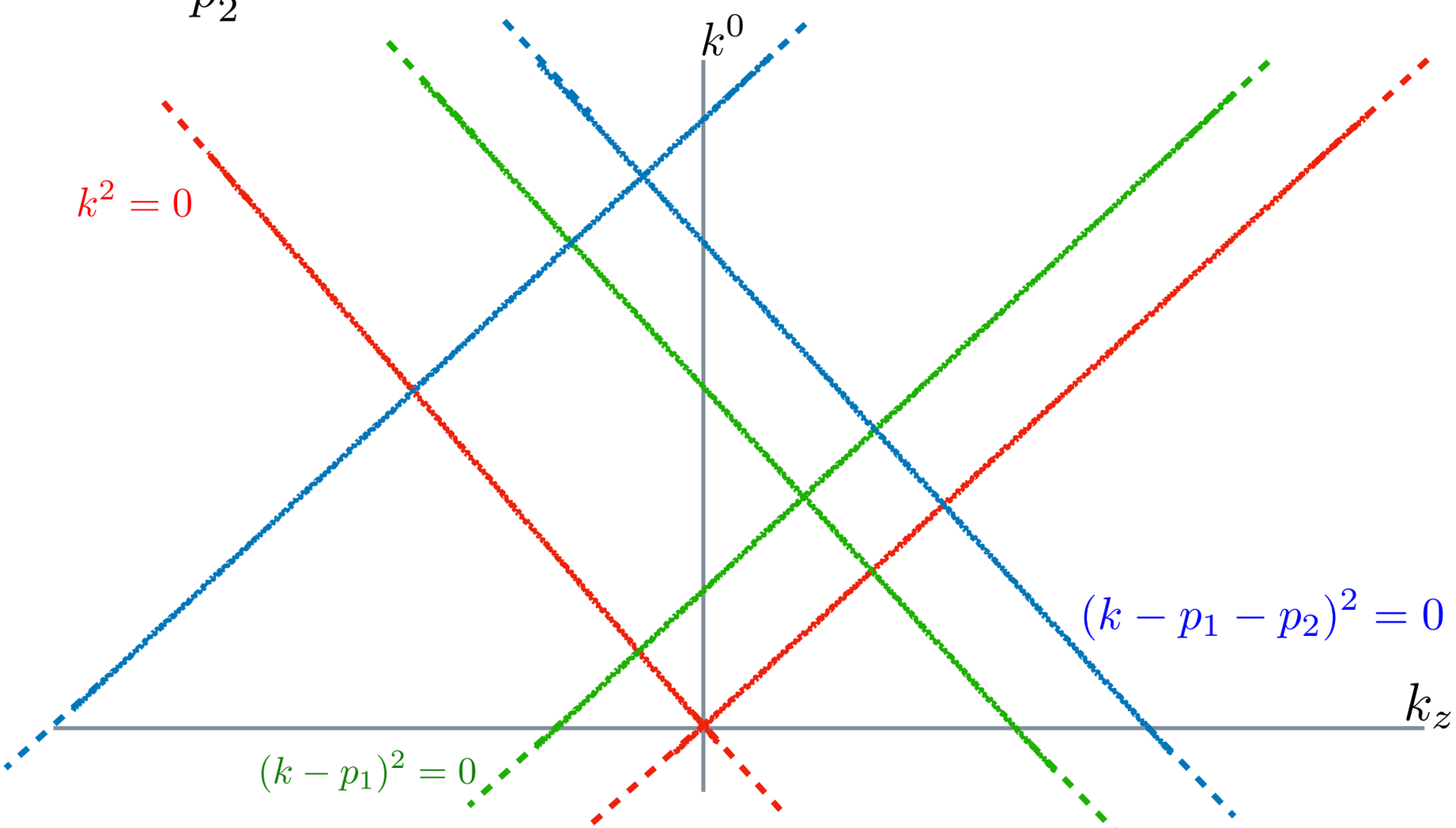
Singular surfaces - 2D ellipsoids

Analytically integrate over the loop energies using Cauchy's theorem (LTD):





$$p_i{}^2 > 0 \quad \forall i$$

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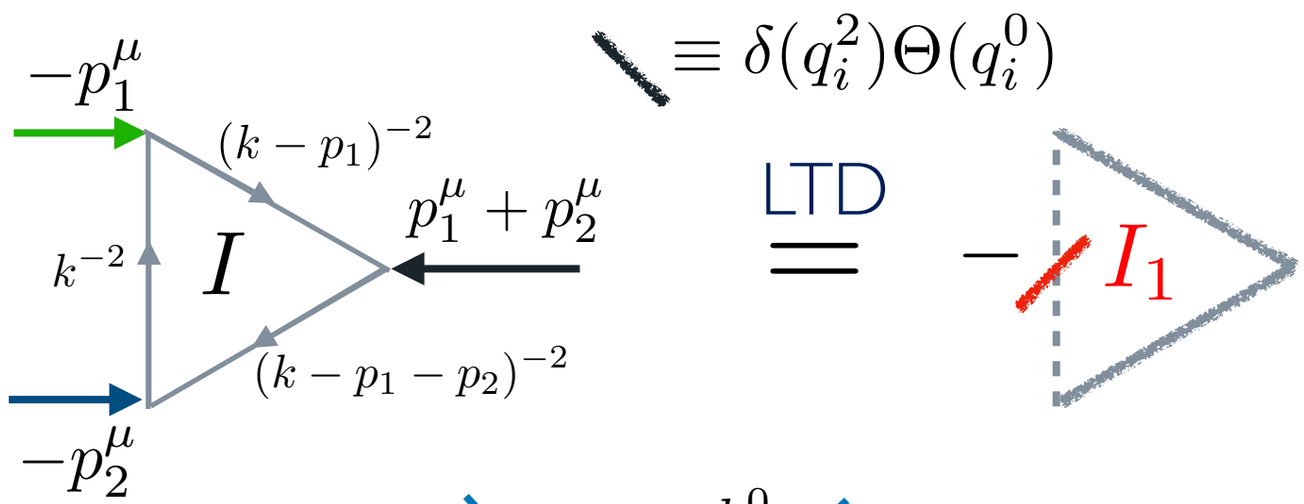

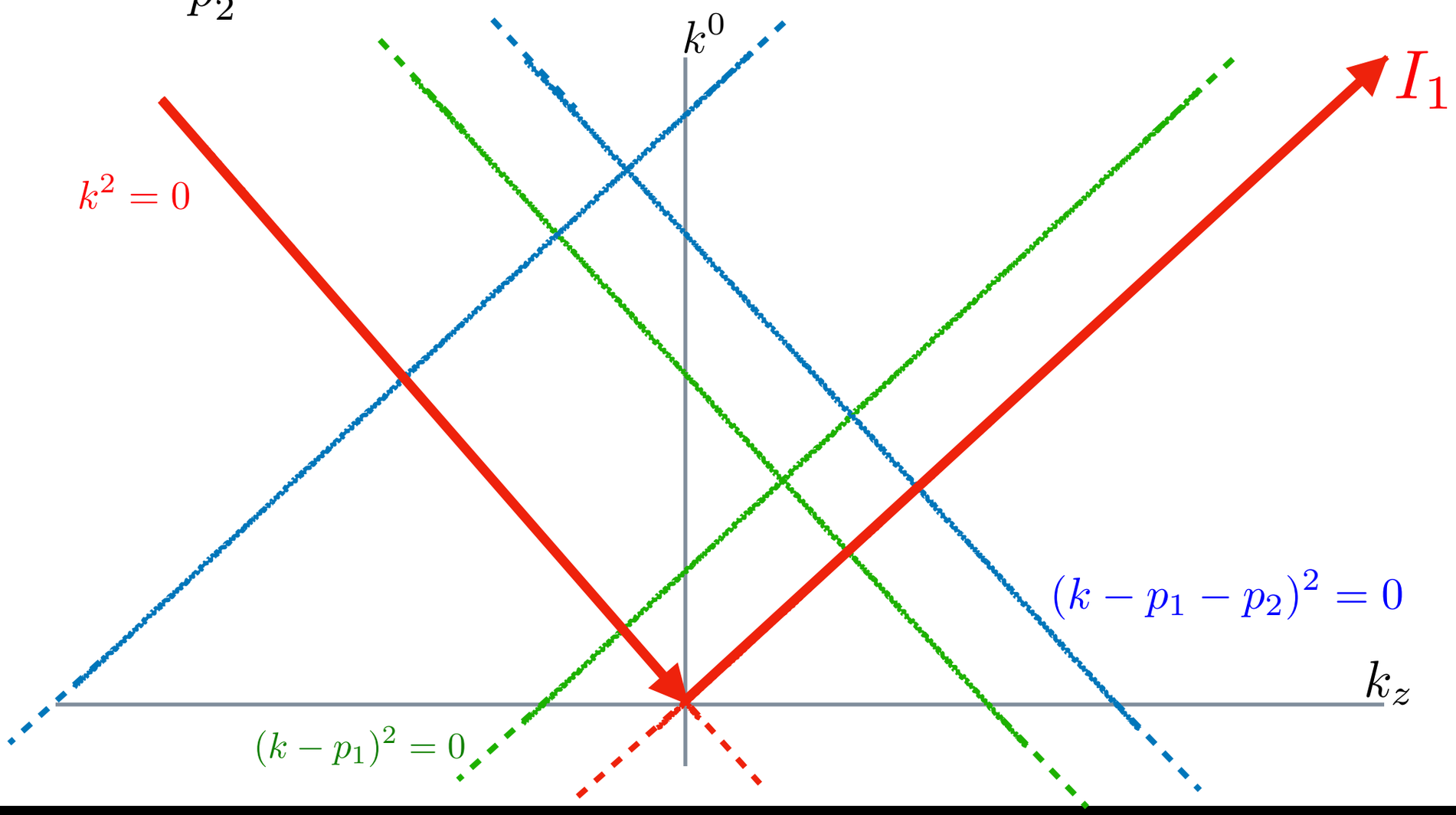

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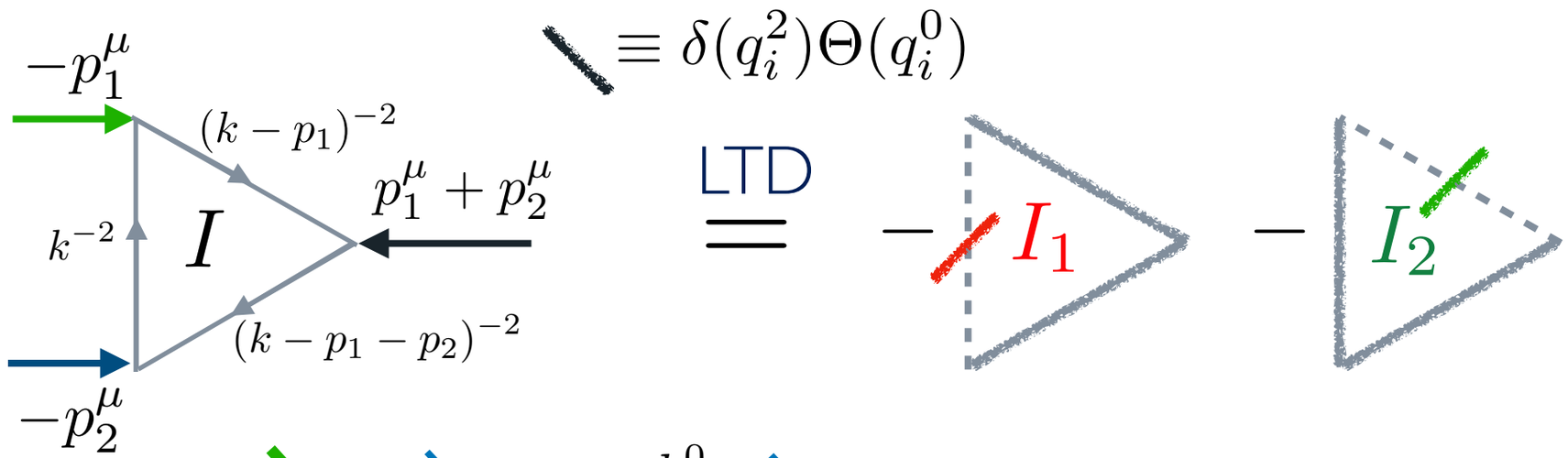

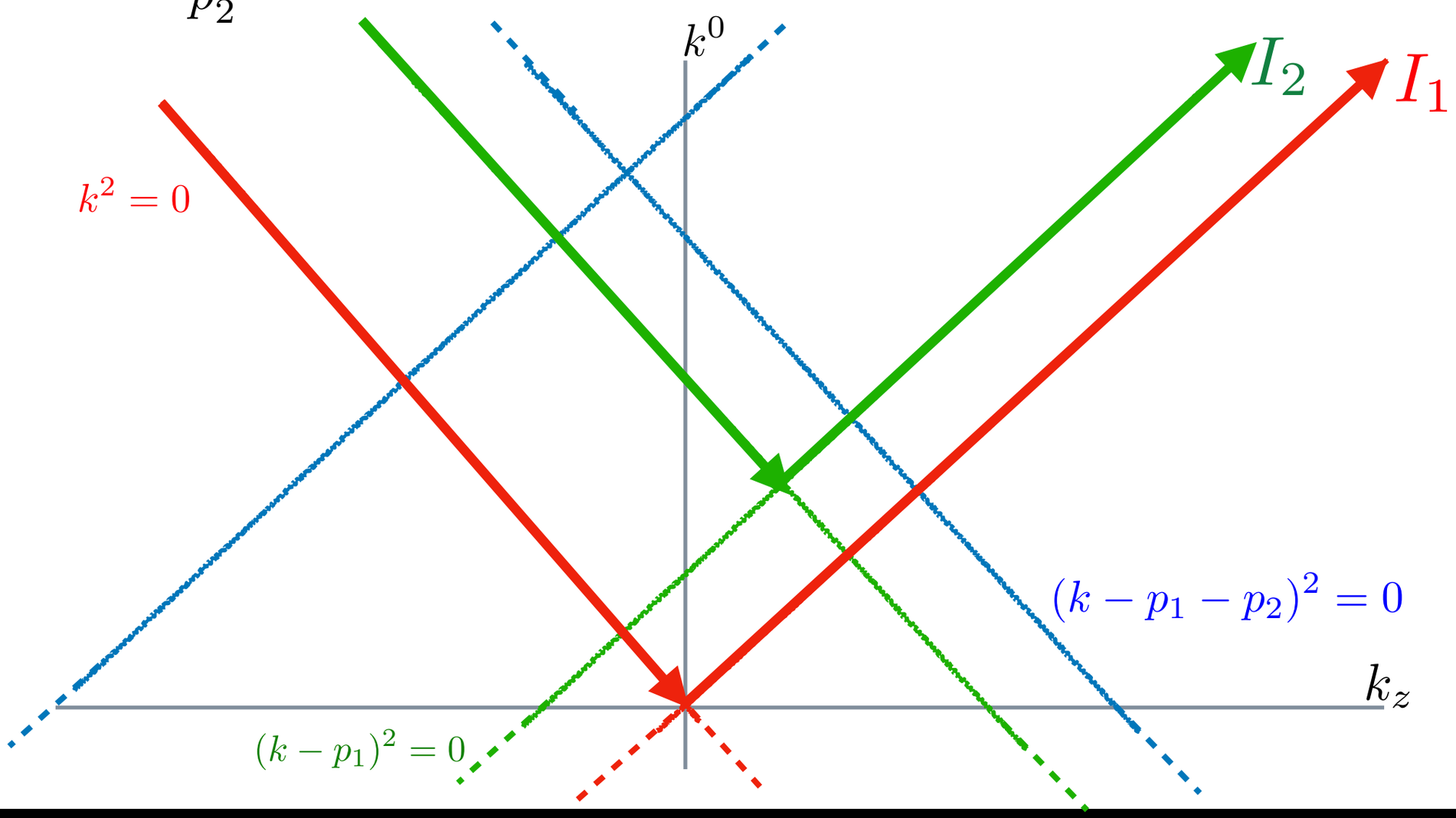

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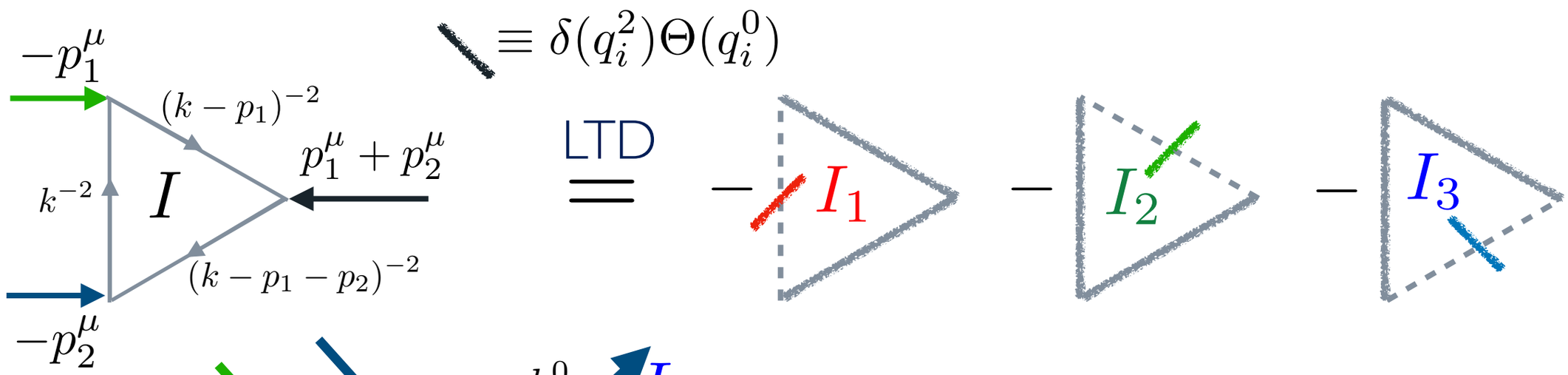

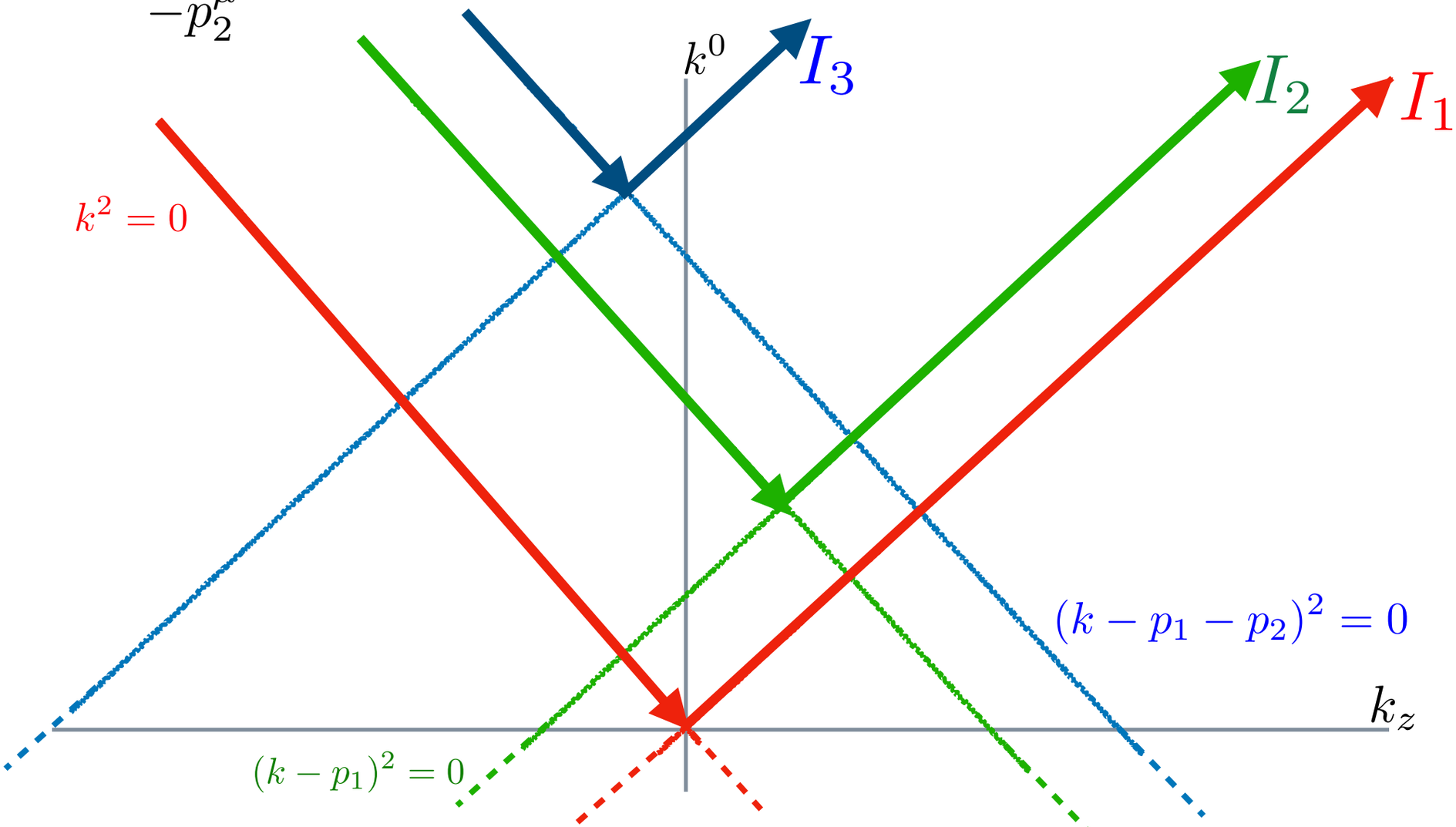

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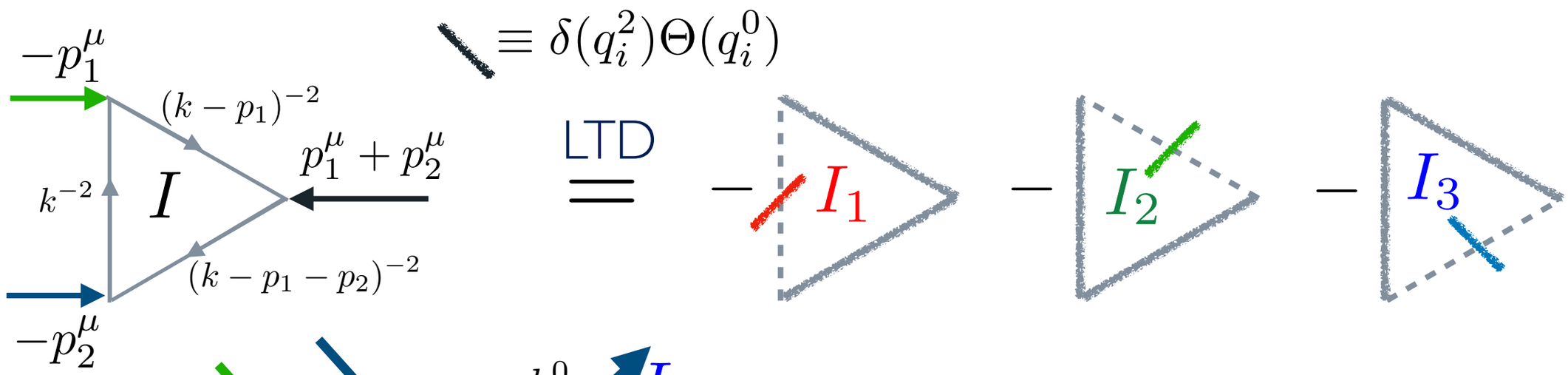

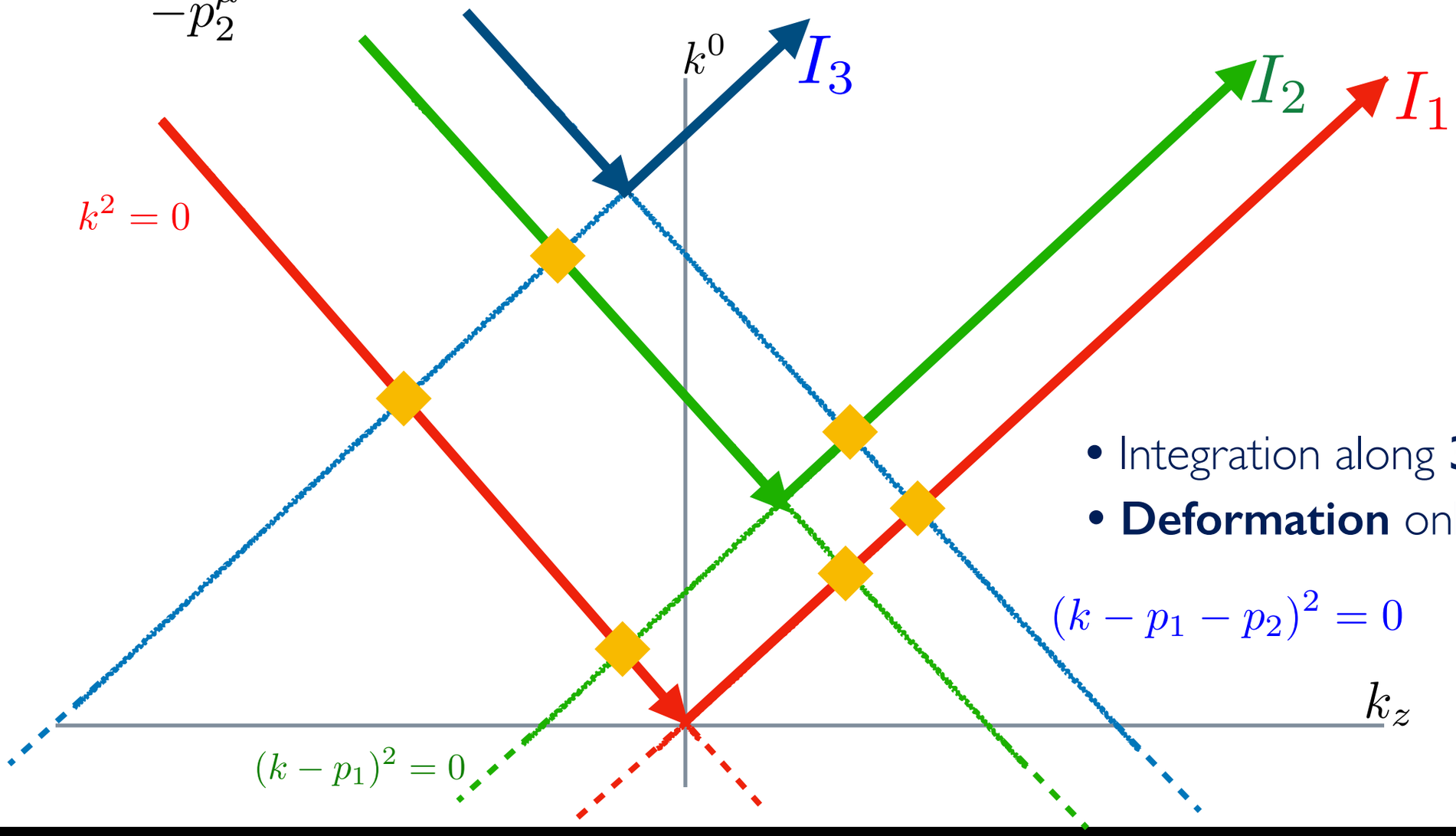

- Integration along **3D euclidian** thick lines
- **Deformation** only needed **on** ◆ !

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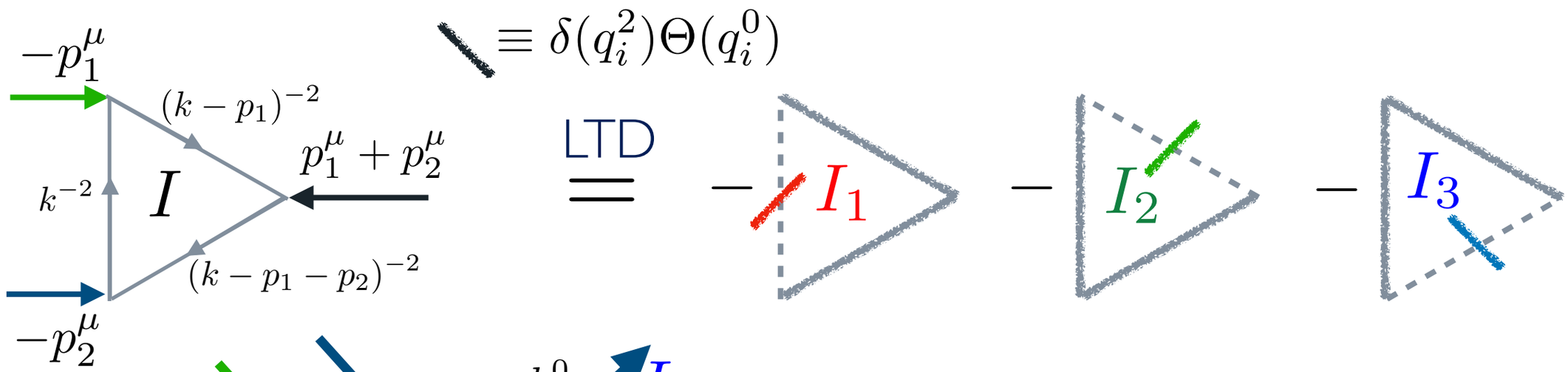

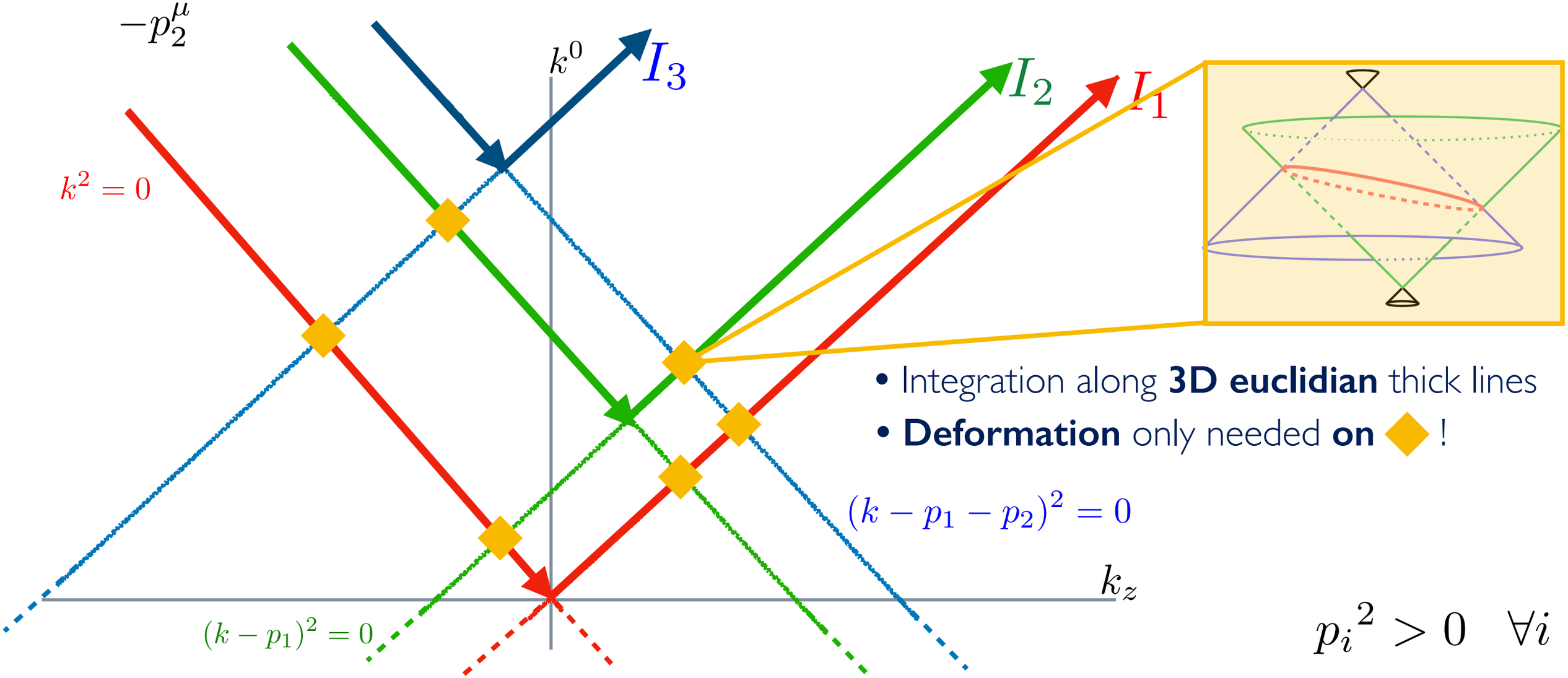

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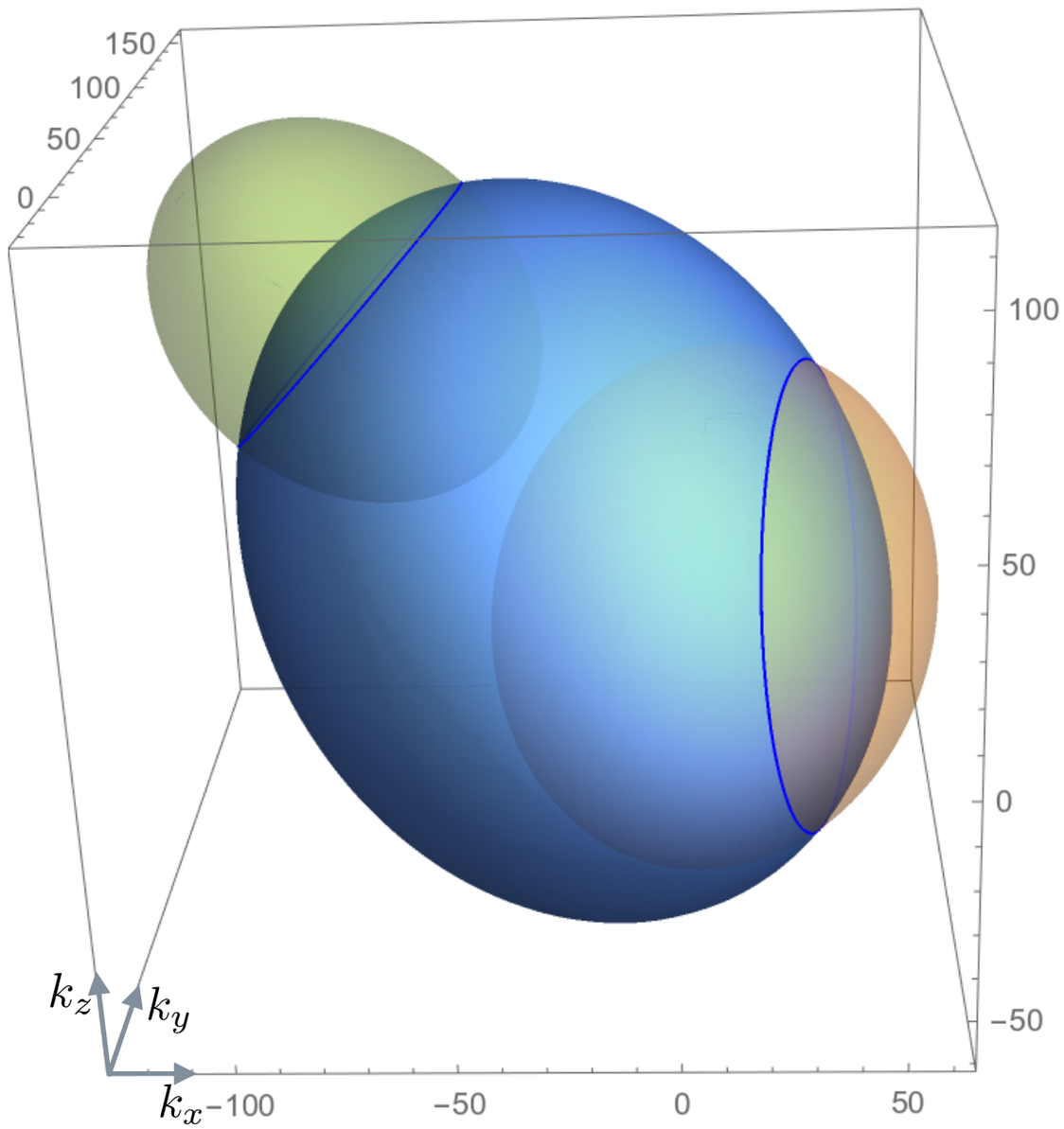


- General **one-loop** ellipsoid **expression**:

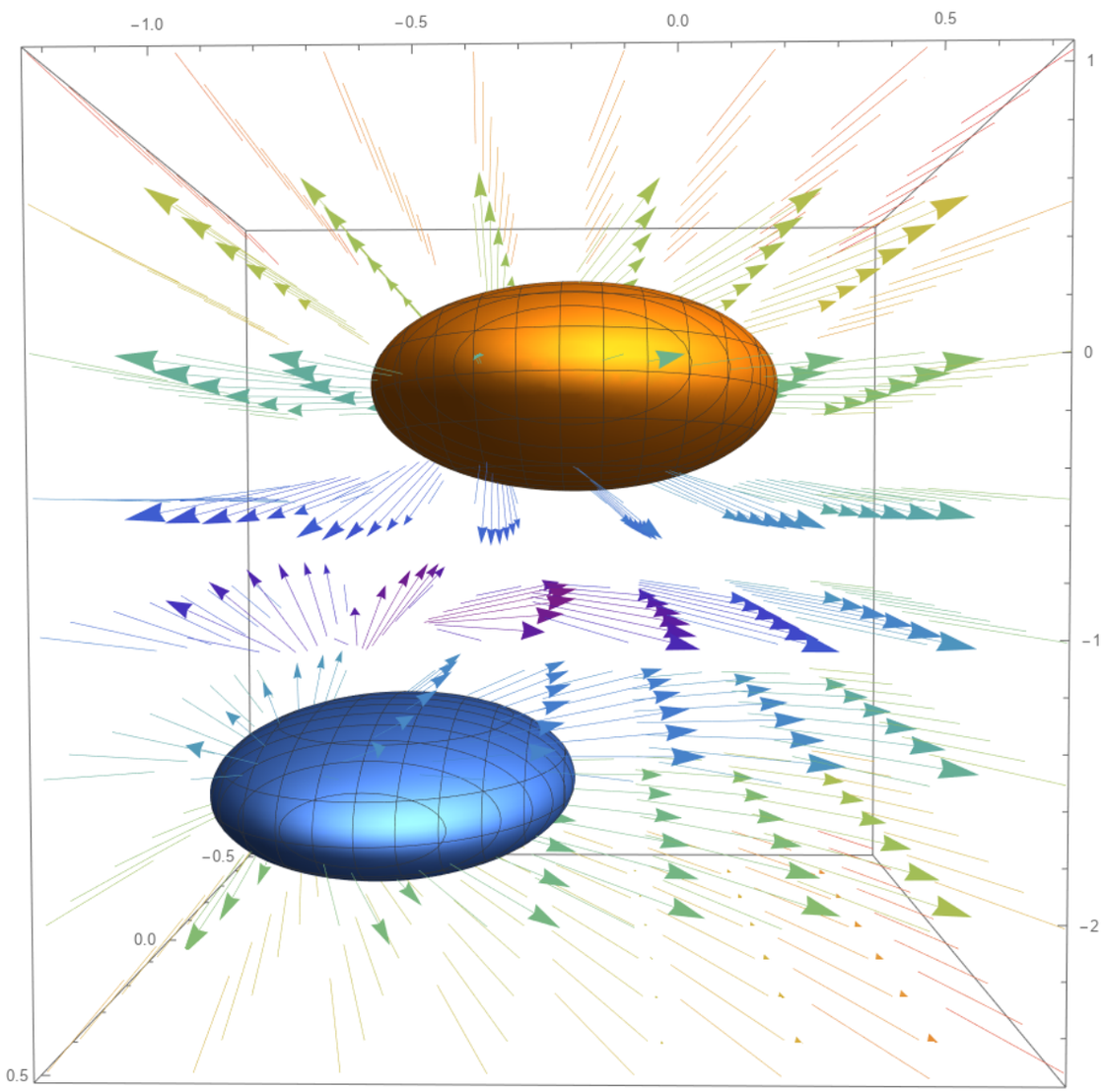
$$E_{ij}(\vec{k}) = \sqrt{\left(\vec{k}+\vec{p_i}\right)^2 + m_i^2 - i\delta} + \sqrt{\left(\vec{k}+\vec{p_j}\right)^2 + m_j^2 - i\delta} - p_i^0 + p_j^0$$

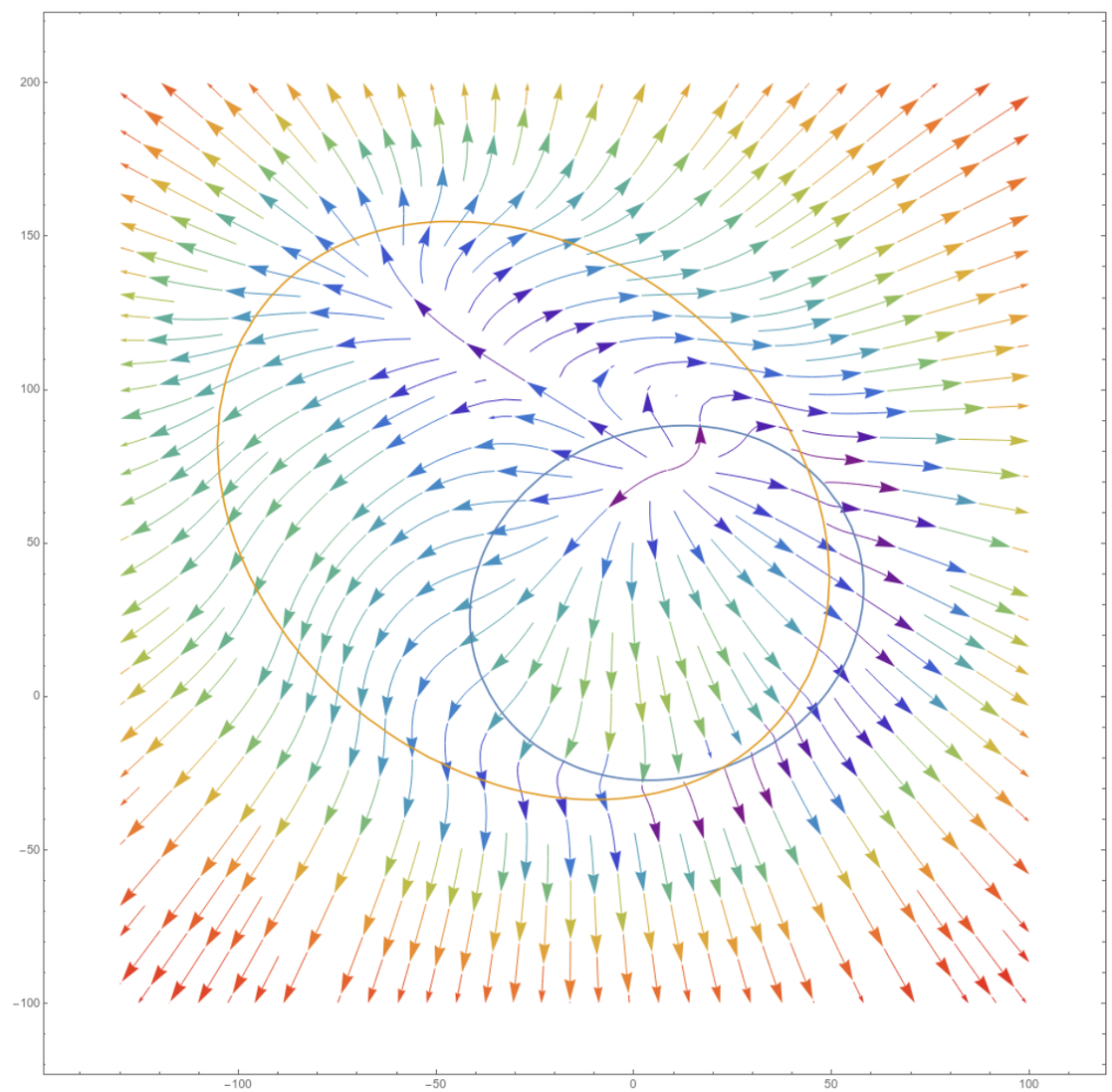
Deforming around singular 2D-ellipsoids

Deformation: $\vec{k} \to \vec{k} - i\vec{\kappa}$

Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$

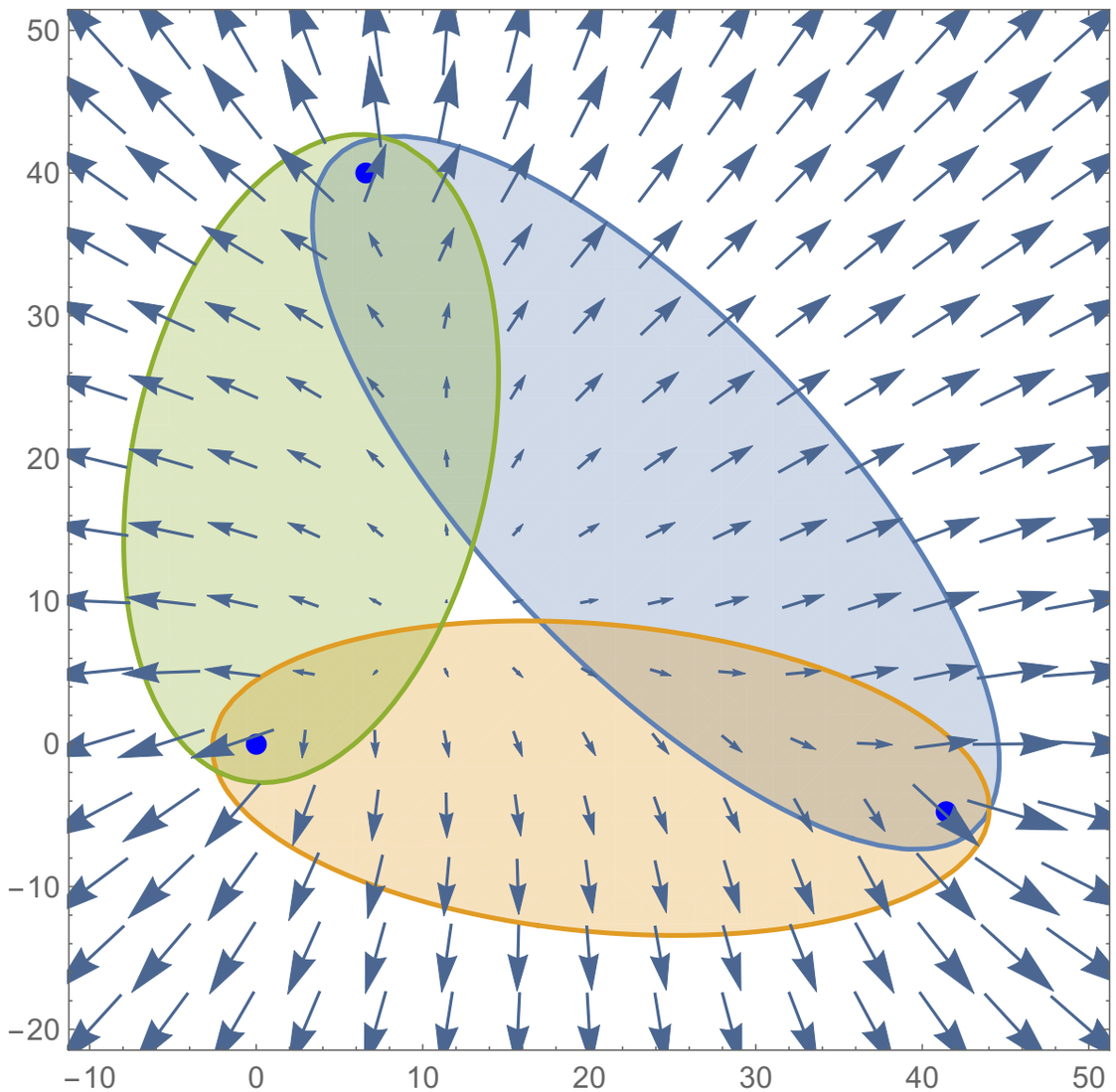
Weighted sum of normals: sometimes working…





Deforming around singular 2D-ellipsoids

… but in many cases not working !



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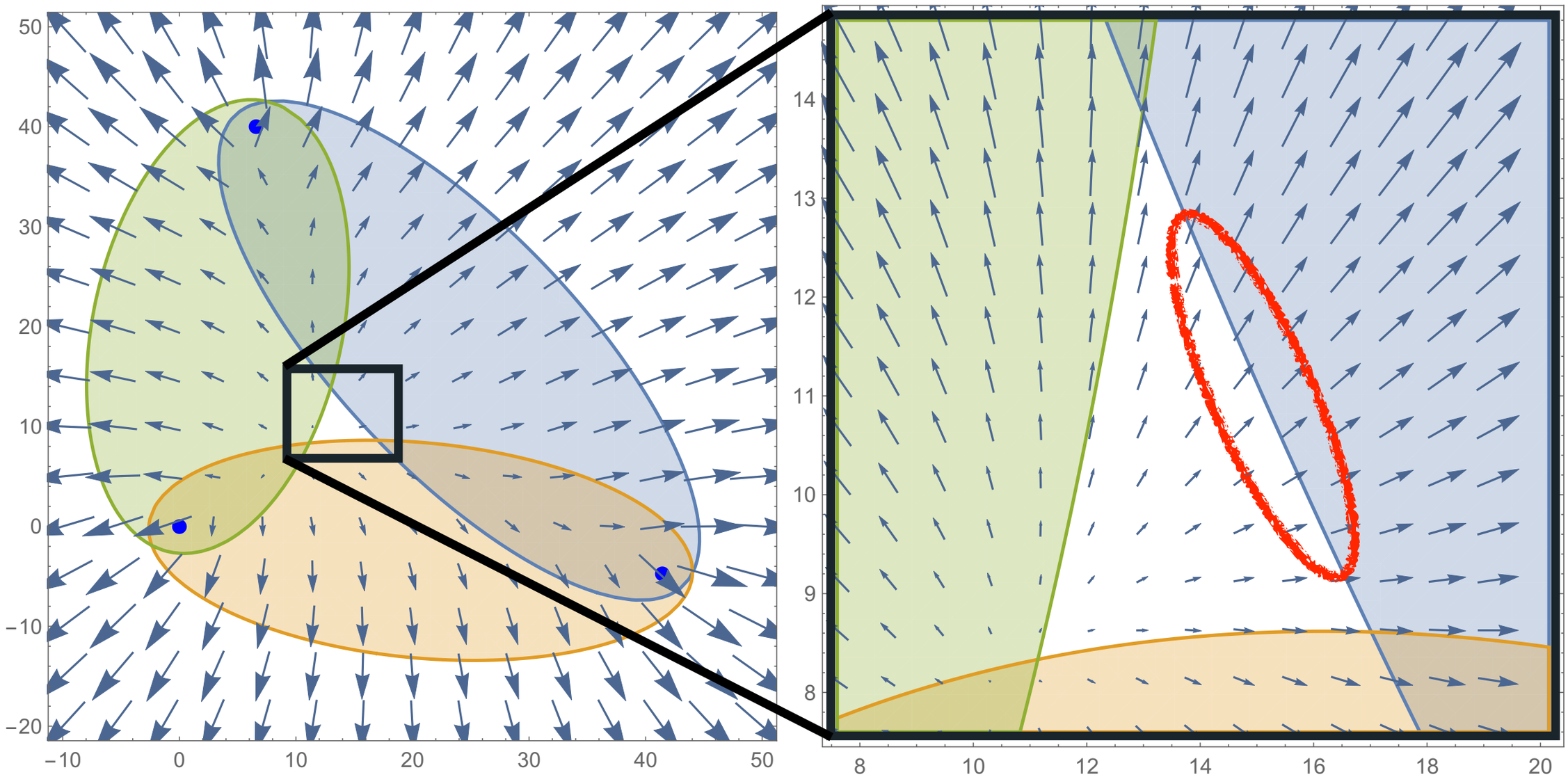

Dampening **help mitigate** this issue but does not remove it, because the problem always remains on **intersections** of ellipsoids.

Deforming around singular 2D-ellipsoids

What is the **solution** then? For example for this case:

Deforming around singular 2D-ellipsoids

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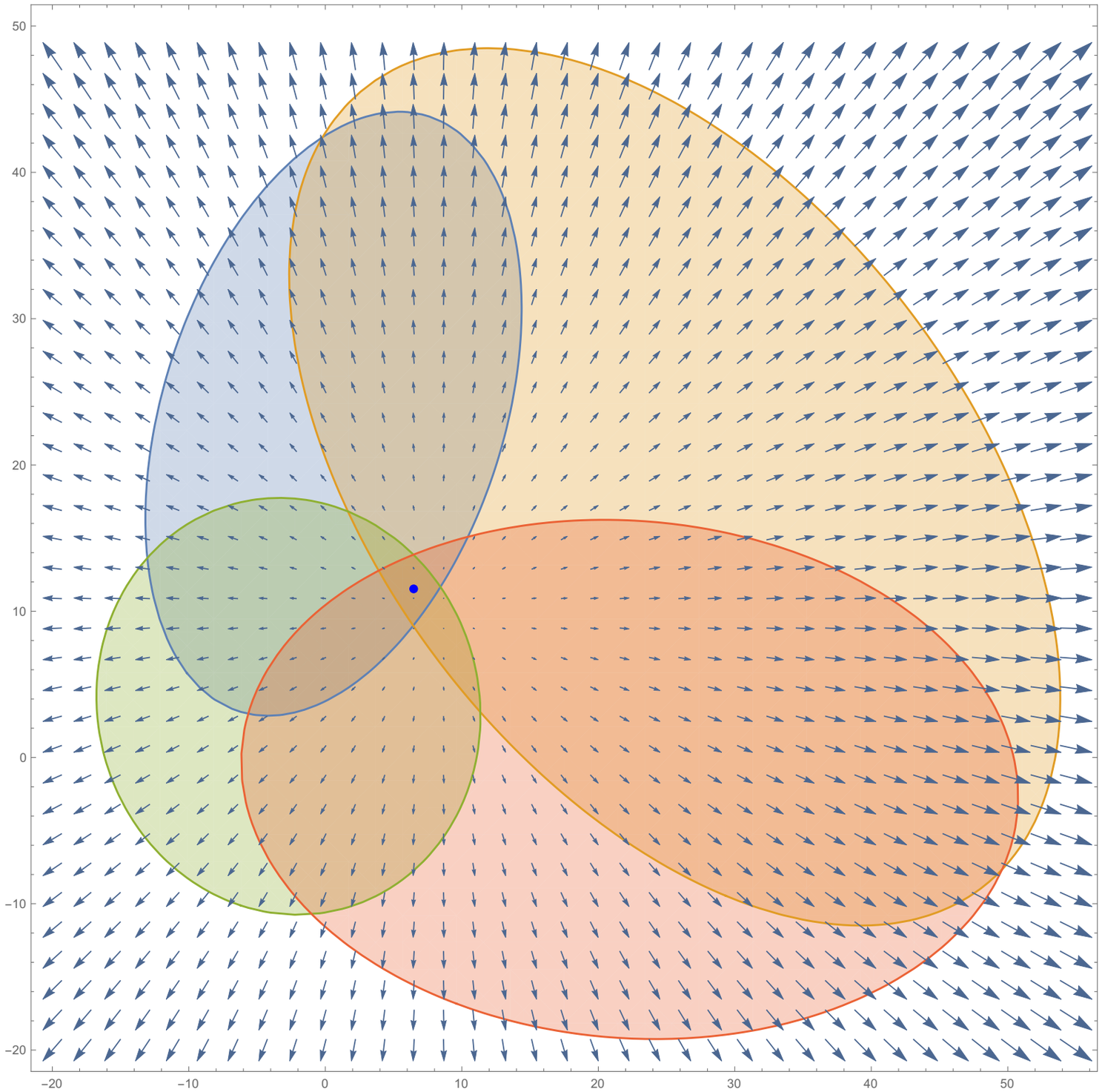
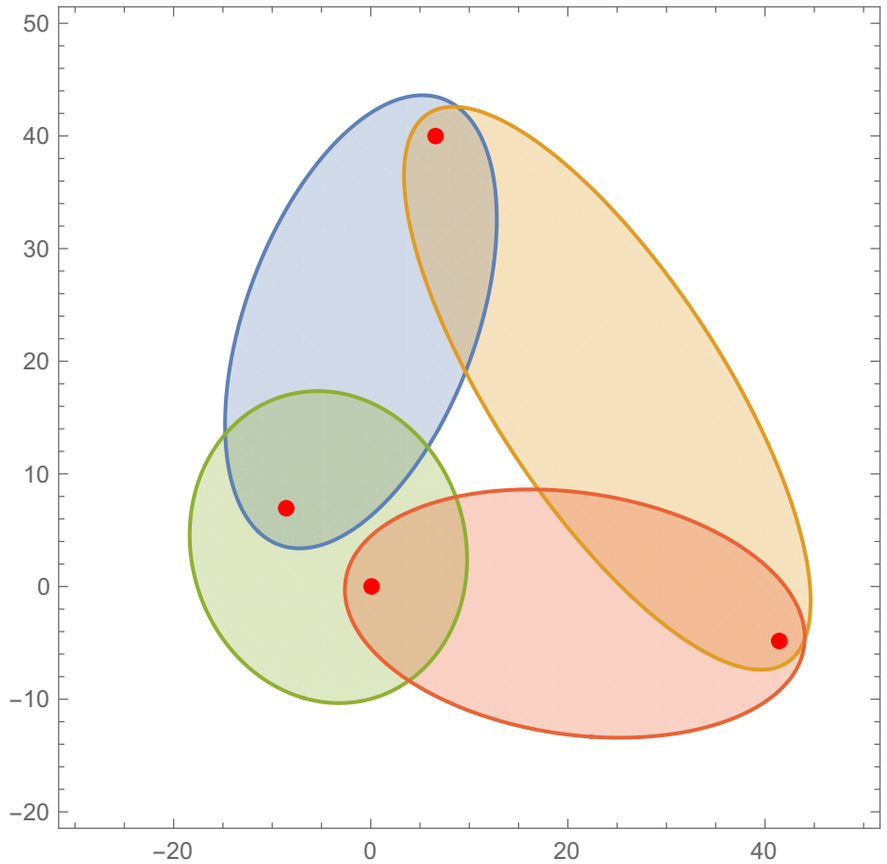


A **radial field** centered in the inside of all ellipsoids!

Deforming around singular 2D-ellipsoids

But then what if there is no point in the inside of **all ellipsoids** (Box4E example) ?



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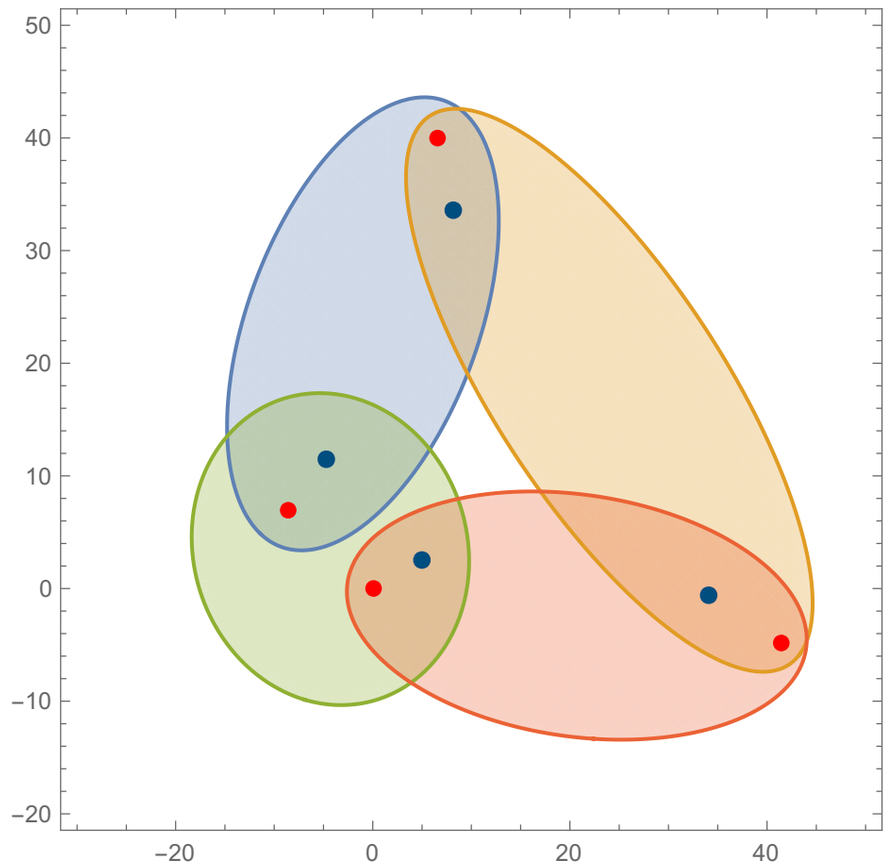

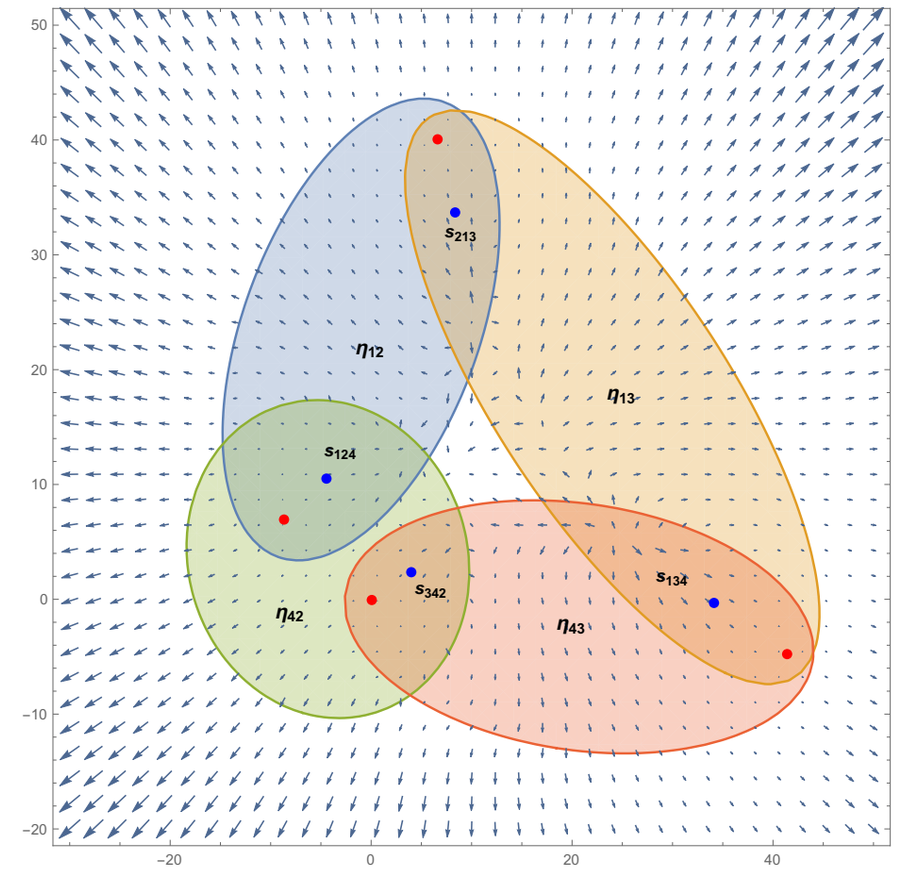

Find the "**maximal overlap structure**" with its four "**centers**" and combine radial fields:

$$\begin{aligned}\vec{\kappa}_{ij} = &(\vec{k} - \vec{s}_{124})\, T(E_{13})\, T(E_{43}) \\ &+ (\vec{k} - \vec{s}_{213})\, T(E_{42})\, T(E_{43}) \\ &+ (\vec{k} - \vec{s}_{134})\, T(E_{42})\, T(E_{12}) \\ &+ (\vec{k} - \vec{s}_{342})\, T(E_{12})\, T(E_{13})\end{aligned}$$

with an **anti-selection** function $T(E_{ij}) \propto E_{ij}(\vec{k})^2$

which satisfies the causal constraint everywhere, e.g:

$$\vec{\kappa}(\vec{k})|_{\vec{k}\in\partial E_{12}\cap\partial E_{13}} \stackrel{!}{=} \left(\vec{k} - \vec{s}_{213}\right) T(E_{42}) T(E_{43})$$

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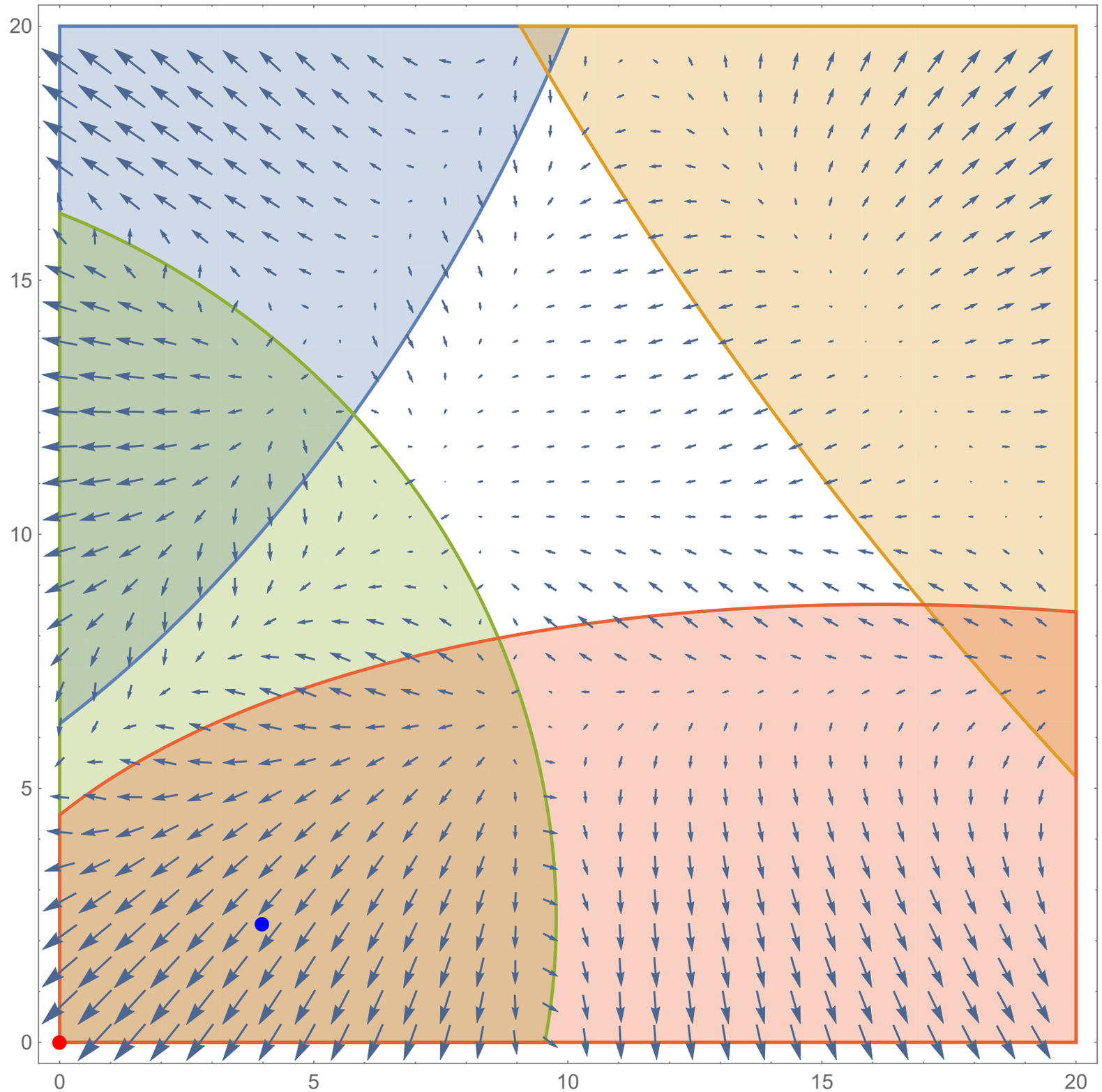
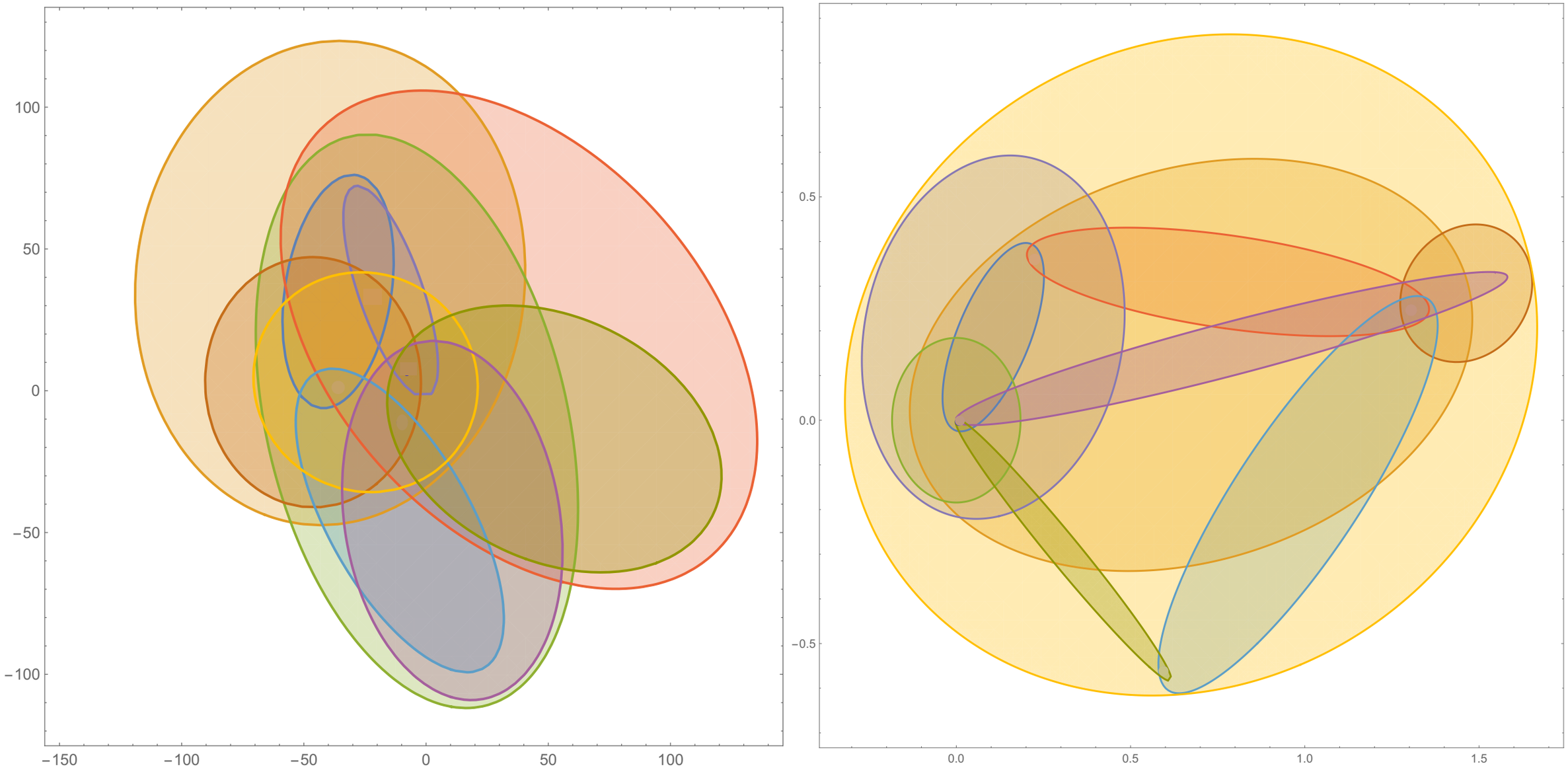


Deforming around singular 2D-ellipsoids

Finding the **maximal overlap structure** and **corresponding centers** is not always simple…
Scalar one-loop **hexagon** with ten E-surface for **two different kinematic configuration**:



This problem can be recast into an **optimisation problem** under **convex constraints**, allowing the use of **efficient standard algorithm** for solving **Second-Order Cone Program**.

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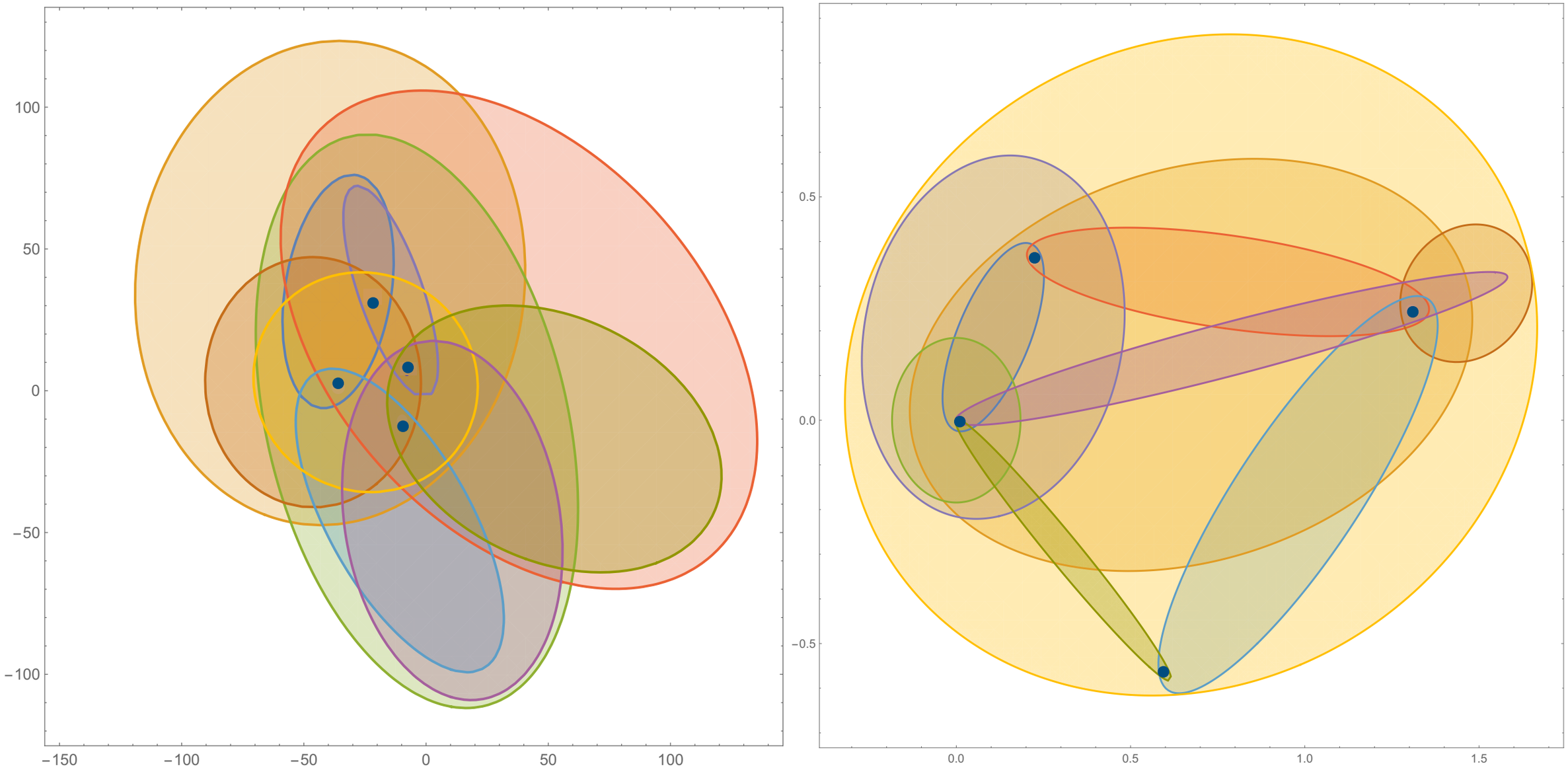

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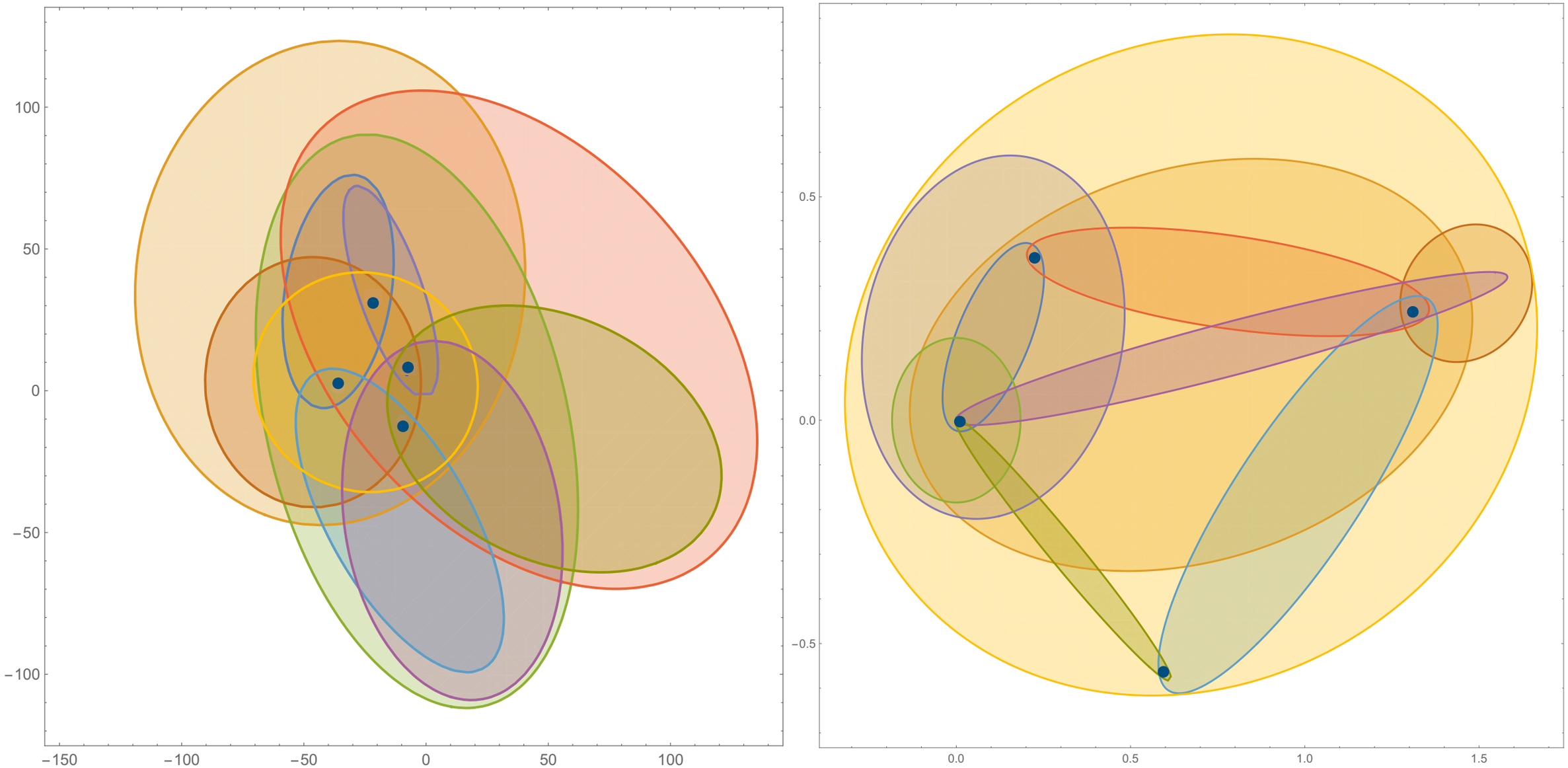

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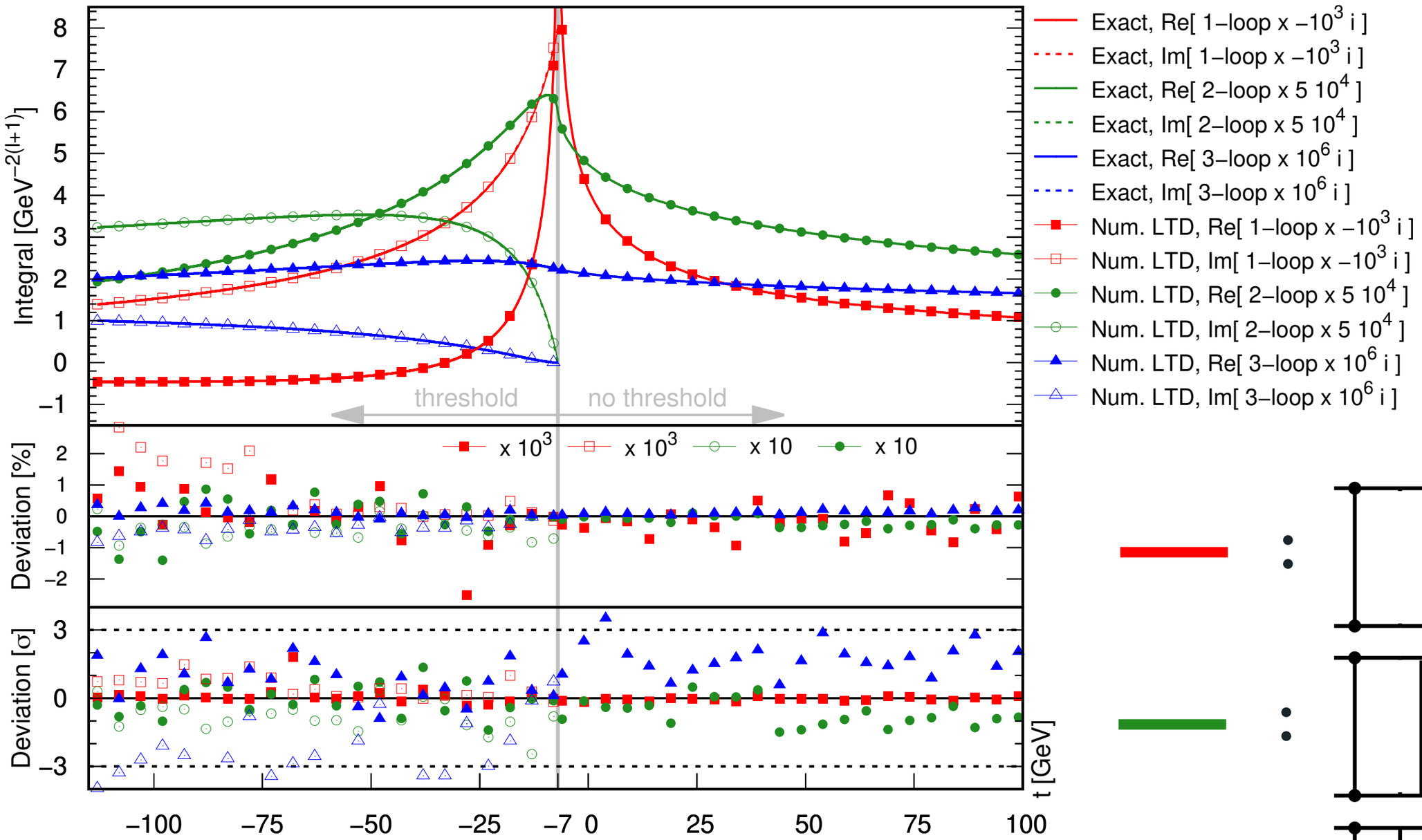
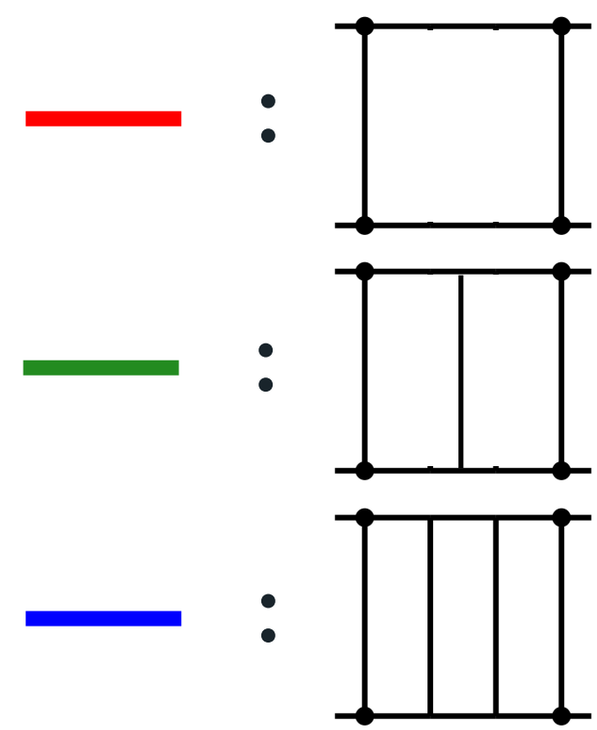
(Additional constraints on the **normalisation** of the **deformation fields** not discussed here **)**

4-point multi-loop scalar integral scan

[Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv:1906.06138]





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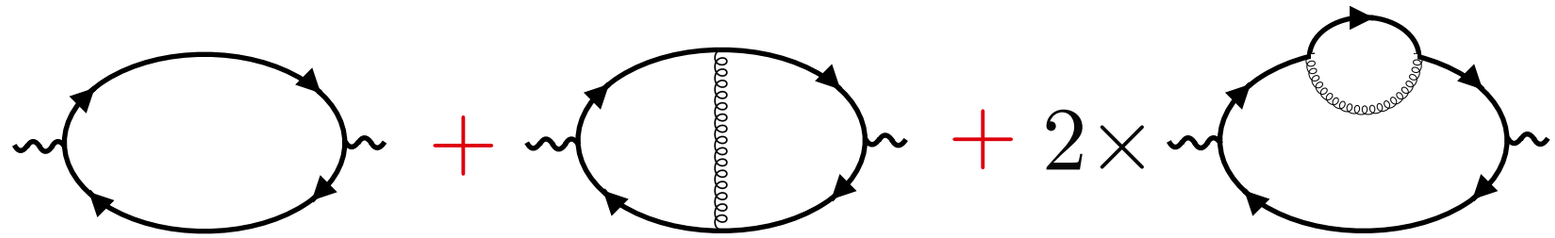
Locality Unitarity

Do not divide and conquer... just conquer

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068] [Soper, arxiv:99.10292] [https://pages.uoregon.edu/soper/beowulf]

$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$



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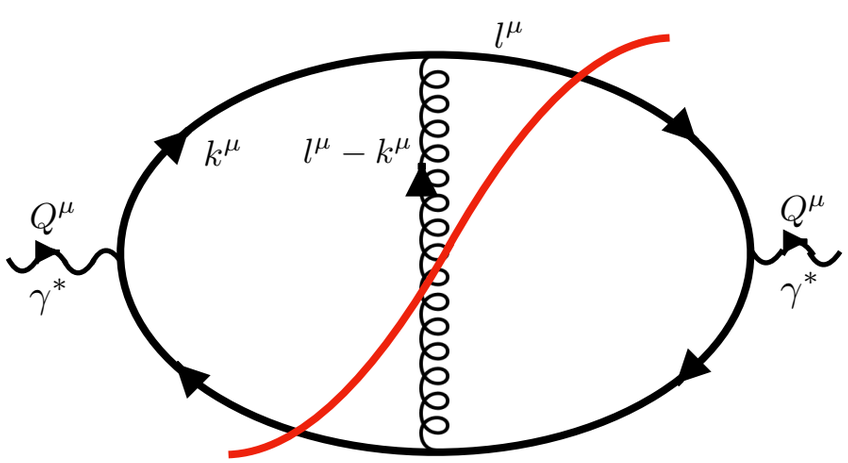
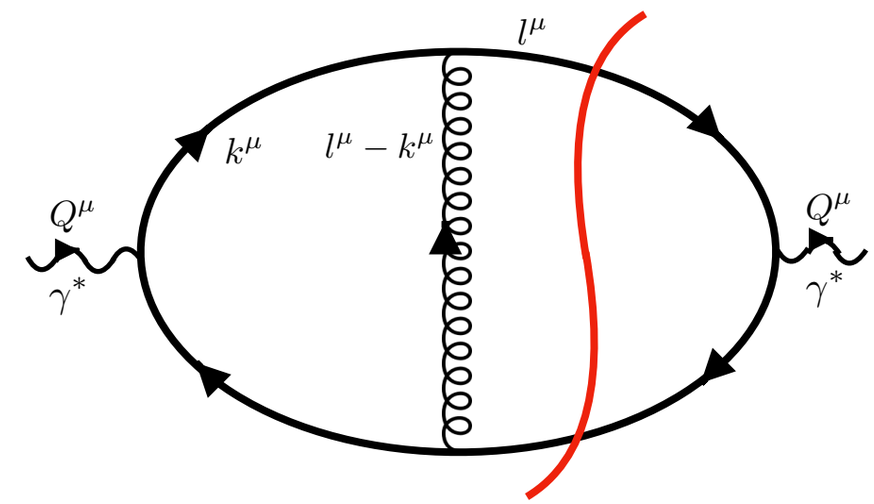


$$\propto \delta^+(l^2)\delta^-((l-k)^2)\delta^-((k-Q)^2)$$

?

LTD !



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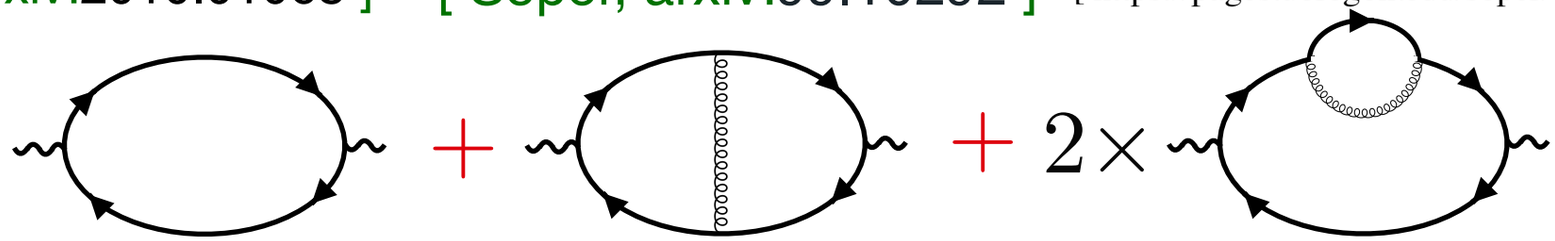

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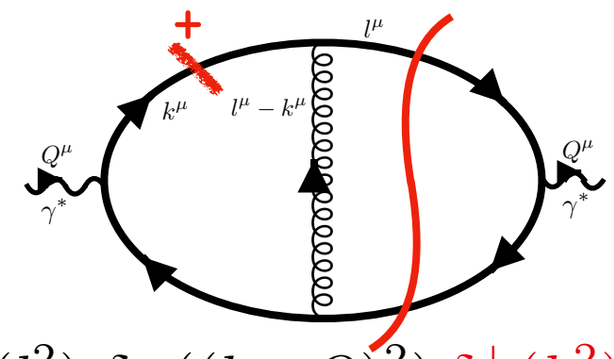

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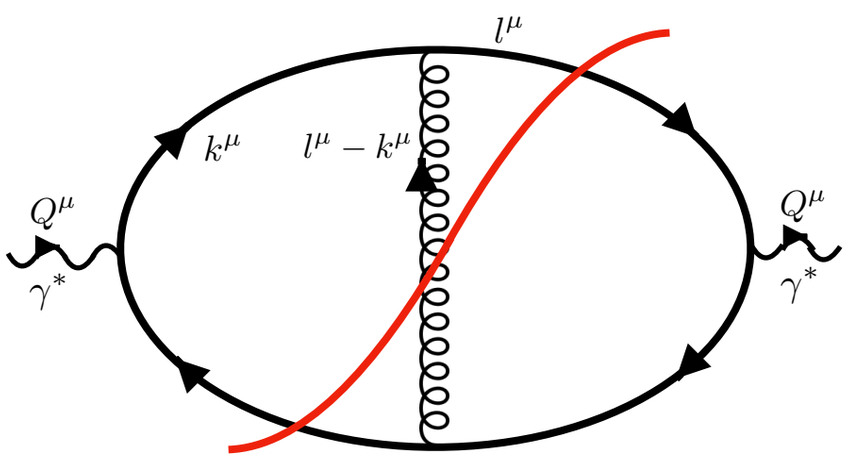

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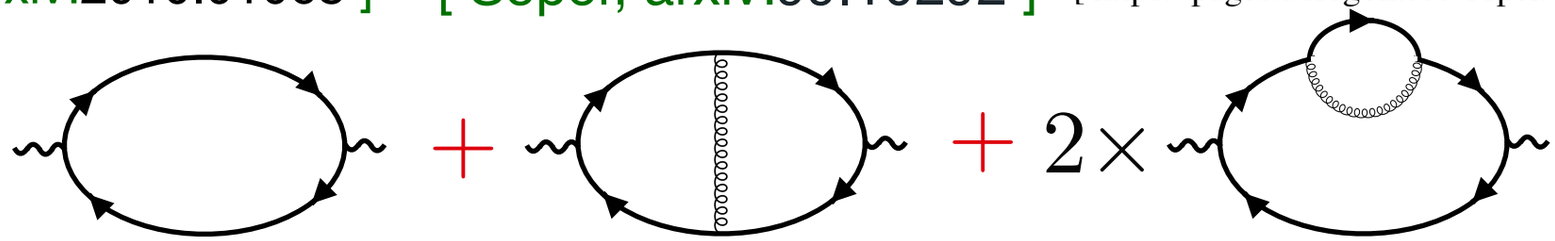
Locality Unitarity

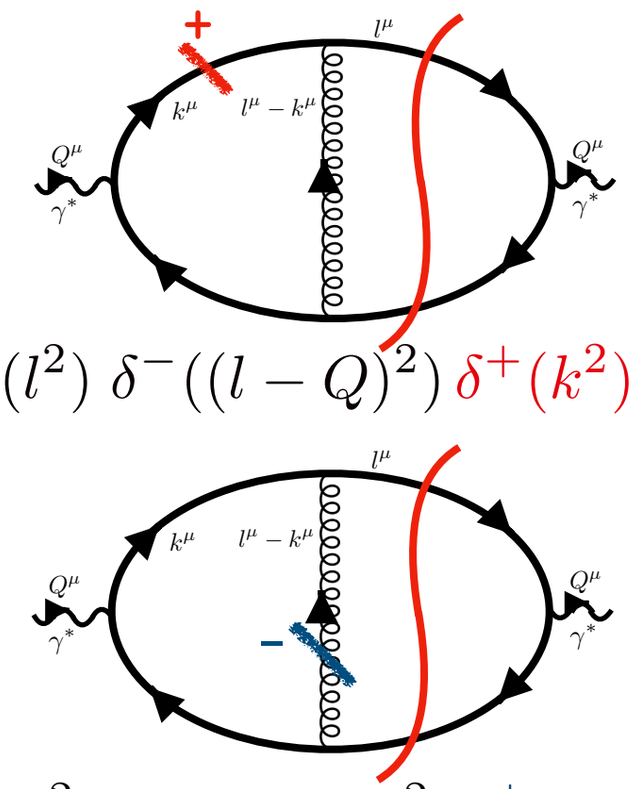
Do not divide and conquer... just conquer

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068] [Soper, arxiv:99.10292] [https://pages.uoregon.edu/soper/beowulf]

$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$



$$+ 2\times \Big]$$



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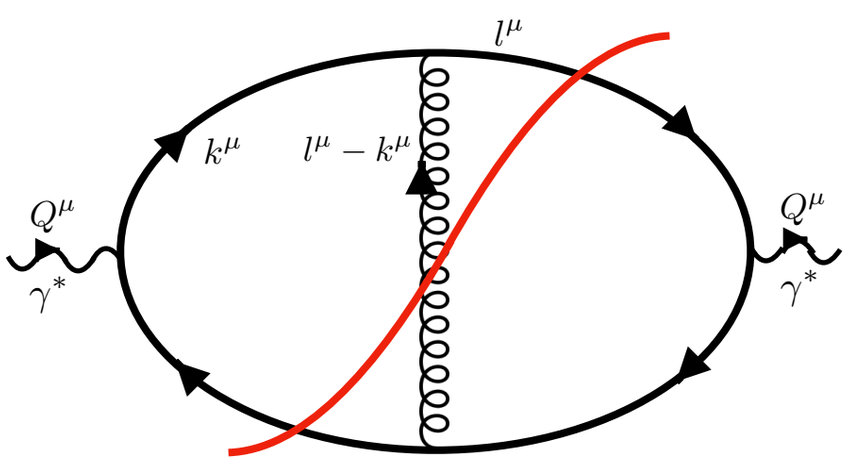

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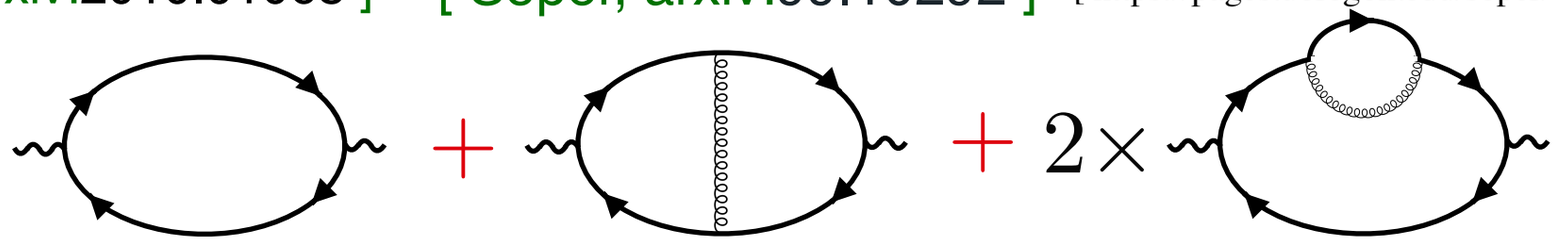

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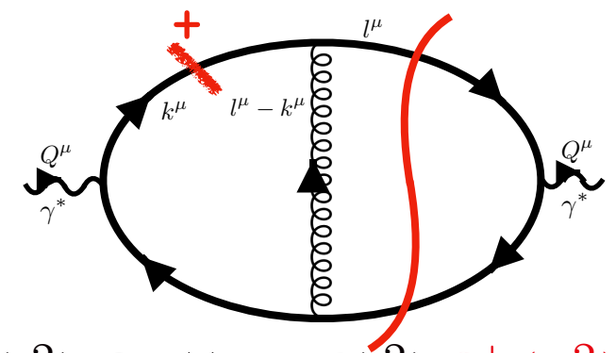

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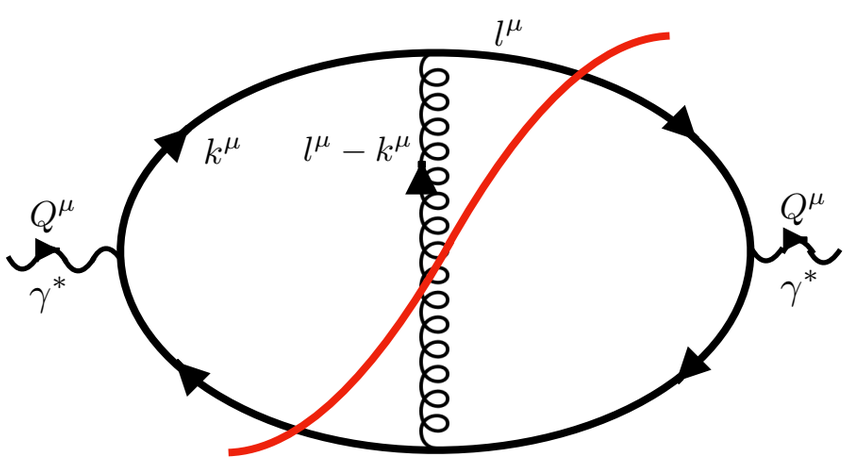

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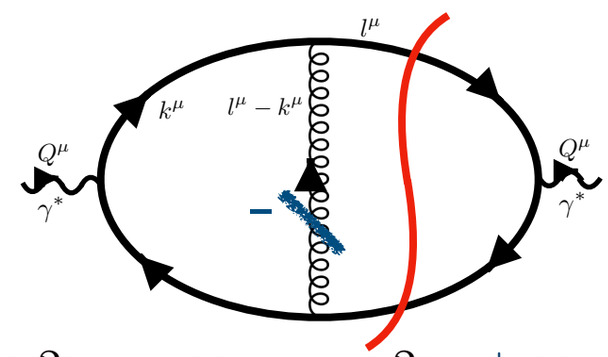

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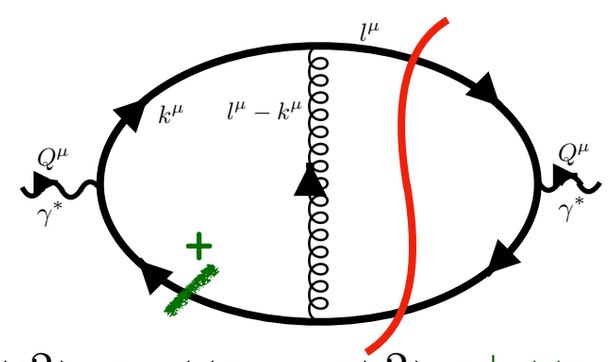

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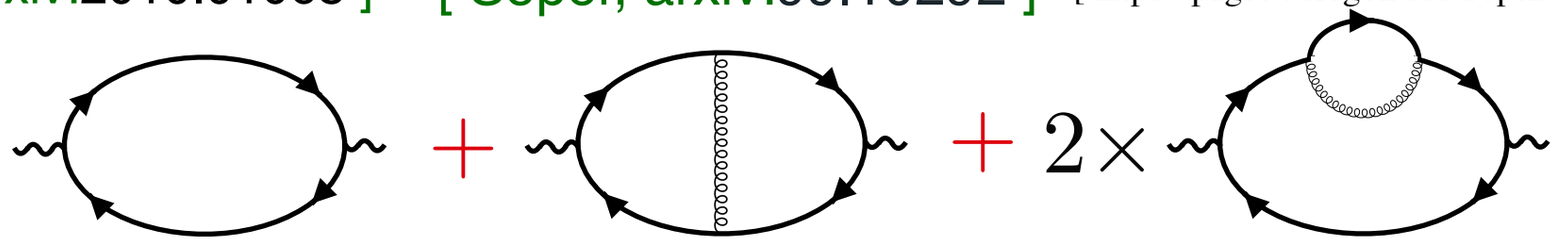

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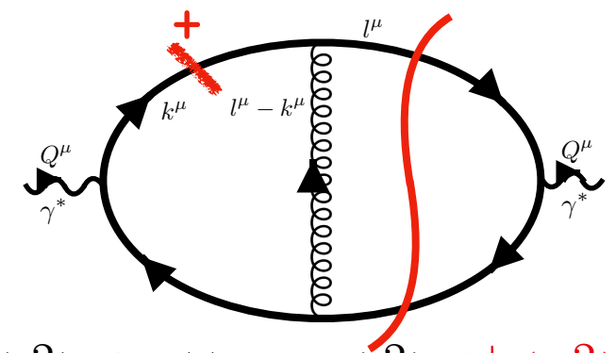

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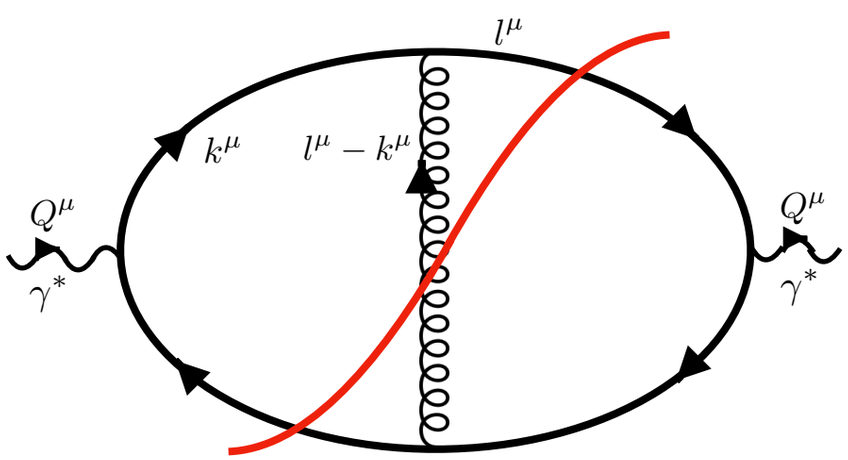

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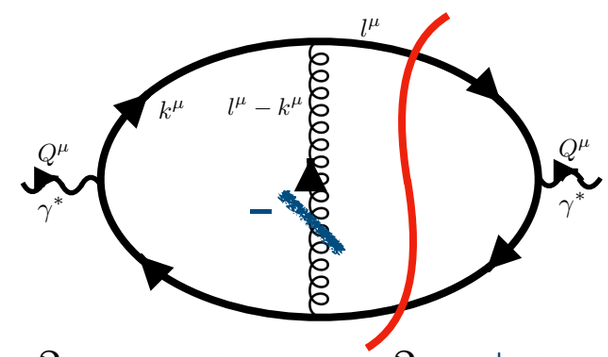

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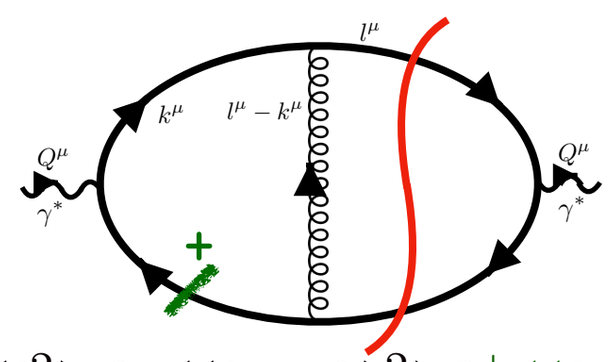

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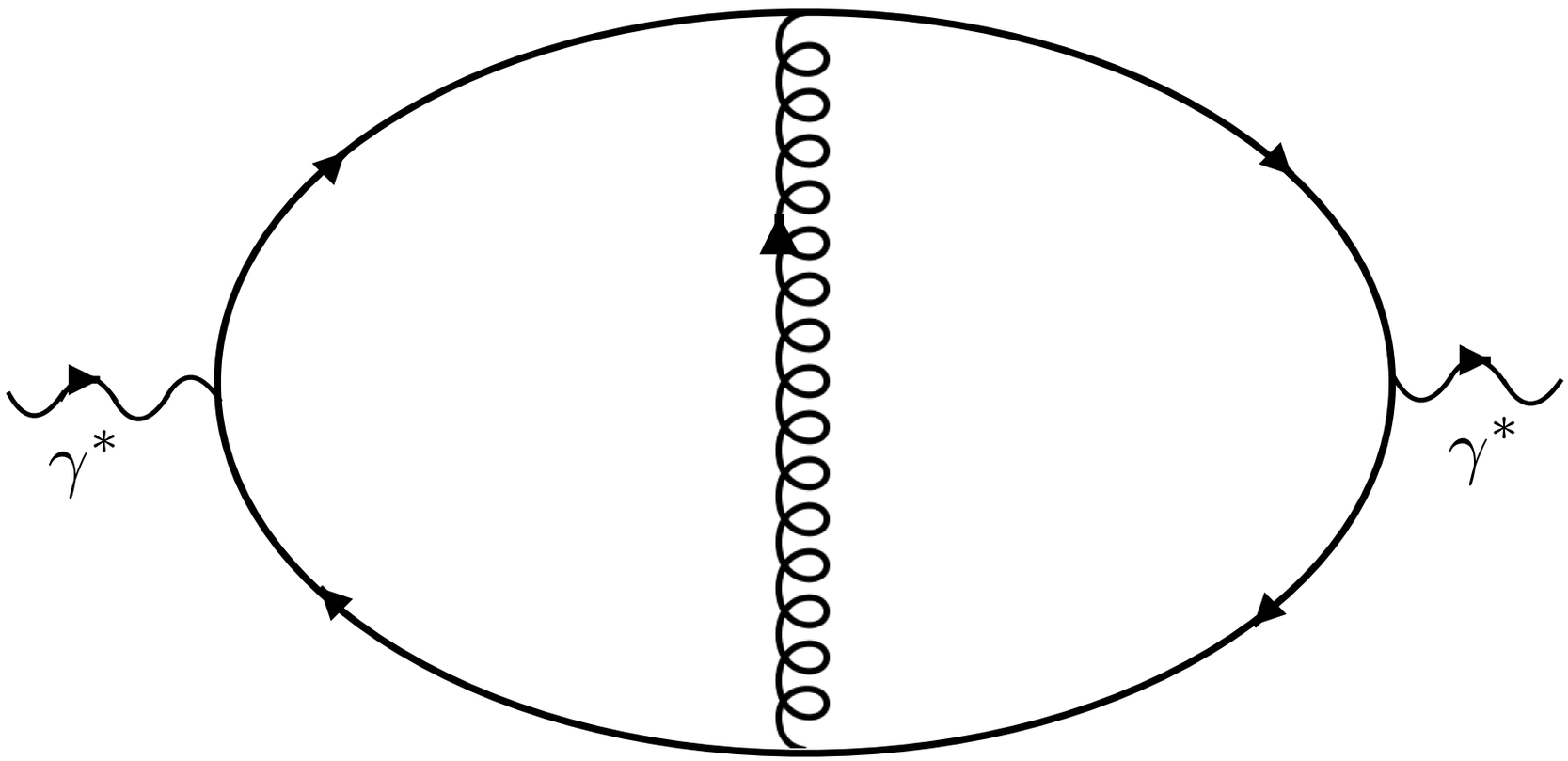
LU !

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]



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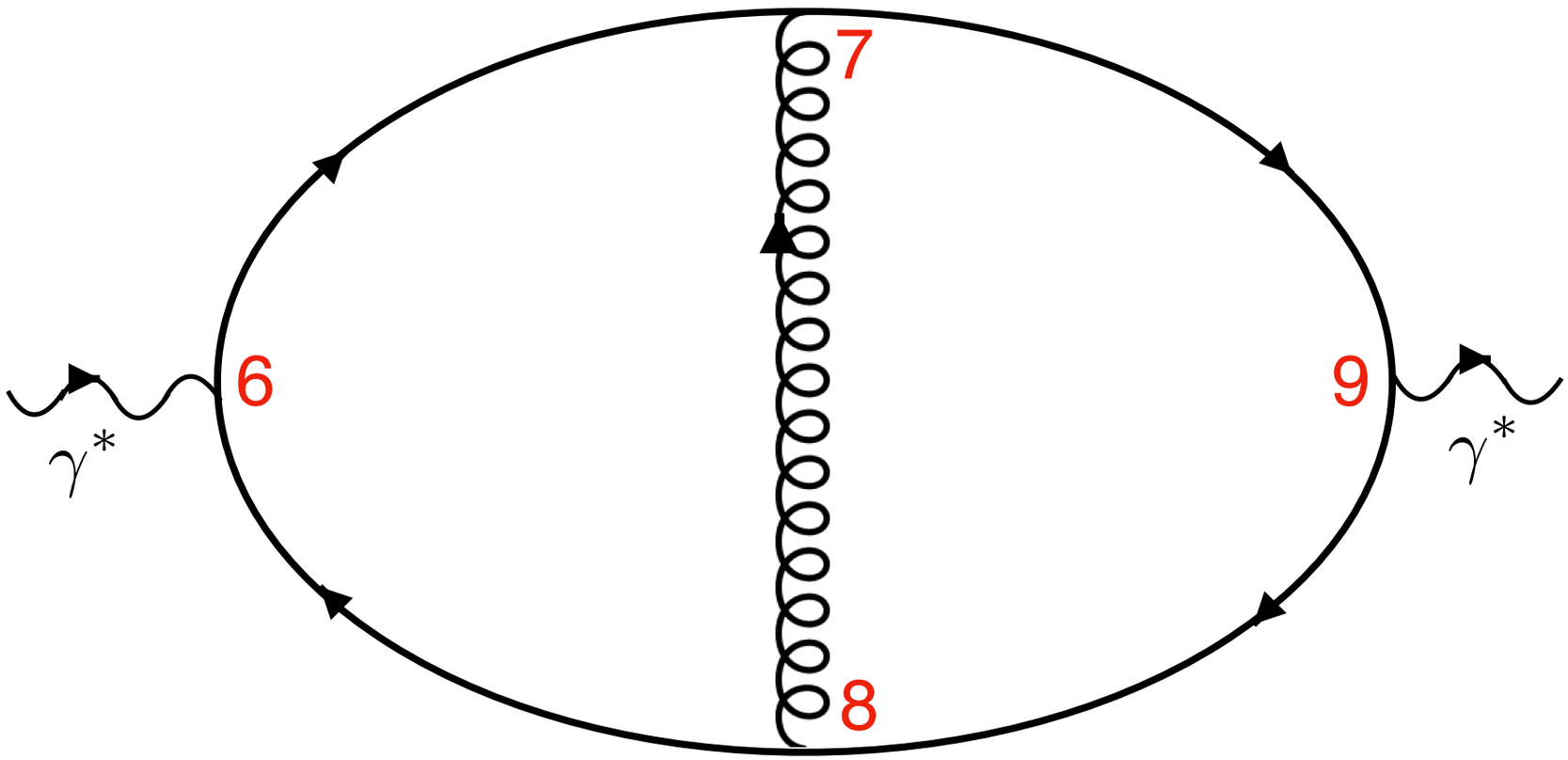

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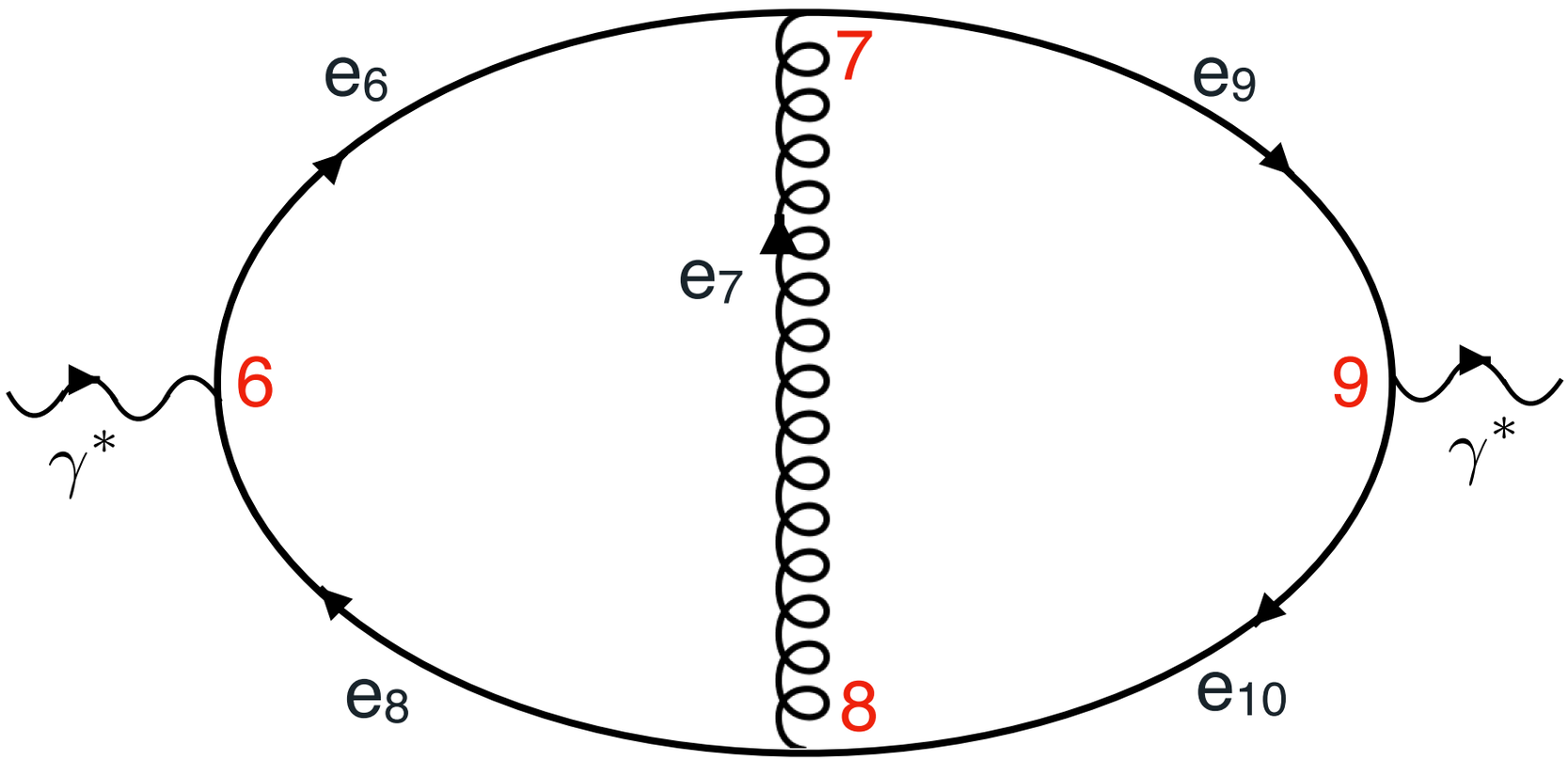

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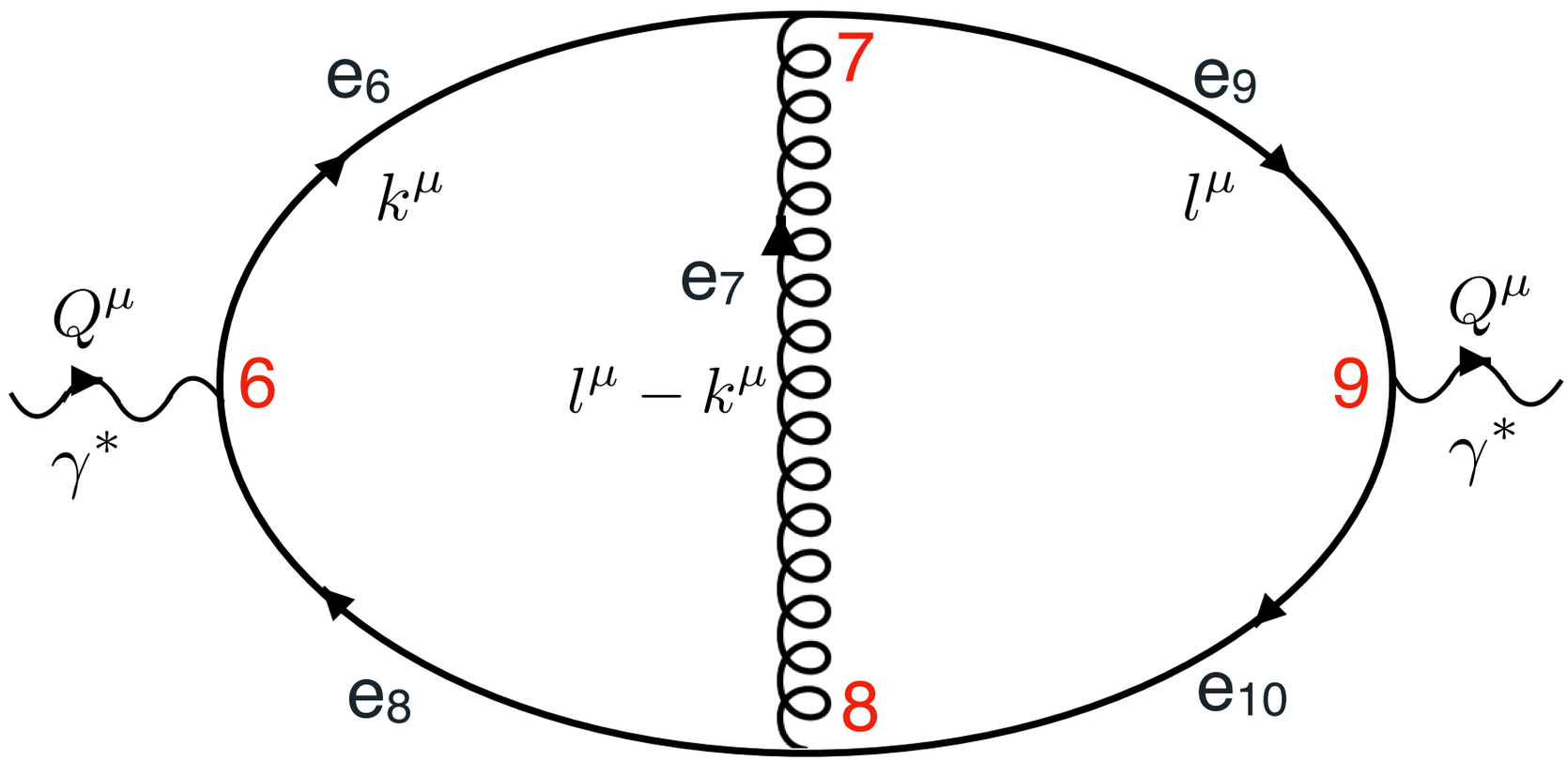

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$$I^{DT}(Q^\mu) = \sum_{\mathbf{b}\in\mathcal{B}} \int \frac{d^4\vec{k}'}{(2\pi)^3} \frac{d^4\vec{l}'}{(2\pi)^3} N \frac{\prod_{e\in\mathbf{b}} \delta^{(\sigma_e^{\mathbf{b}})}(q_e^2 - m_e^2)}{\prod_{e\in\mathbf{e}\setminus\mathbf{b}}(q_e^2 - m_e^2)}$$

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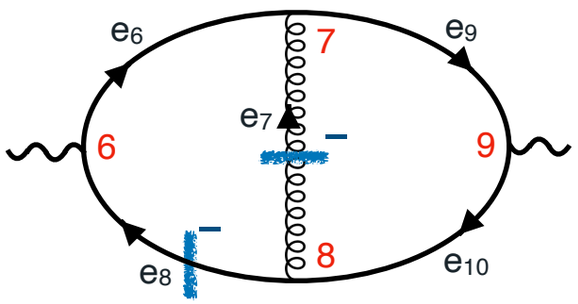

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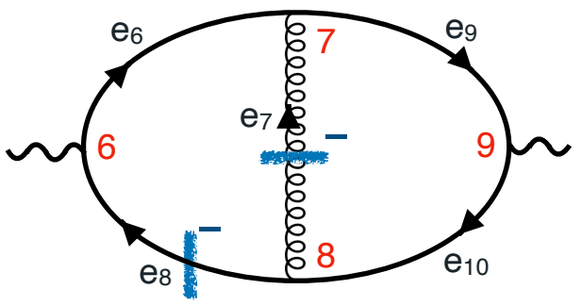

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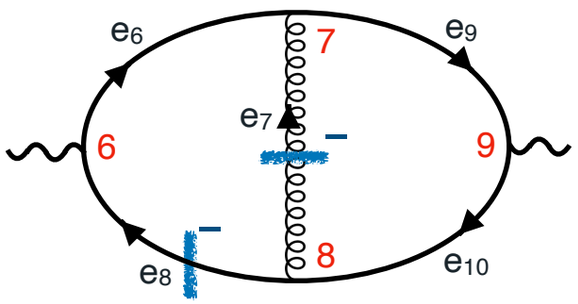

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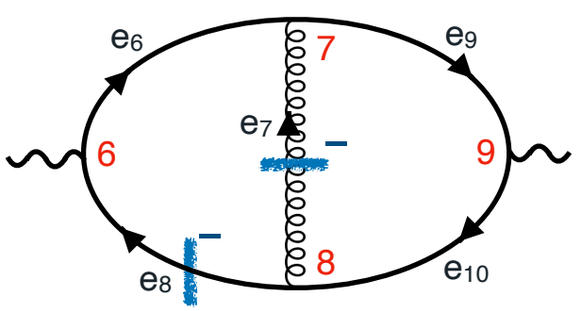

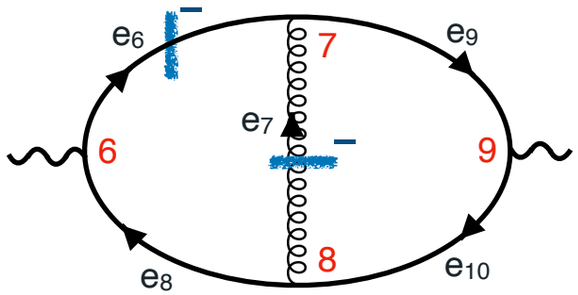

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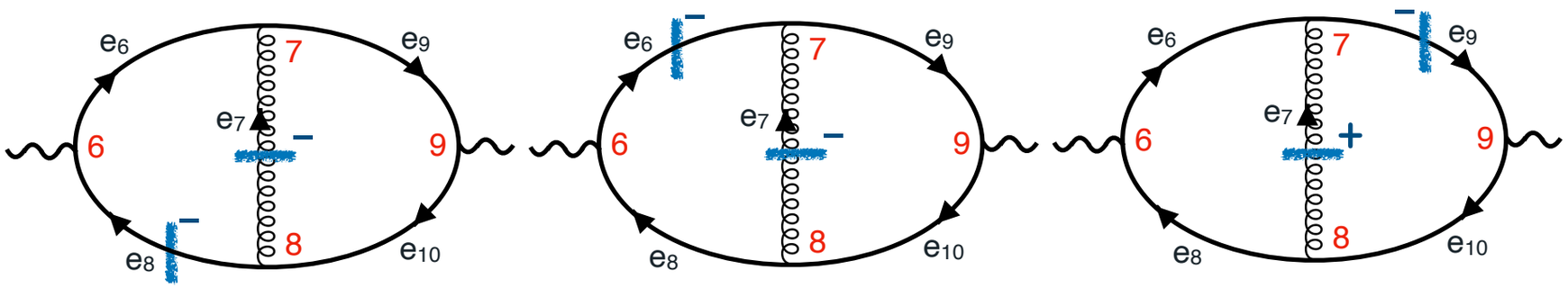

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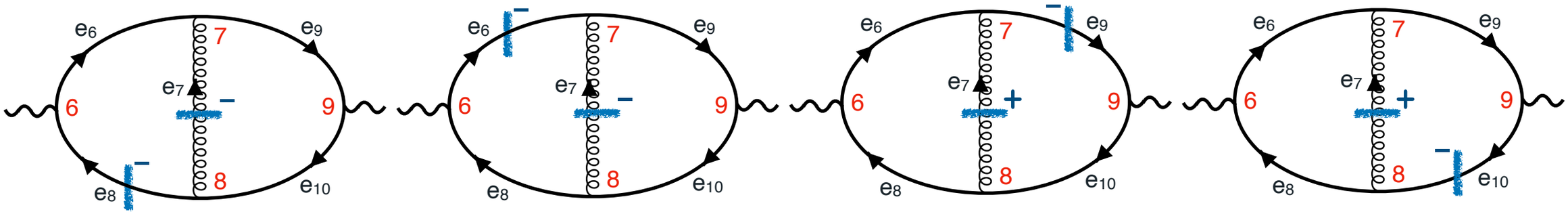

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$\mathbf{b}_3 = \{e_7, e_9\},$ $\sigma^{\mathbf{b}_3} = \{+,-\}$

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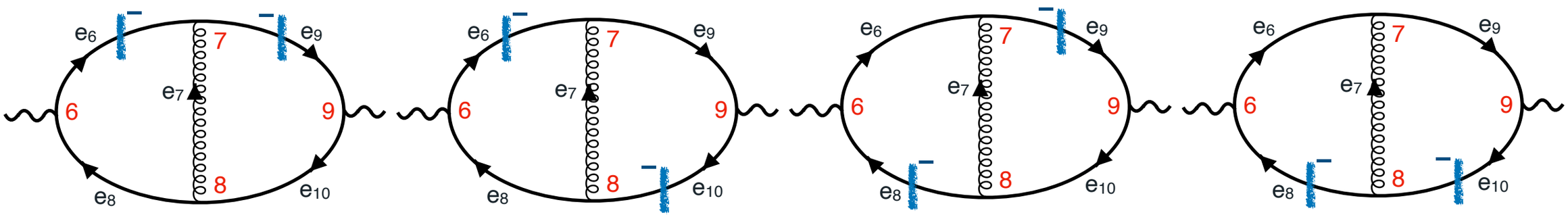


$\mathbf{b}_5 = \{e_6, e_9\},$ $\sigma^{\mathbf{b}_5} = \{-,-\}$

$\mathbf{b}_6 = \{e_6, e_{10}\},$ $\sigma^{\mathbf{b}_6} = \{-,-\}$

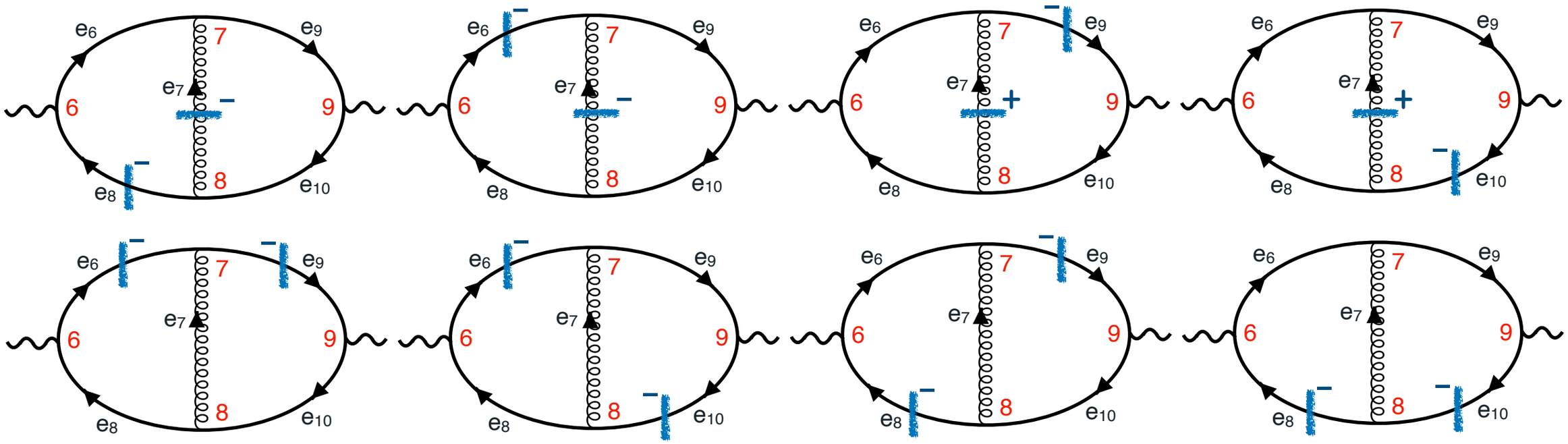
$\mathbf{b}_7 = \{e_8, e_9\},$ $\sigma^{\mathbf{b}_7} = \{-,-\}$

$\mathbf{b}_8 = \{e_8, e_{10}\}\ \}$ $\sigma^{\mathbf{b}_8} = \{-,-\}$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

The **threshold singularities** of the LTD representation are the **Cutkosky Cuts** :



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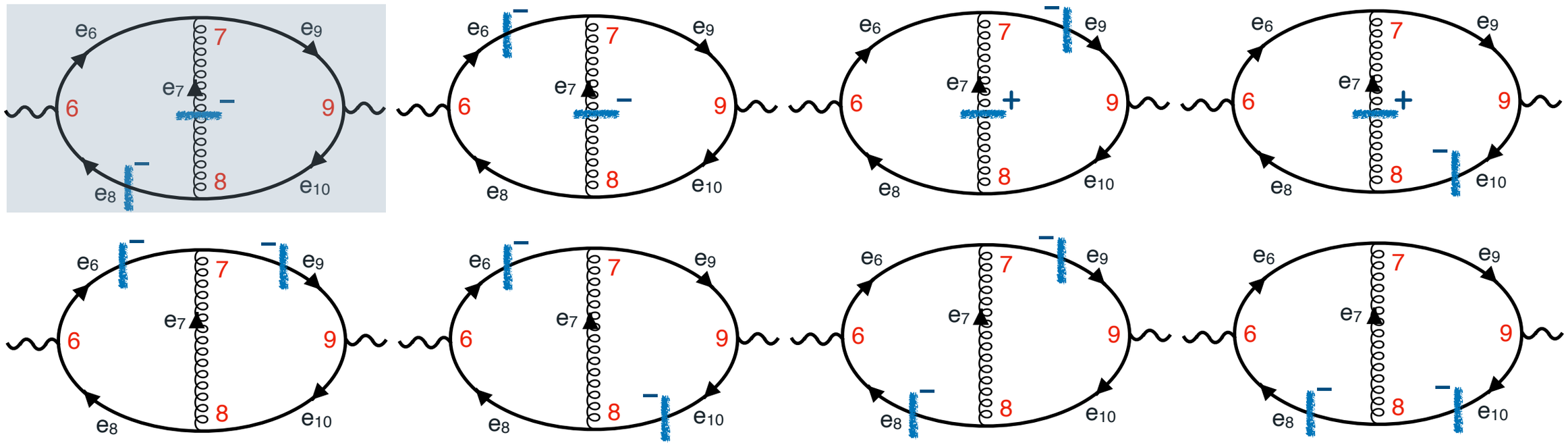

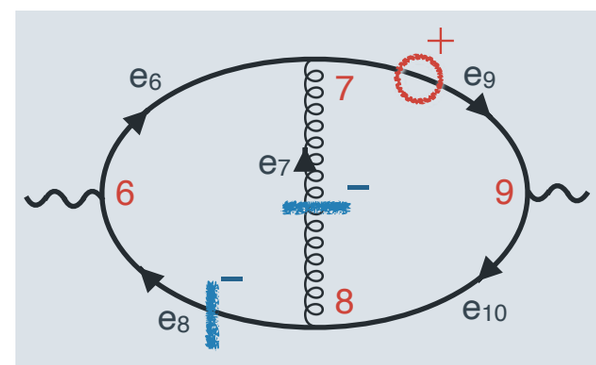

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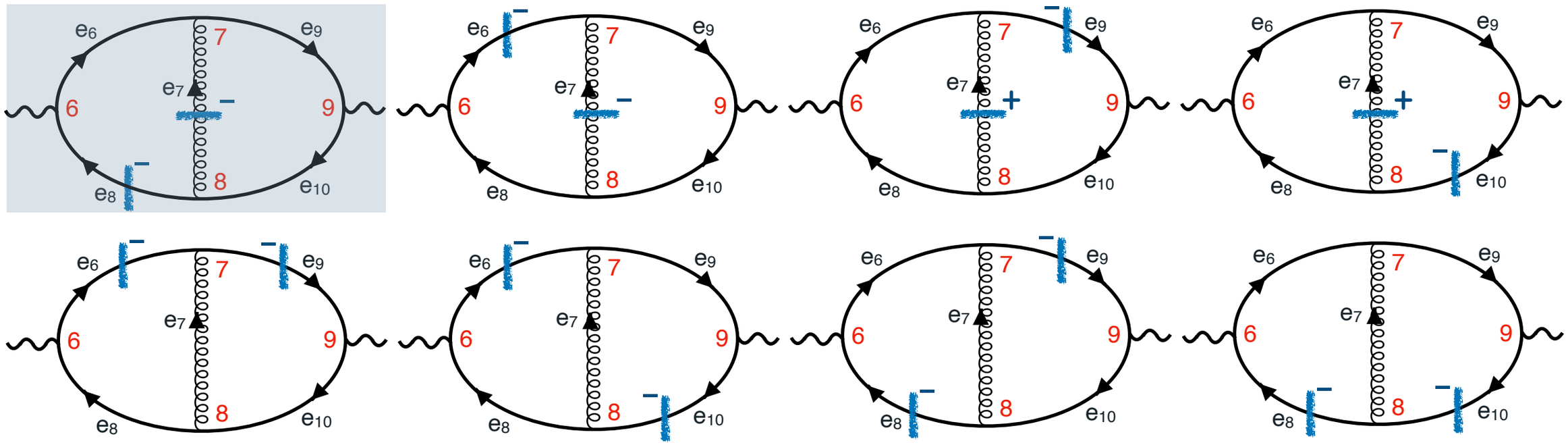

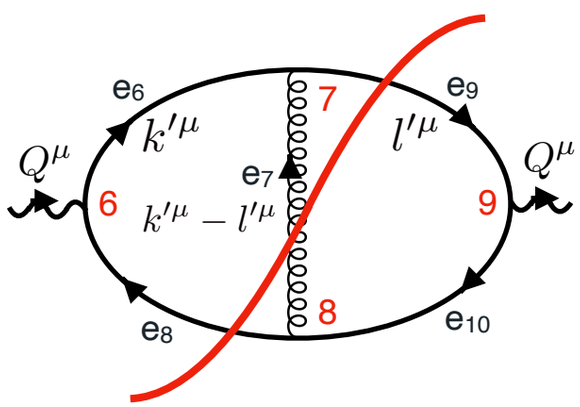

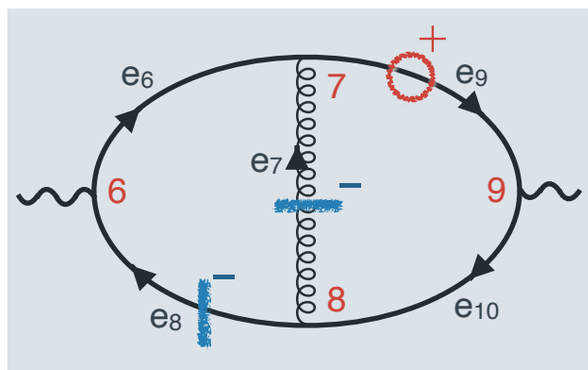

$$\mathbf{s}_2^r = \{6, 7\} \quad \mathbf{c}_{\mathbf{s}_2^r} = \{e_8, e_7, e_9\}$$

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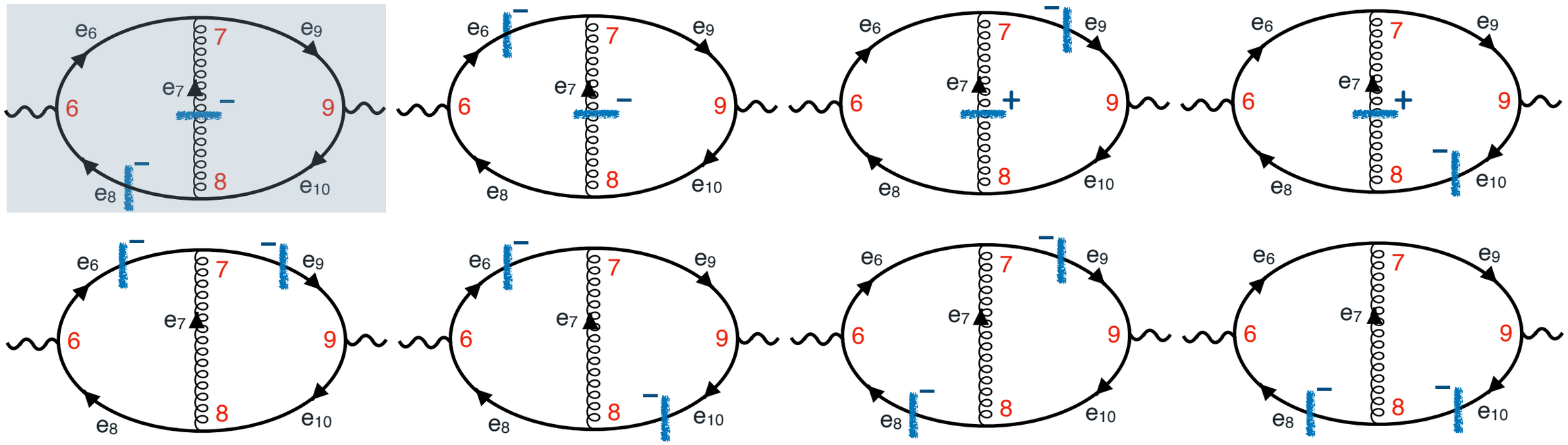

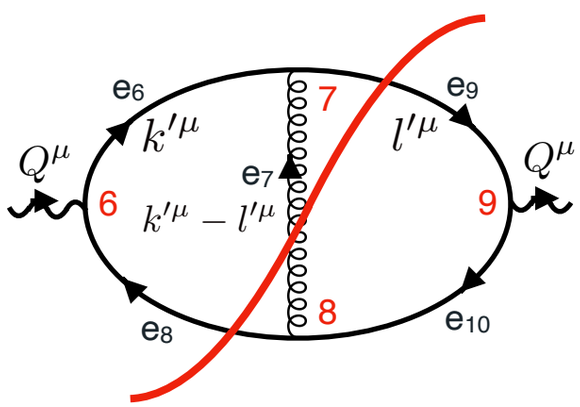

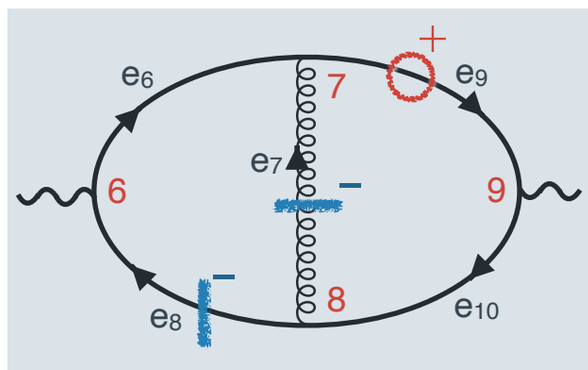

$$\mathbf{s}_2^r = \{6, 7\} \quad \mathbf{c}_{\mathbf{s}_2^r} = \{e_8, e_7, e_9\} \quad \subset \quad \delta^-(q^2_{e_7}) \quad \delta^-(q^2_{e_8}) \quad \delta^+(q^2_{e_9})$$

Locality Unitarity

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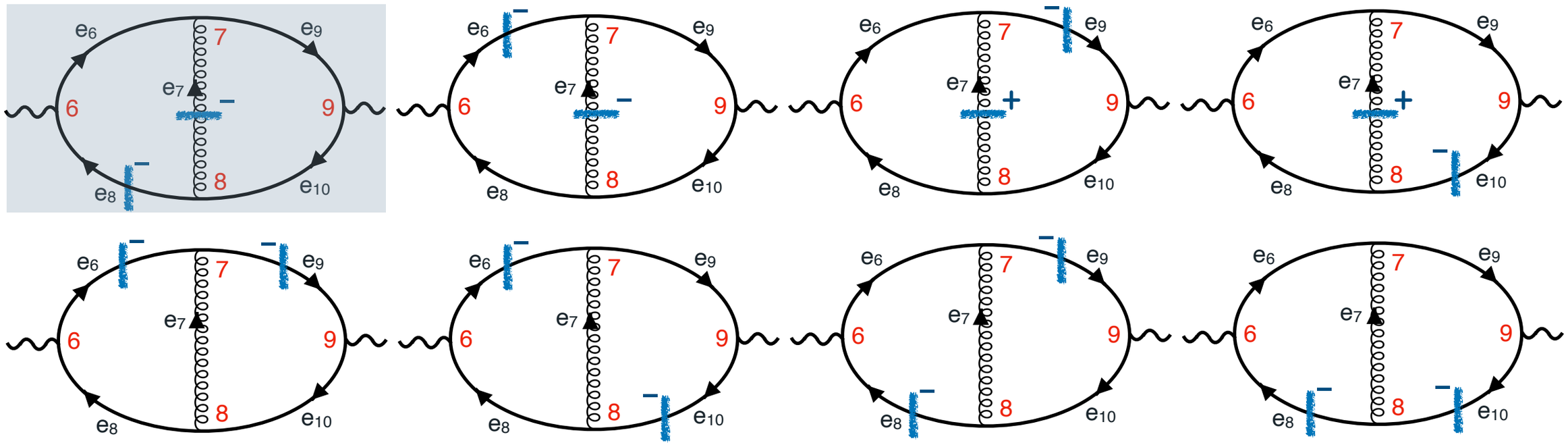

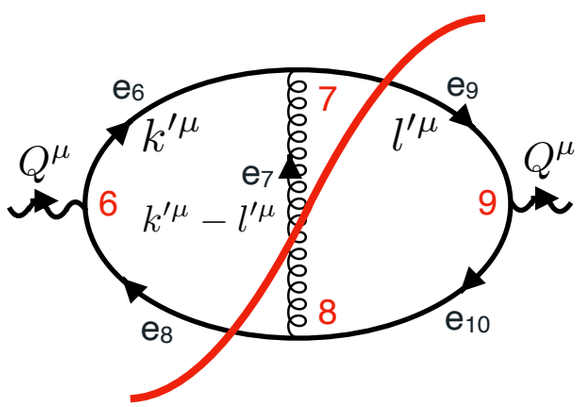

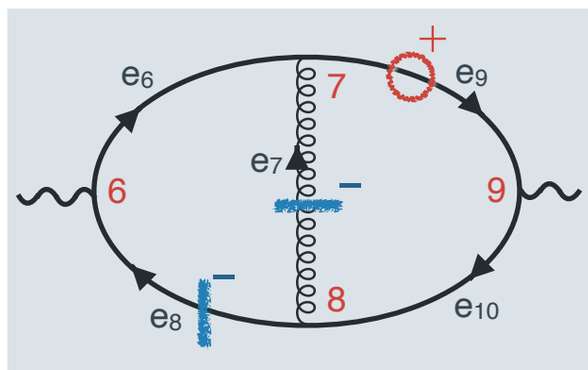

$$\mathbf{s}_2^r = \{6, 7\} \quad \mathbf{c}_{\mathbf{s}_2^r} = \{e_8, e_7, e_9\} \quad \subset \quad \begin{array}{c|c|c} \delta^-(q_{e_7}^2) & \delta^-(q_{e_8}^2) & \delta^+(q_{e_9}^2) \\ \downarrow & \downarrow & \downarrow \\ -q_{e_7}^0 = E_{e_7} & -q_{e_8}^0 = E_{e_8} & +q_{e_9}^0 = E_{e_9} \end{array}$$

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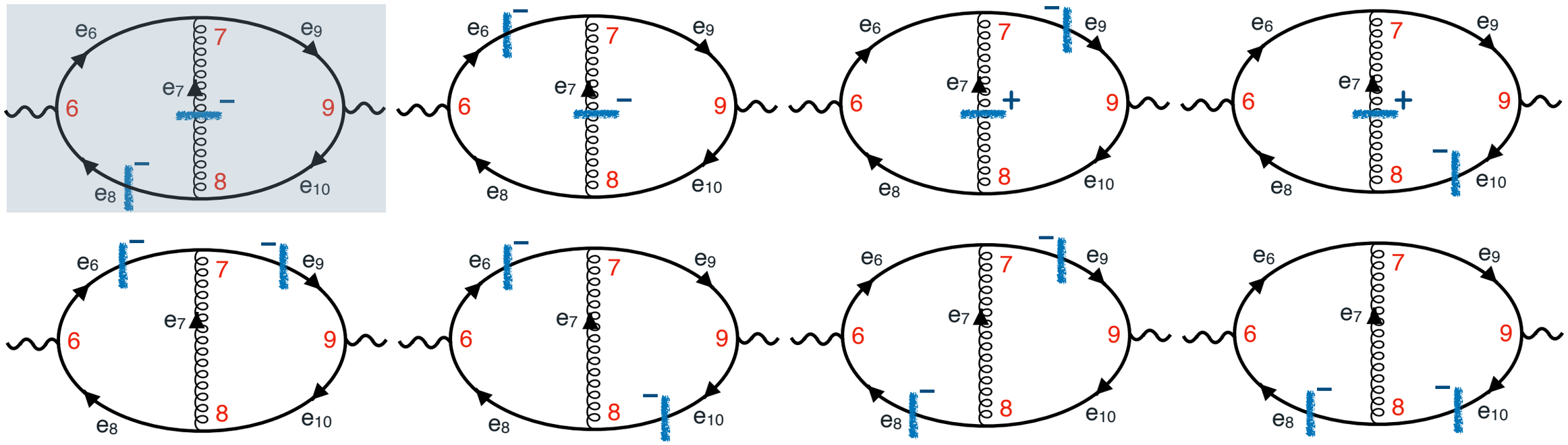

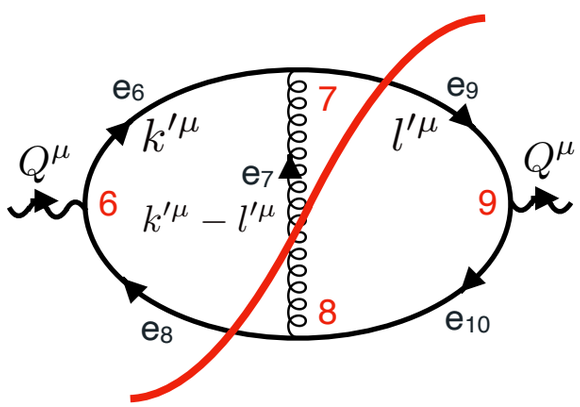

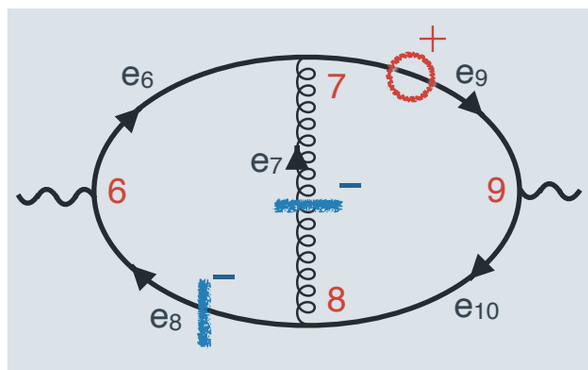

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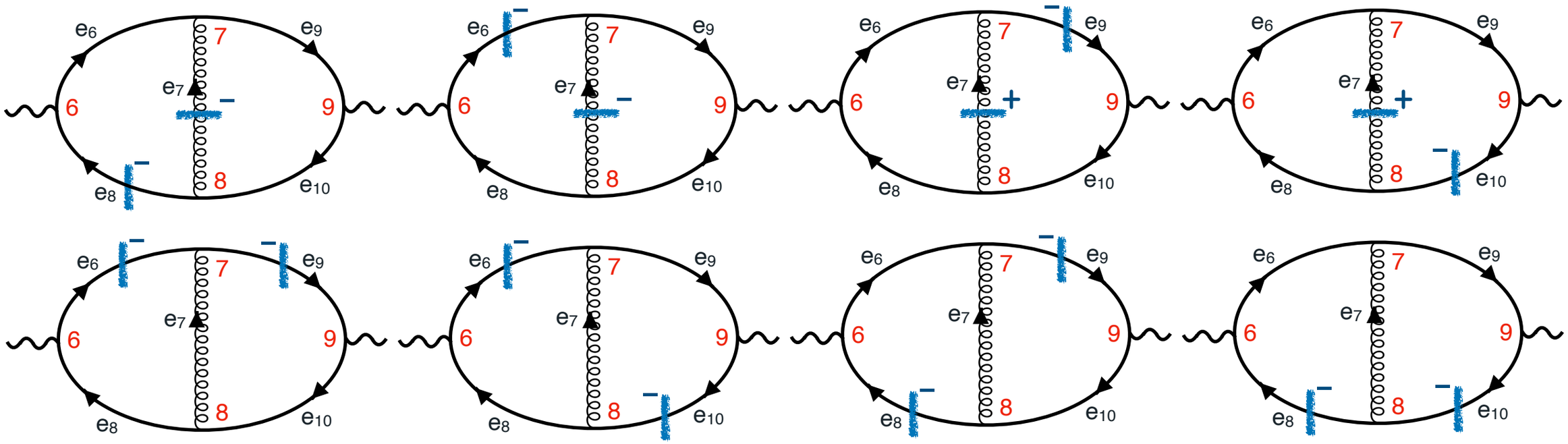

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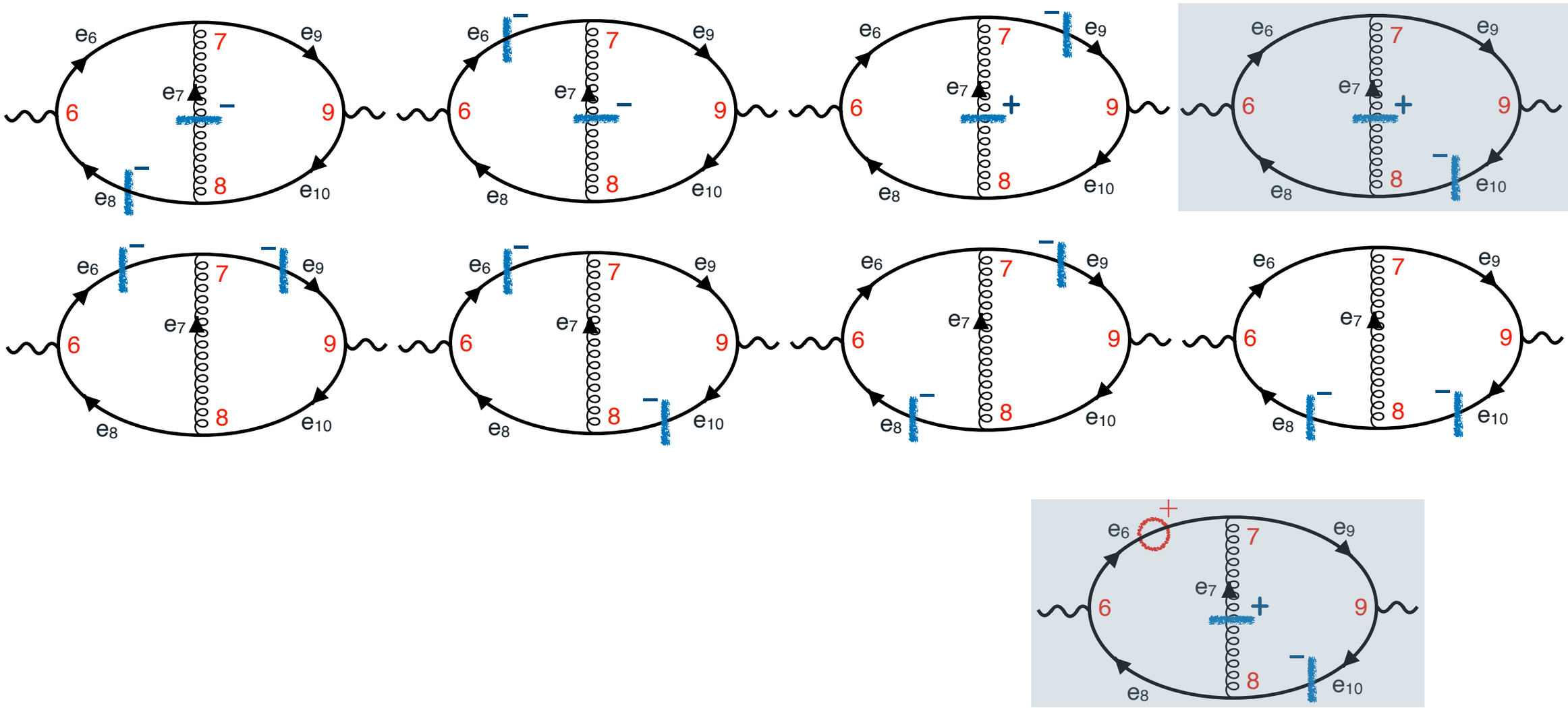

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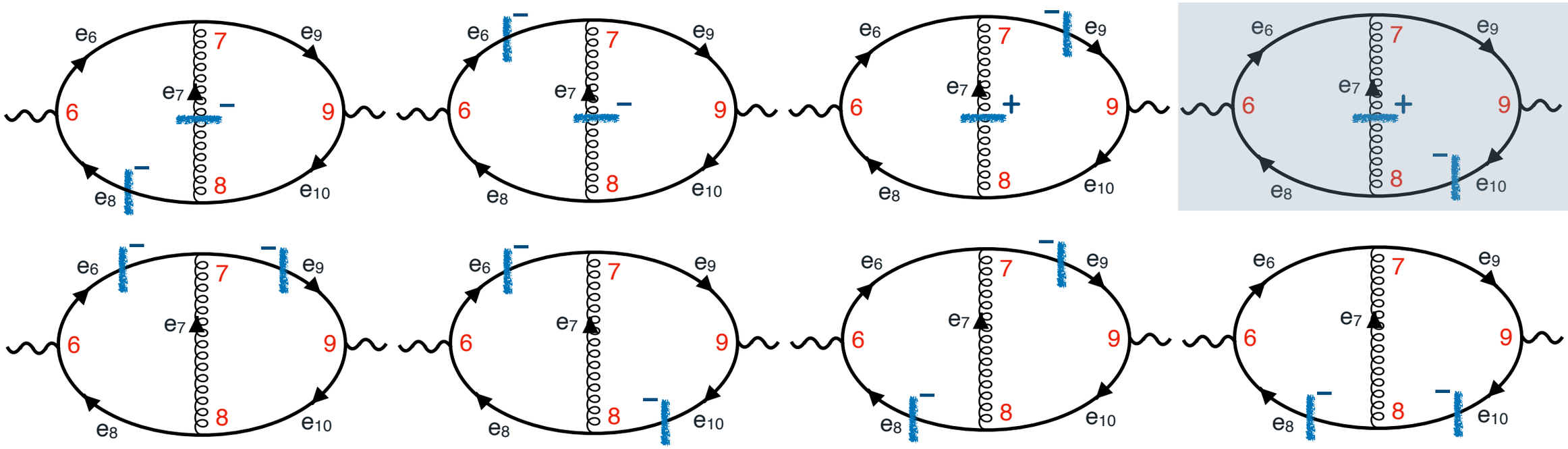

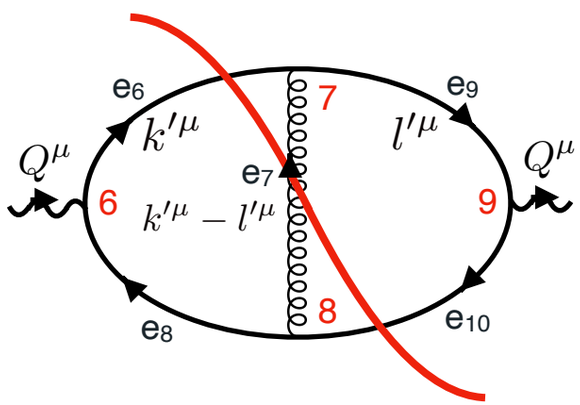

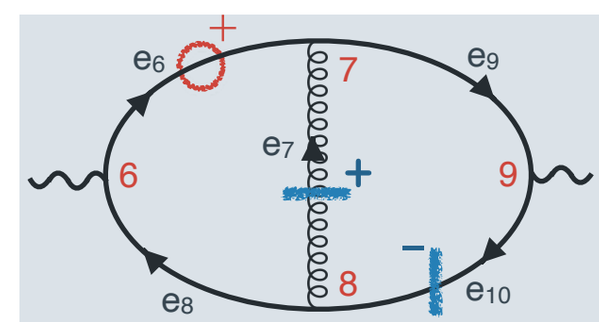

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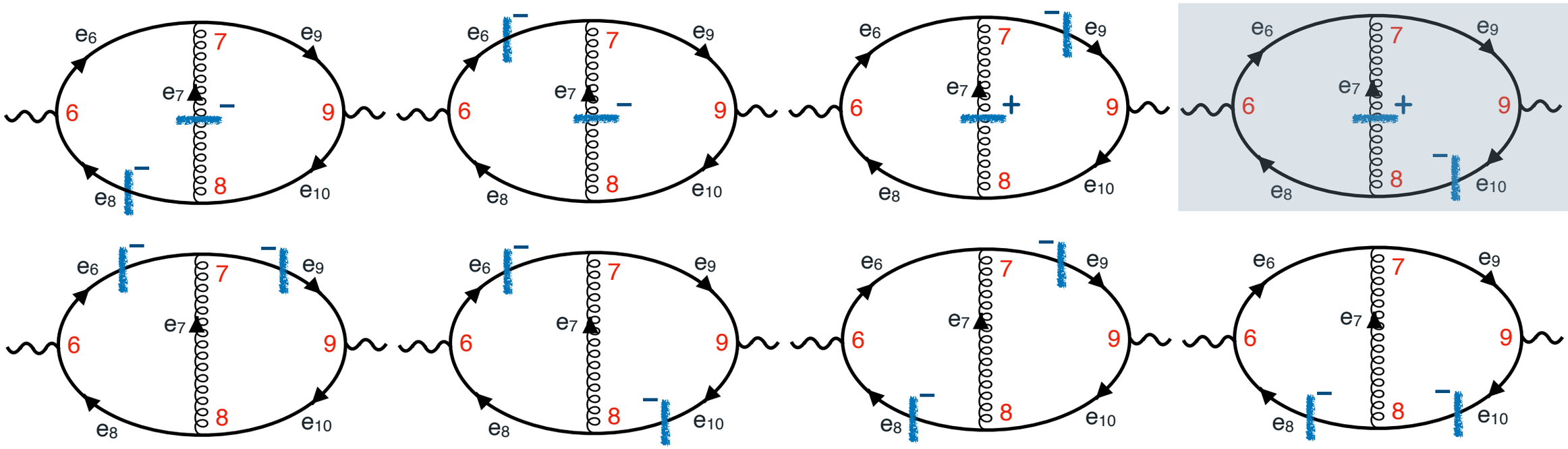

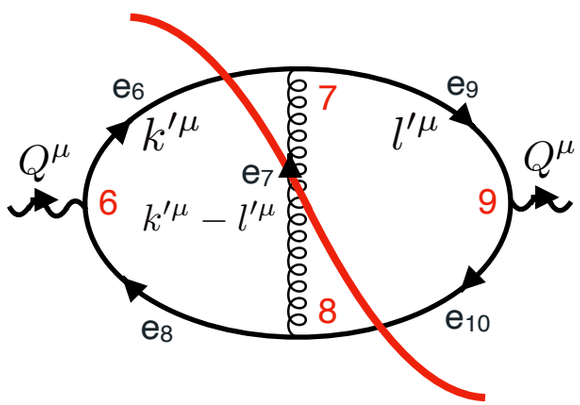

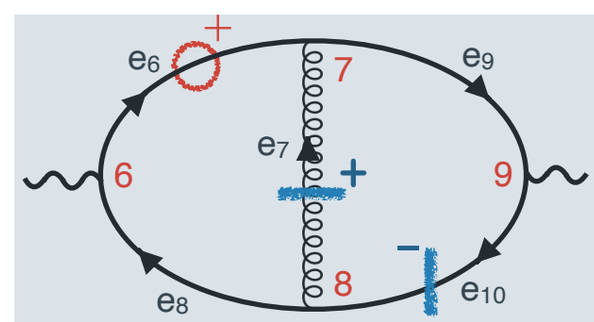

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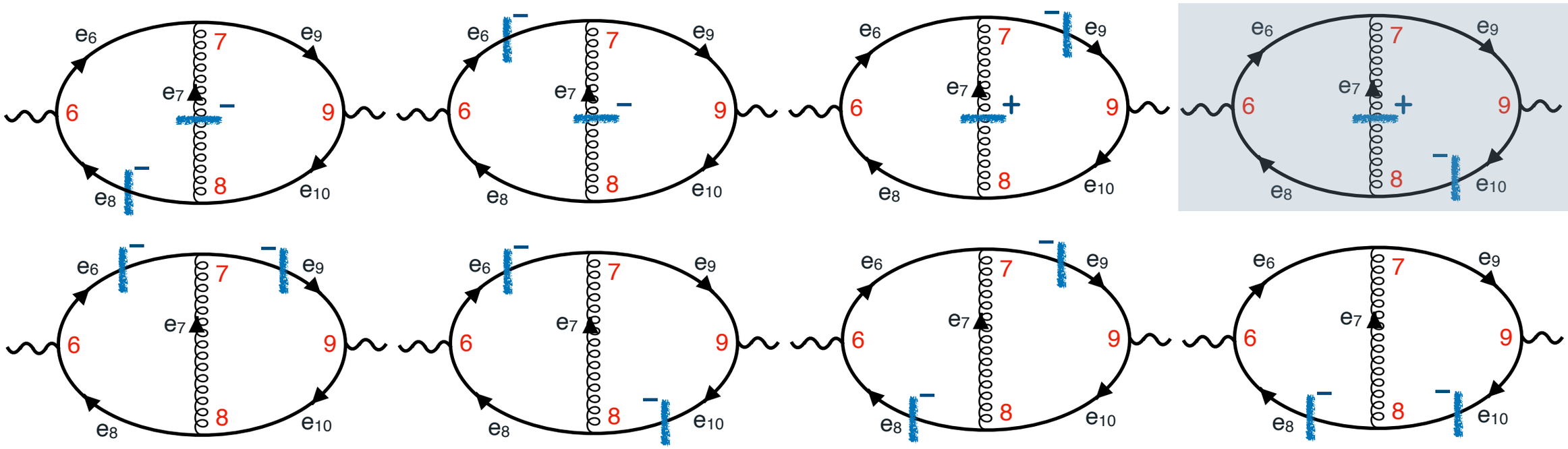

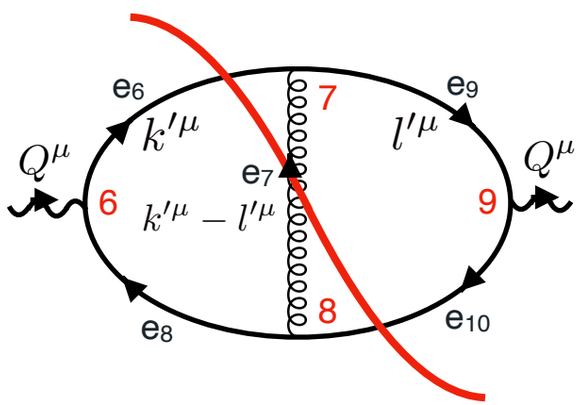

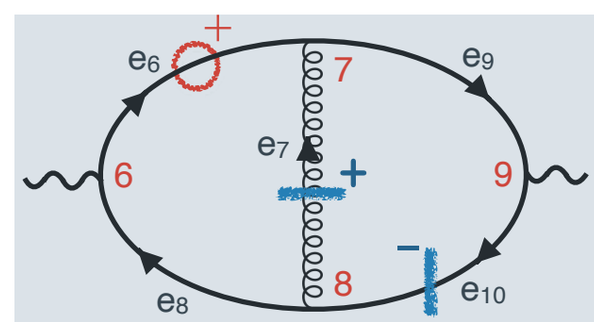

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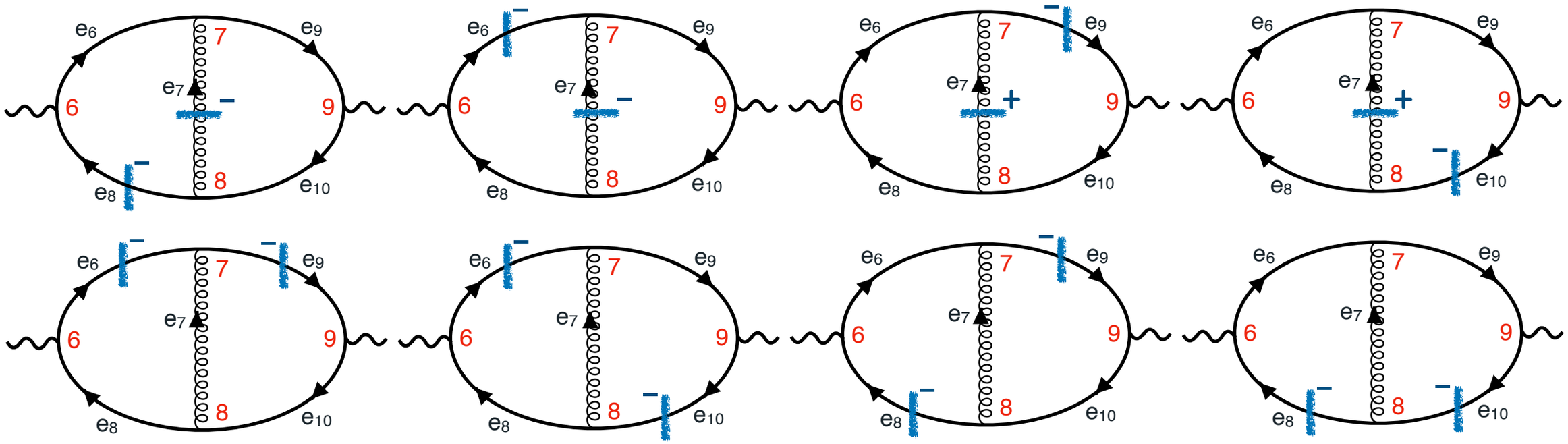

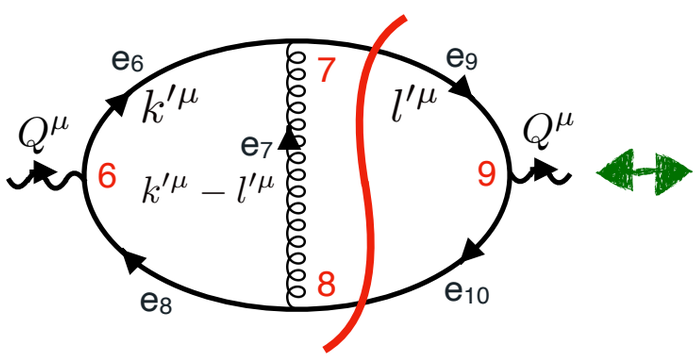

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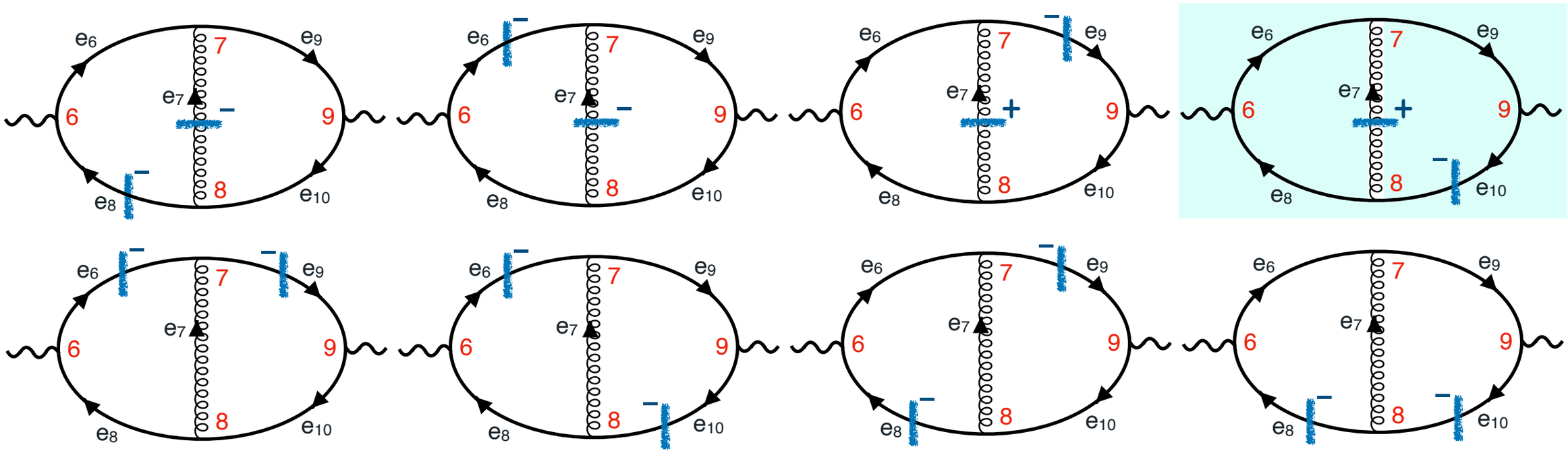

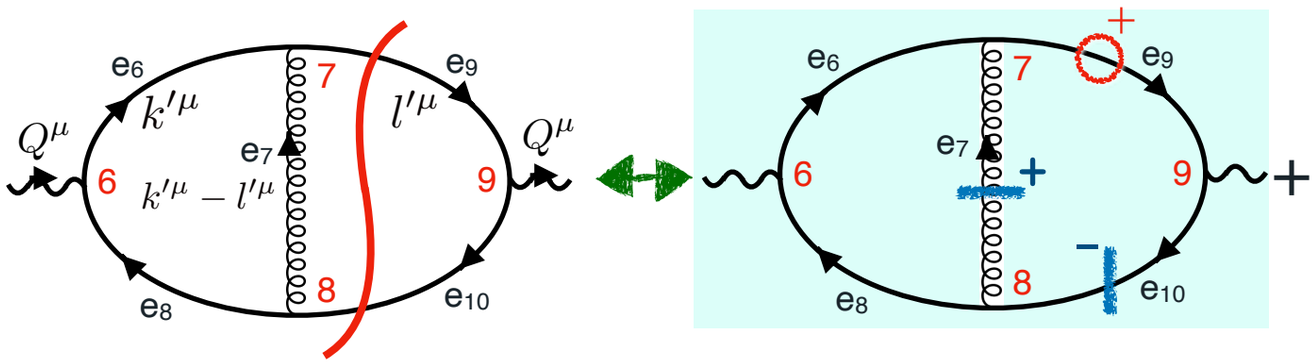

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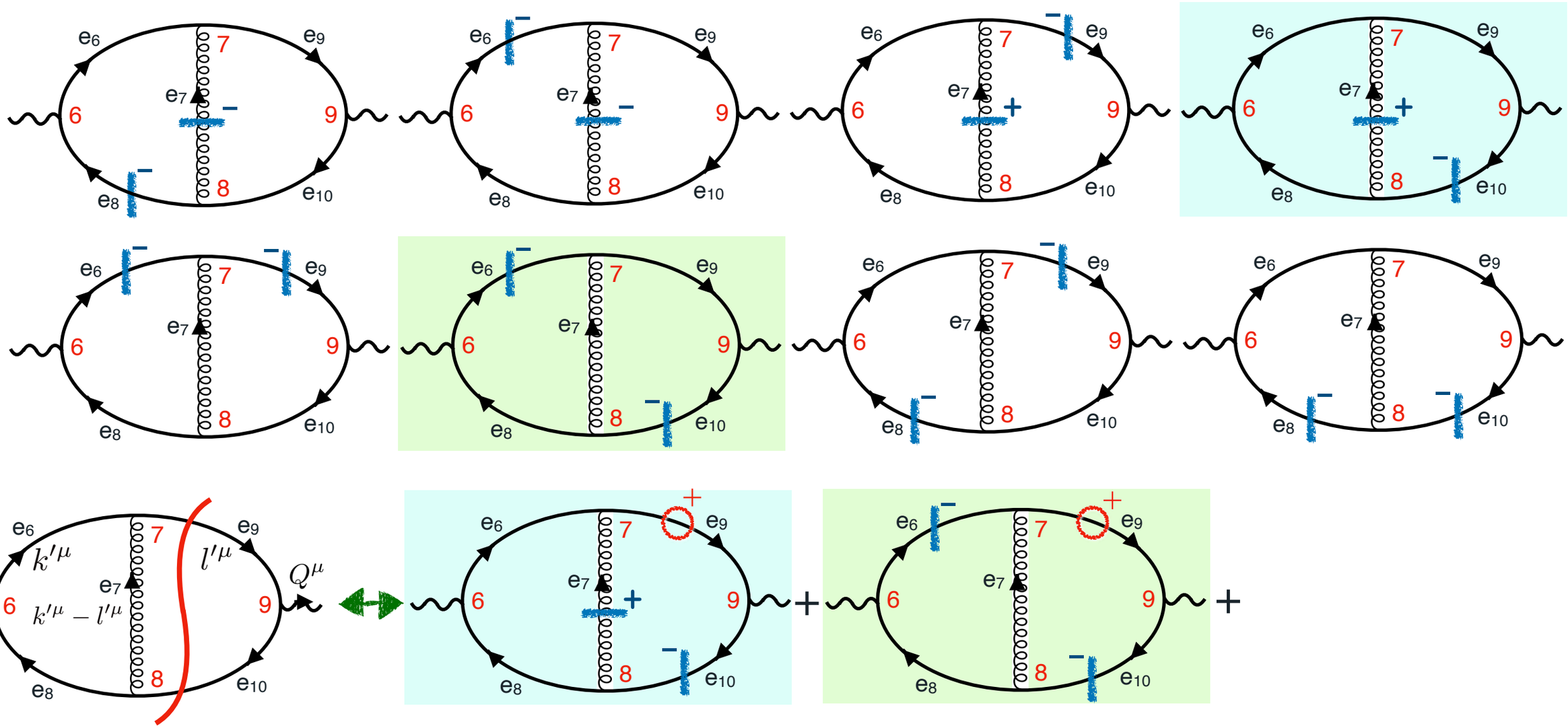

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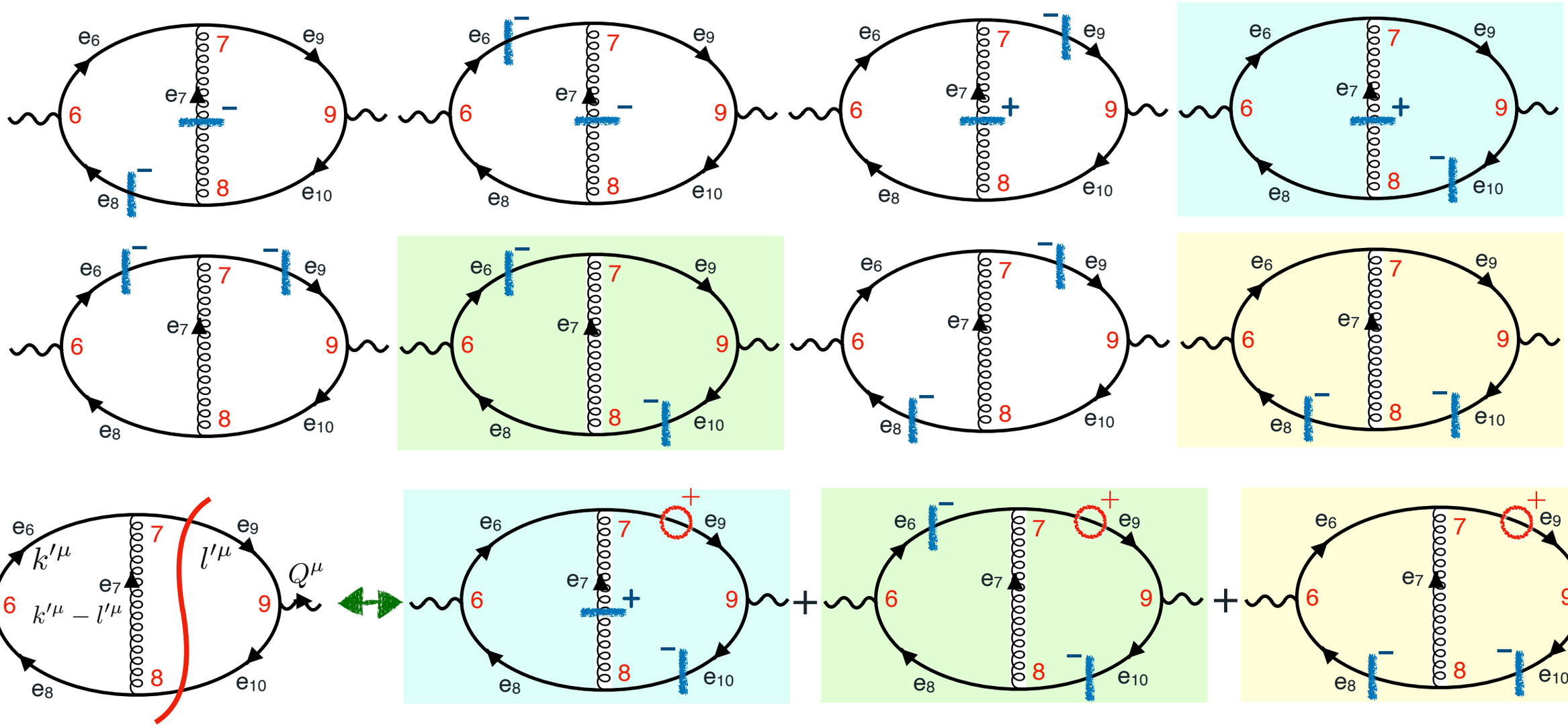

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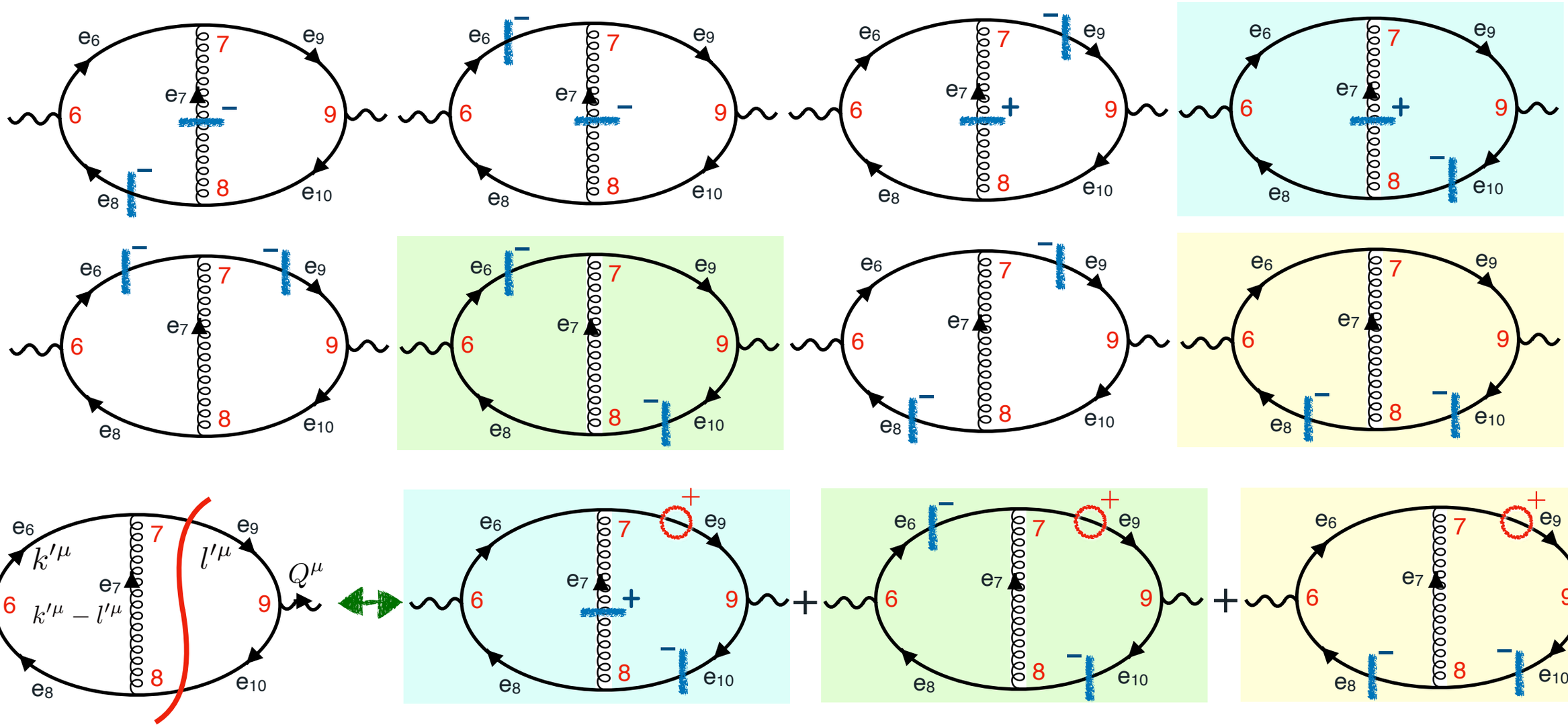

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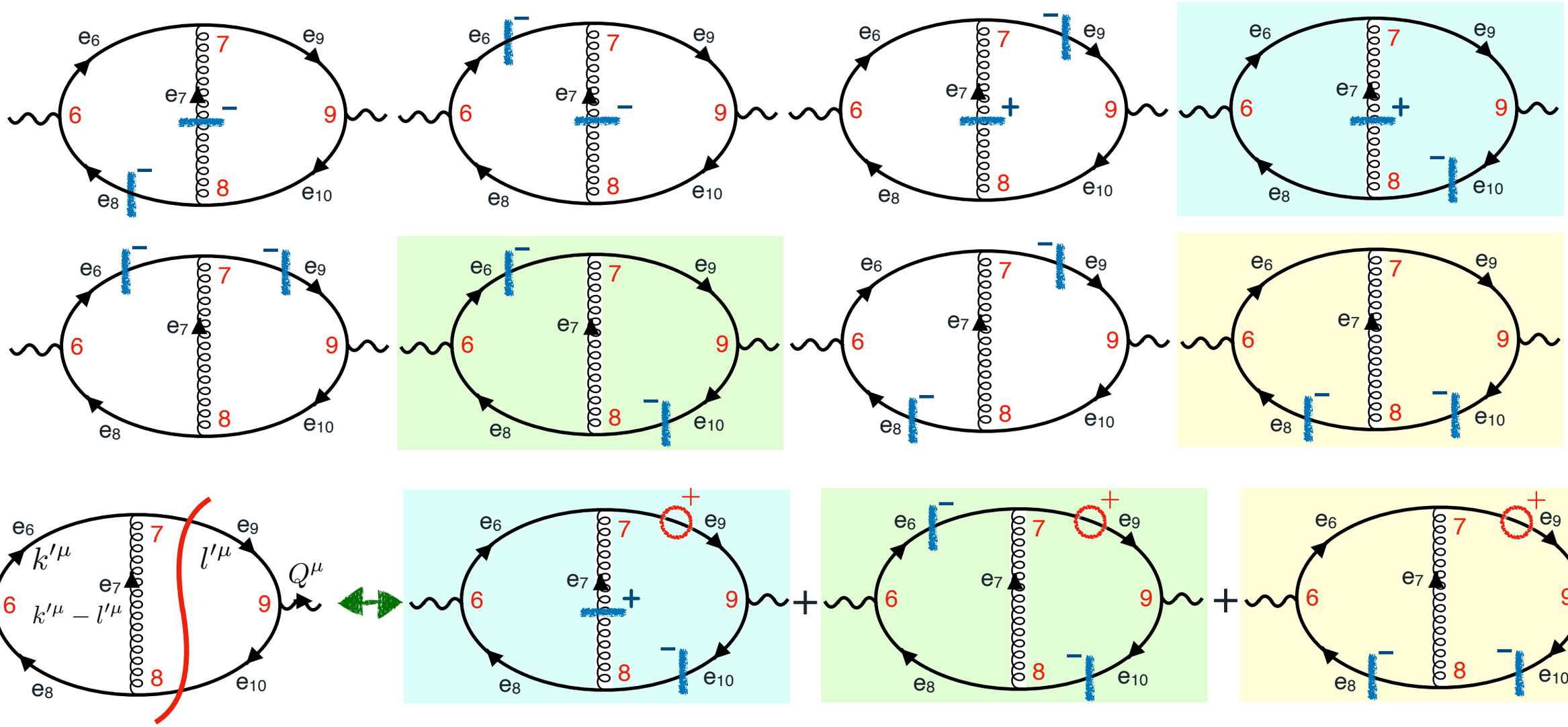

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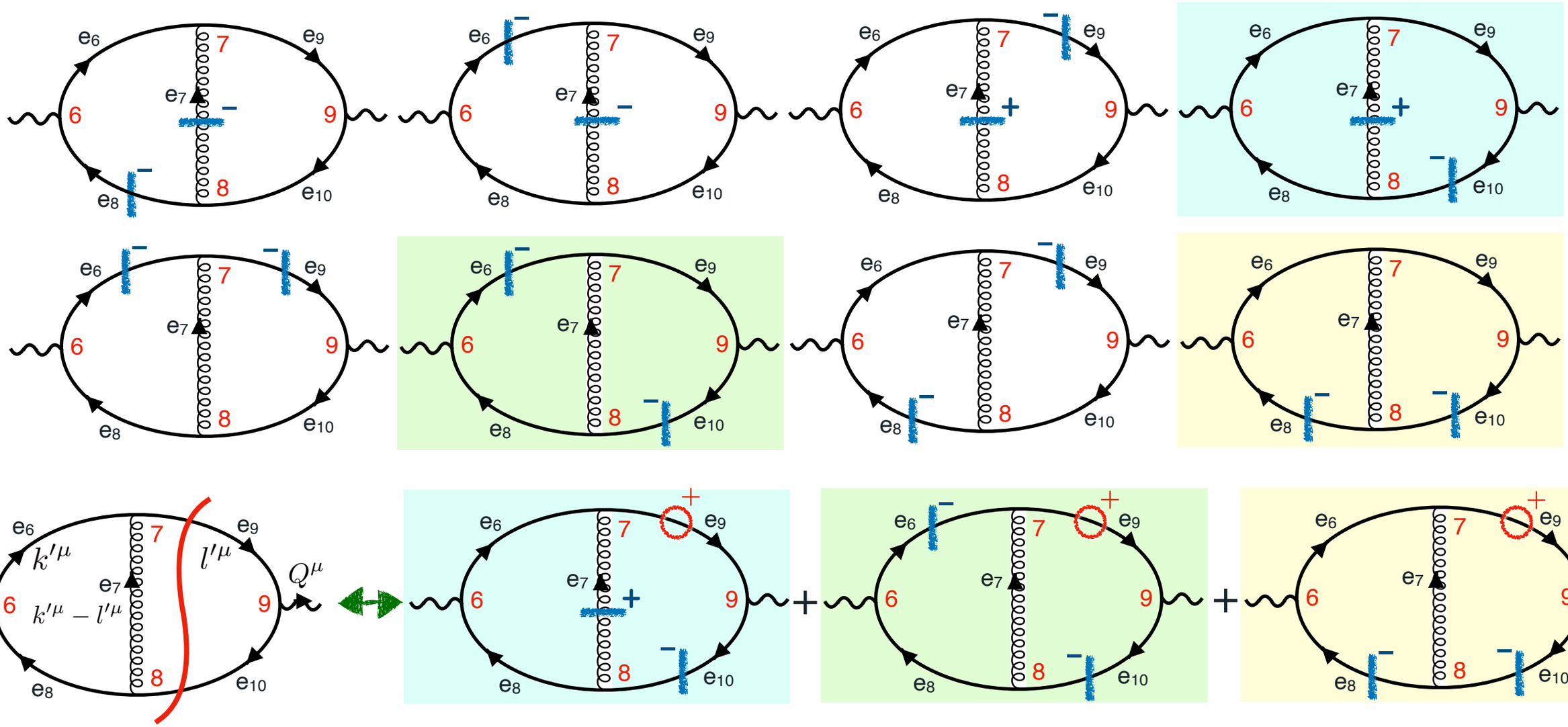

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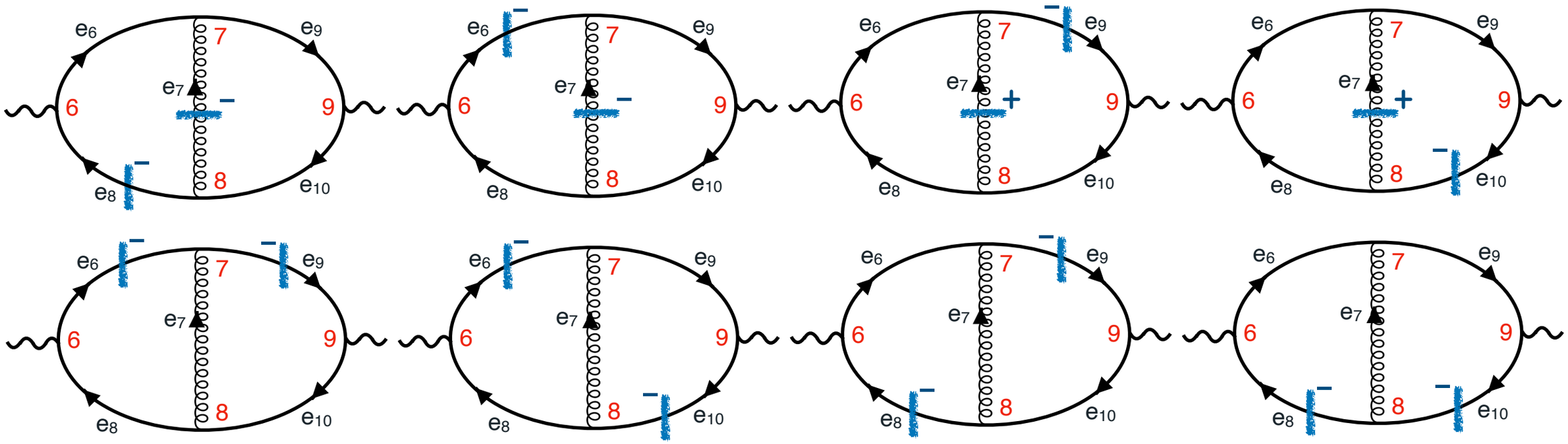

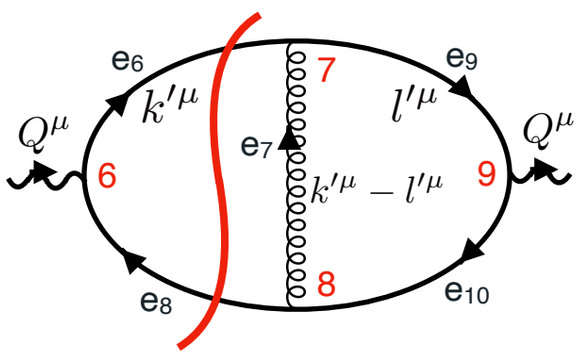

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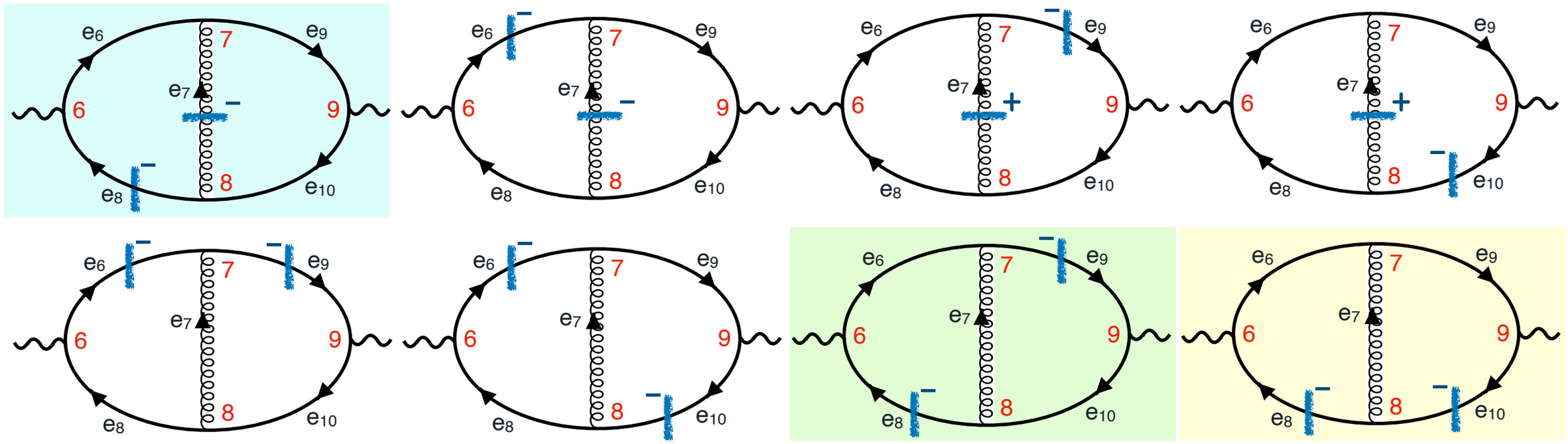

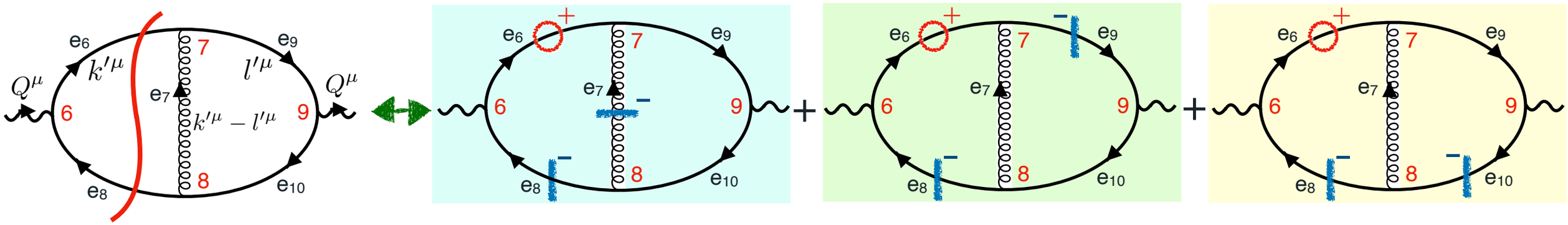

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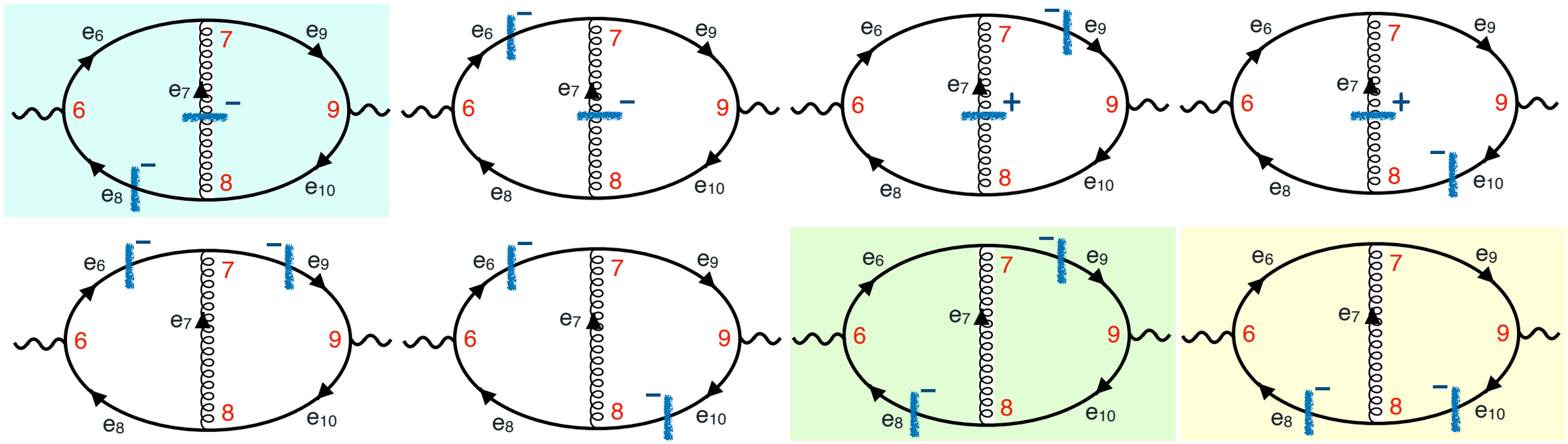

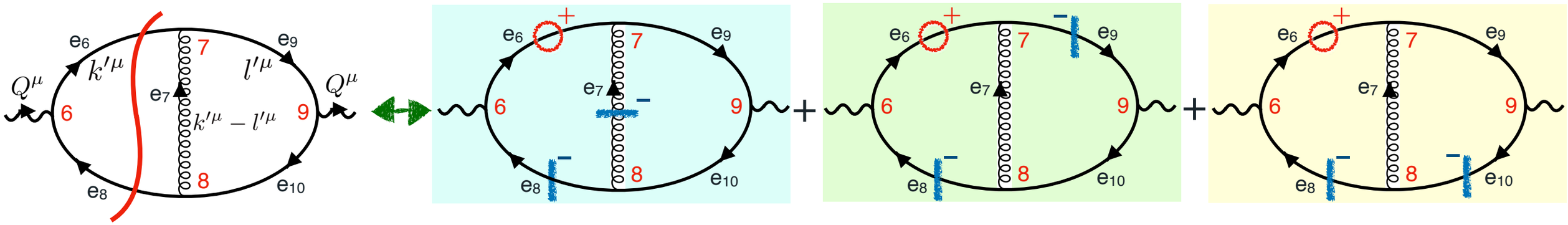

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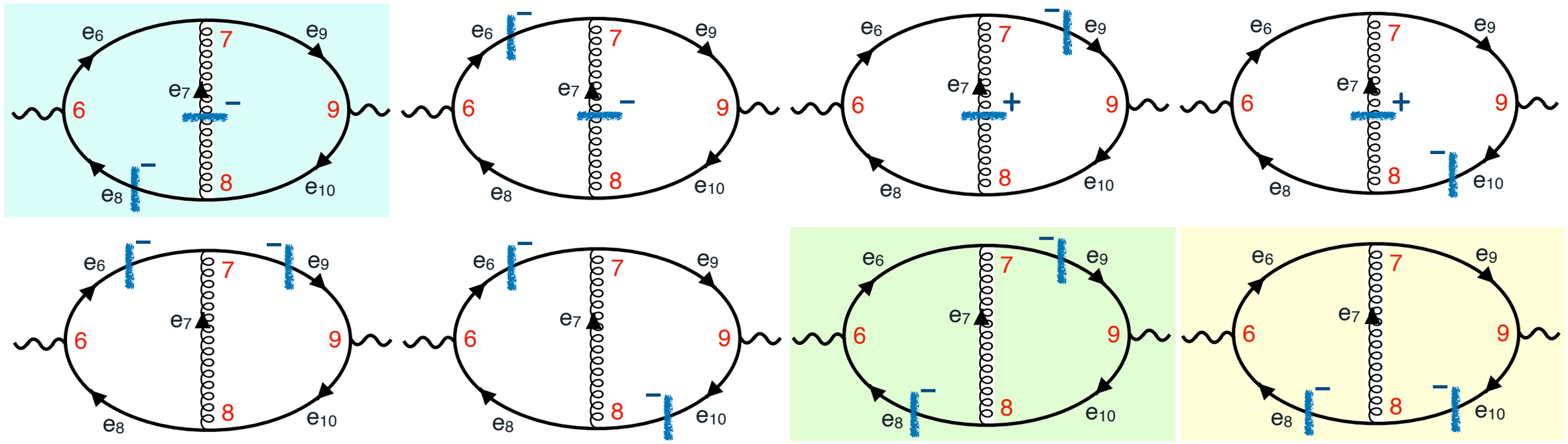

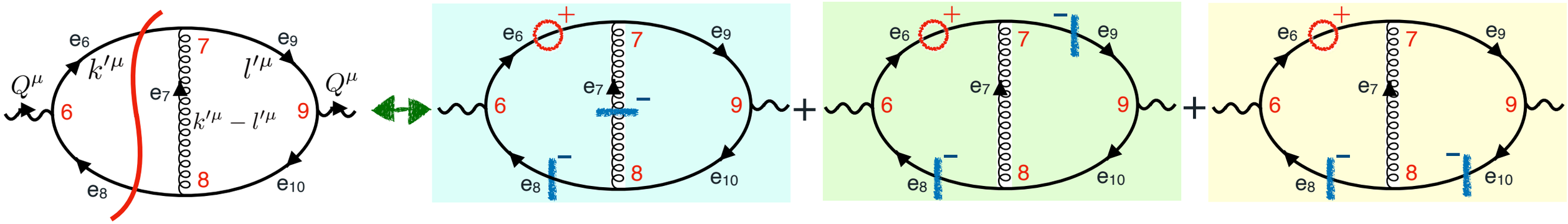

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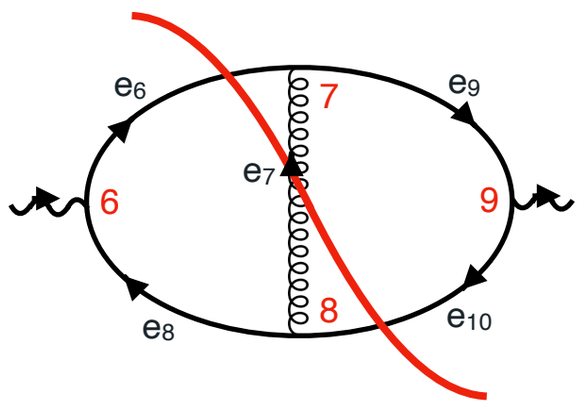
How to **align** the integration measure for all the contributing **Cutkosky Cuts** ?

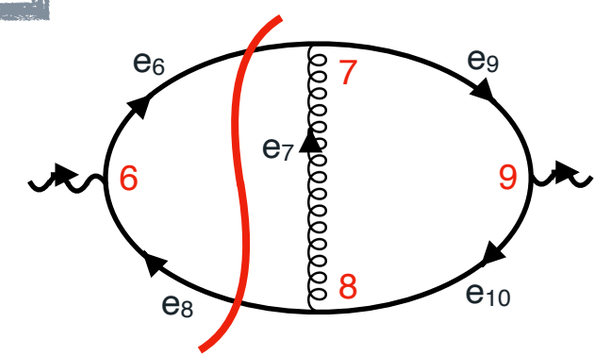
$\mathbf{s}_1^{\mathrm{r}}$

$\sigma_{\mathrm{LU}}^{(DT)} =$



$+$

$\mathbf{s}_1^{\mathrm{v}}$

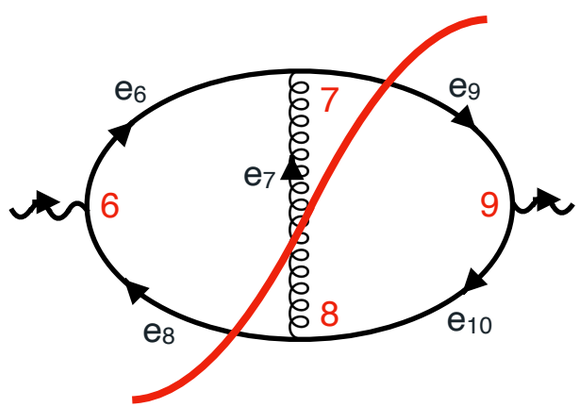


$$\sim \delta\Big(\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}', Q^0) = E_{e_7} + E_{e_{10}} + E_{e_6} - Q^0\Big)$$

$$\sim \delta\Big(\eta_{\mathbf{s}_1^{\mathrm{v}}}(\vec{k}', \vec{l}', Q^0) = E_{e_6} + E_{e_8} - Q^0\Big)$$

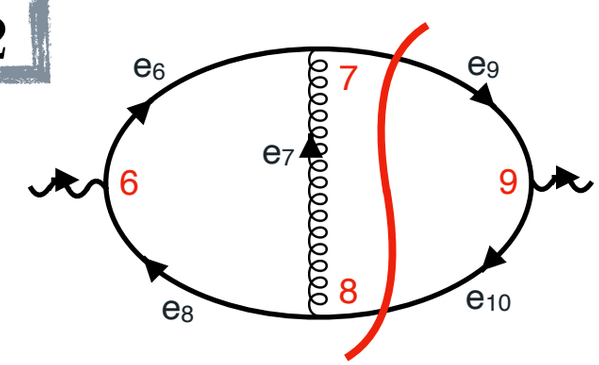
$\mathbf{s}_2^{\mathrm{r}}$

$+$



$+$

$\mathbf{s}_2^{\mathrm{v}}$



$$\sim \delta\Big(\eta_{\mathbf{s}_2^{\mathrm{r}}}(\vec{k}', \vec{l}', Q^0) = E_{e_7} + E_{e_8} + E_{e_9} - Q^0\Big)$$

$$\sim \delta\Big(\eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}', Q^0) = E_{e_9} + E_{e_{10}} - Q^0\Big)$$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

$$\sigma^{DT}_{e^+e^-\to d\bar{d}} = \sum_{\mathbf{s}\in\{\mathbf{s}_1^{\mathrm{r}},\mathbf{s}_2^{\mathrm{r}},\mathbf{s}_1^{\mathrm{v}},\mathbf{s}_2^{\mathrm{v}}\}} \int \frac{d^4k'}{(2\pi)^4}\frac{d^4l'}{(2\pi)^4}\,\mathcal{I}^{DT}_{\mathbf{s}} \prod_{I\in\mathbf{b}\cup\mathbf{b}'\cup\mathbf{c}_{\mathbf{s}}} \delta^{\sigma_i^{\mathbf{sbb}'}}(q_i^2-m_i^2)$$

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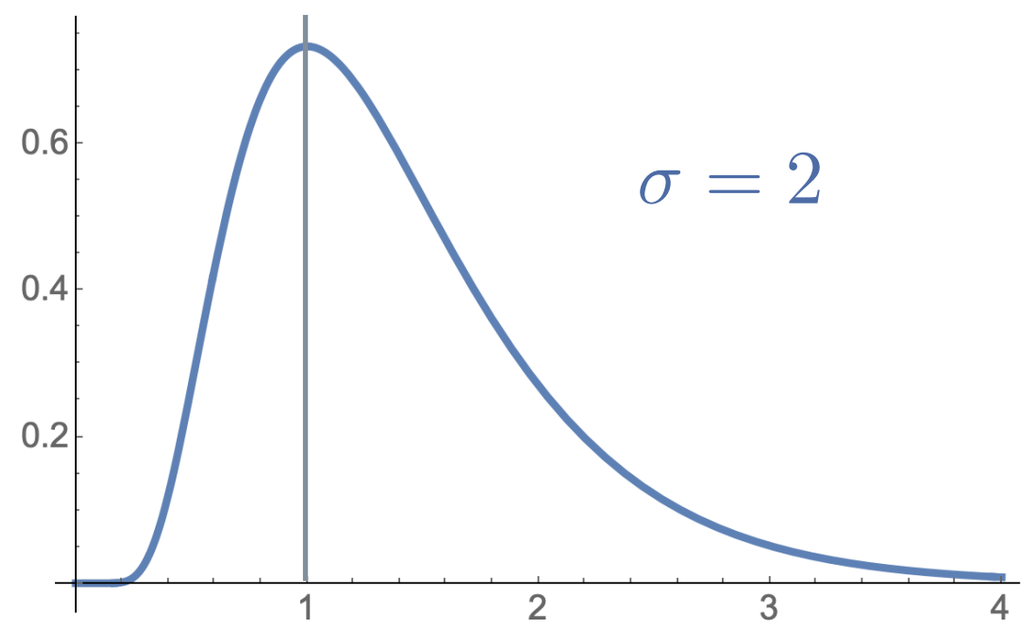
e.g. $h(t) = \dfrac{e^{-\sigma\frac{t^2+1}{t}}}{2K_1(2\sigma)}$



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Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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Solve the **energy conservation** constraint in the **variable t** : $\eta_{\mathbf{s}}(t^\star_{\mathbf{s}}\vec{k},t^\star_{\mathbf{s}}\vec{l},Q^0) = 0$

$$= \int d^3\vec{k}\; d^3\vec{l} \sum_{\mathbf{s}\in\{\mathbf{s}^{\mathrm{r}}_1,\mathbf{s}^{\mathrm{r}}_2,\mathbf{s}^{\mathrm{v}}_1,\mathbf{s}^{\mathrm{v}}_2\}} h(t^\star_{\mathbf{s}})\;(t^\star_{\mathbf{s}})^6\; I^{DT}_{\mathbf{s}}\big|_{t=t^\star_{\mathbf{s}}} \frac{1}{\left(\left|\partial_t\eta_{\mathbf{s}}(t\vec{k},t\vec{l},Q^0)\right|\right)_{t=t^\star_{\mathbf{s}}}}$$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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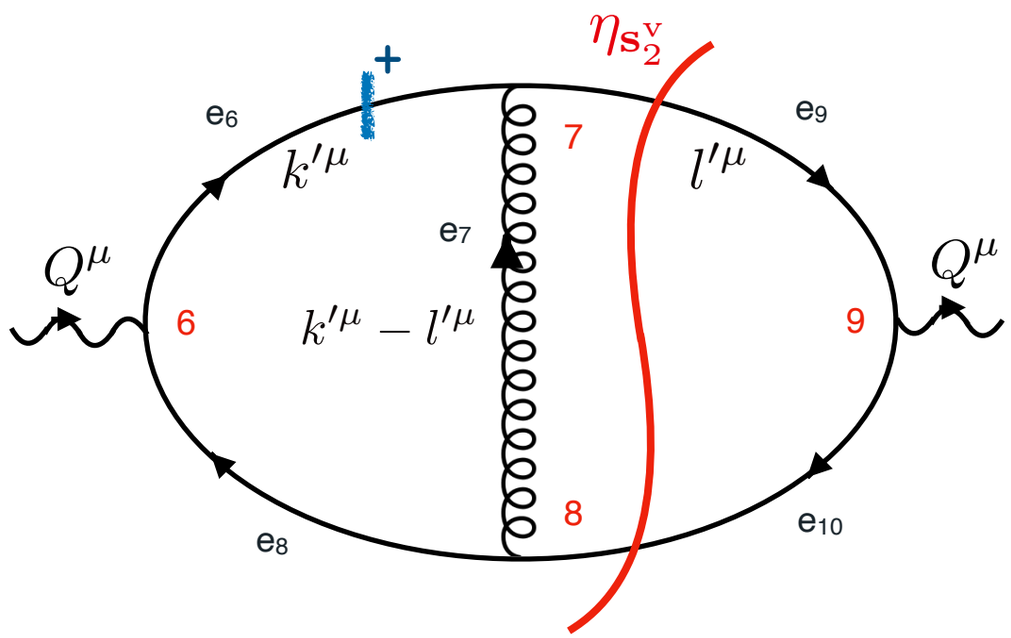
$$\boxed{= \int d^3\vec{k}\; d^3\vec{l}\;\sigma^{DT}_{\mathrm{d}}}$$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

Concrete example :



Locality Unitarity

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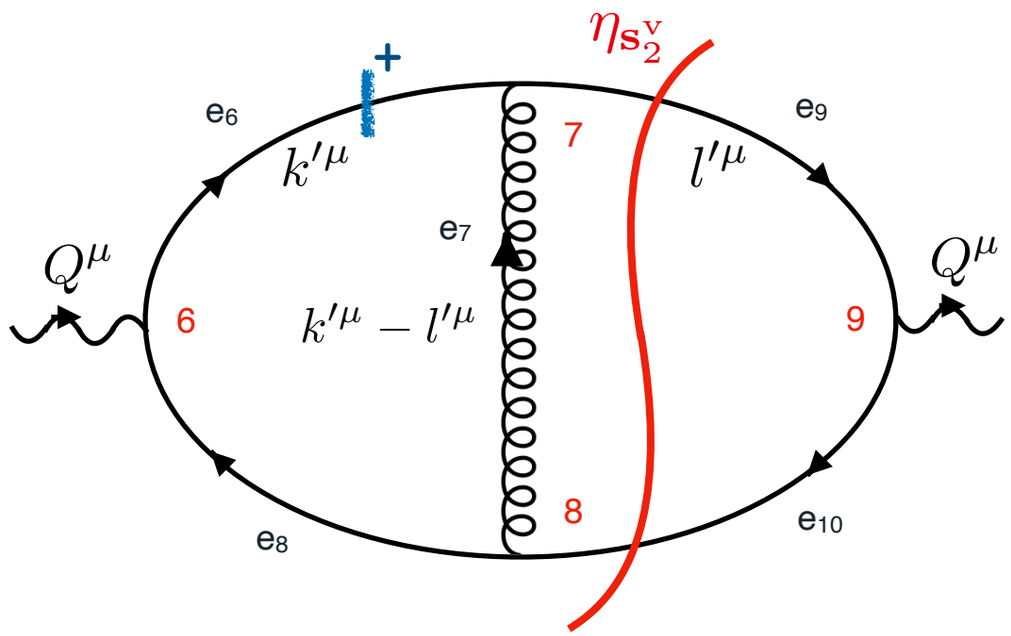

Fix the collision energy…

$$Q^\mu = (100, 0, 0, 0)$$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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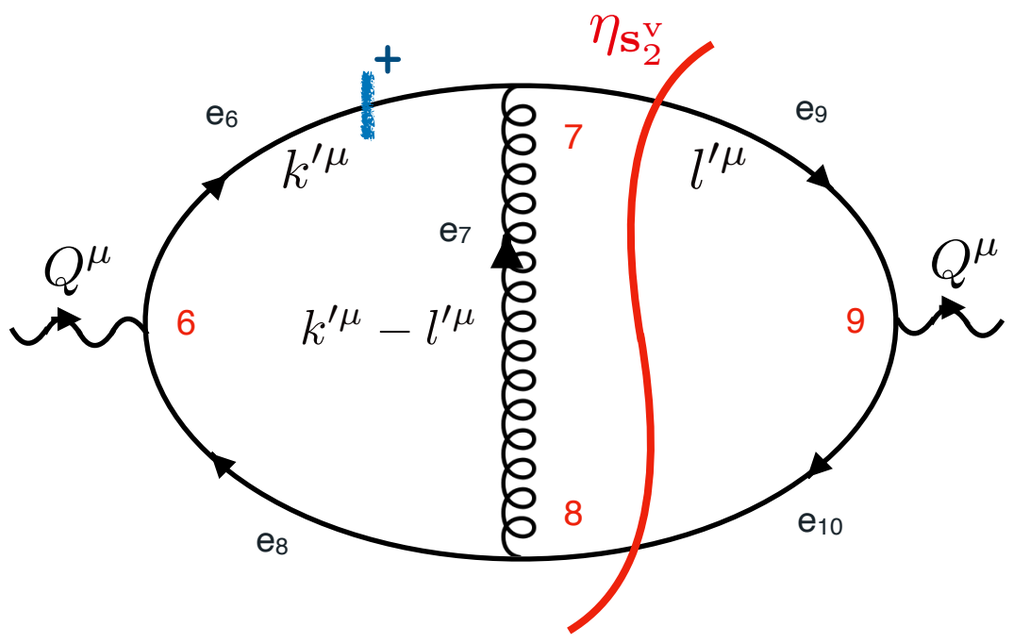

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… and **a sample point**:

$$\vec{k} = (0, 0, 2)$$

$$\vec{l} = (0, 3, 4)$$

Locality Unitarity

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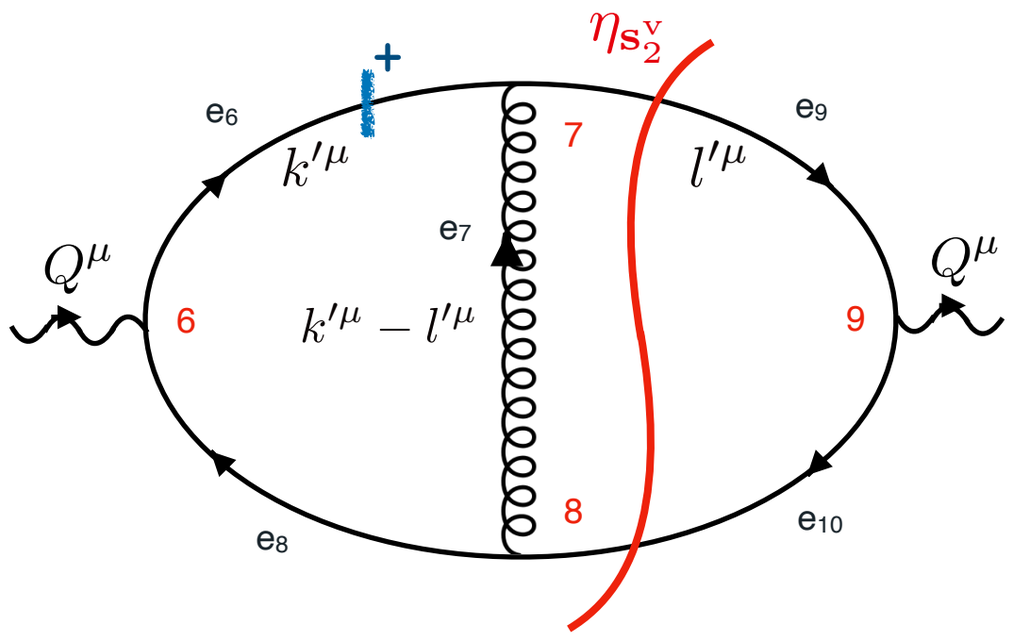

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Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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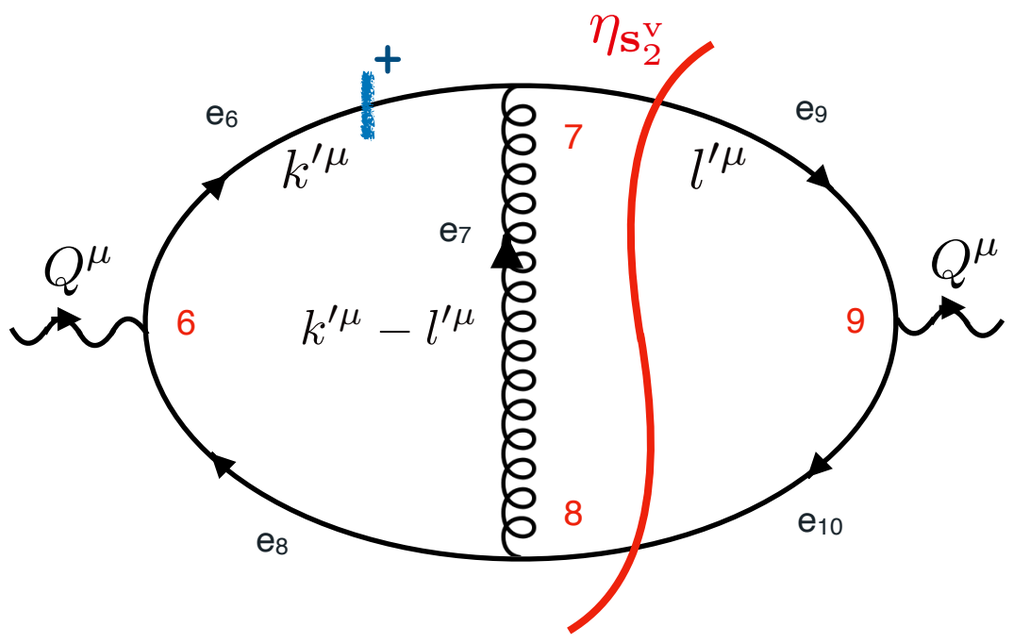

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Locality Unitarity

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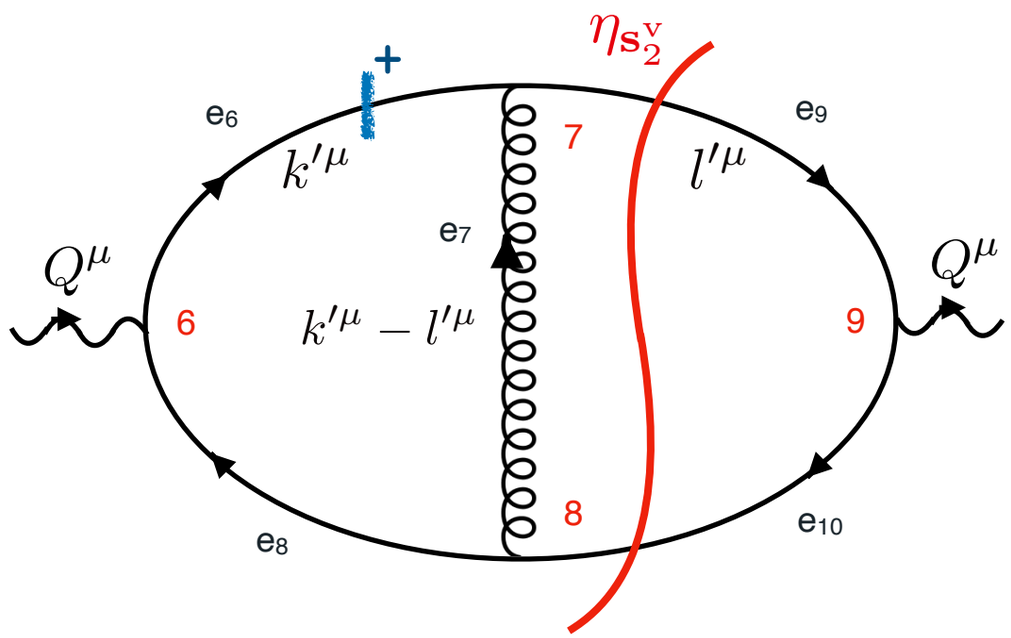

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$$\eta_{\mathbf{s}_2^{\mathrm{v}}}(t^\star_{\mathbf{s}_2^{\mathrm{v}}}\vec{k}, t^\star_{\mathbf{s}_2^{\mathrm{v}}}\vec{l}, Q^0) = 0 \quad \rightarrow \quad t^\star_{\mathbf{s}_2^{\mathrm{v}}} = \frac{Q^0}{2|\vec{l}|} = 10$$

Locality Unitarity

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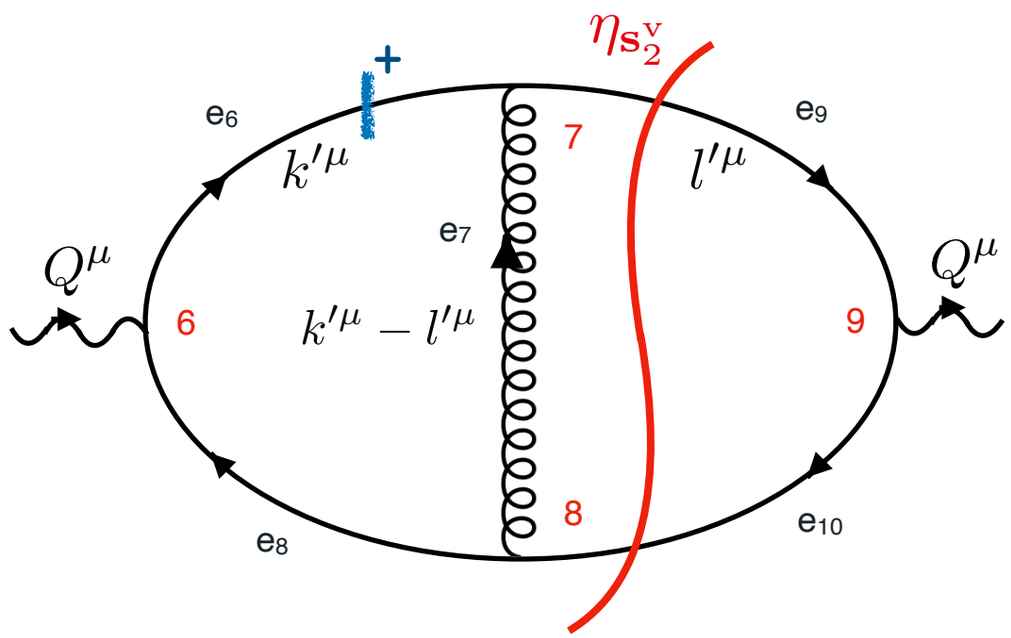

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Our **LU** integrand $I_{\mathbf{s}_2^{\mathrm{v}}}$ and observable $\mathcal{O}_{\mathbf{s}_2^{\mathrm{v}}}$ will be evaluated for :

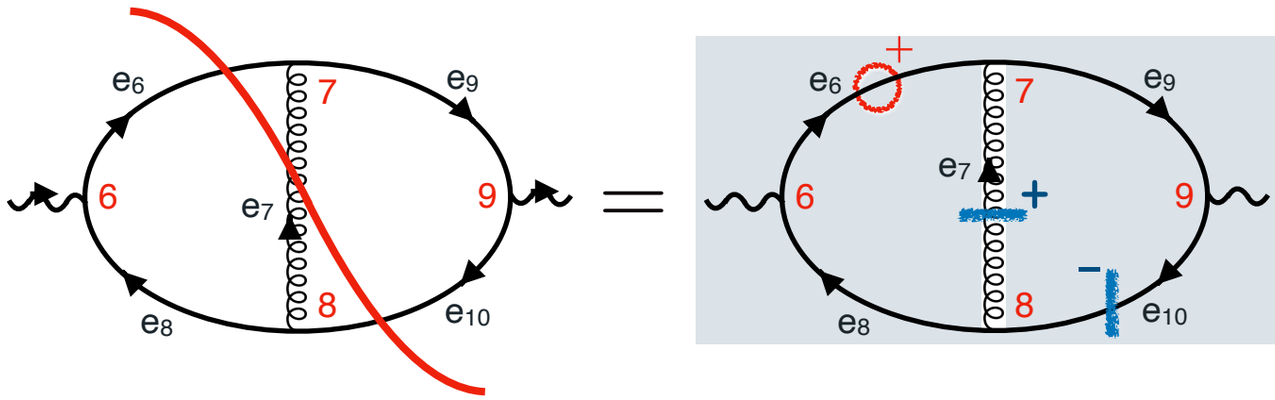
$$k'^\mu = (|t^\star_{\mathbf{s}_2^{\mathrm{v}}}\vec{k}|, t^\star_{\mathbf{s}_2^{\mathrm{v}}}\vec{k}) = (20, 0, 0, 20)$$

$$l'^\mu = (|t^\star_{\mathbf{s}_2^{\mathrm{v}}}\vec{l}|, t^\star_{\mathbf{s}_2^{\mathrm{v}}}\vec{l}) = (50, 0, 30, 40)$$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

Is $\int d^3\vec{k}\, d^3\vec{l}\, \sigma_{\mathrm{d}}^{DT}$ **finite**? Let us investigate one **pair-wise cancelation**:



Real-emission contrib. : $\mathbf{s}_1^{\mathrm{r}}$

Evaluated on: $\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Locality Unitarity

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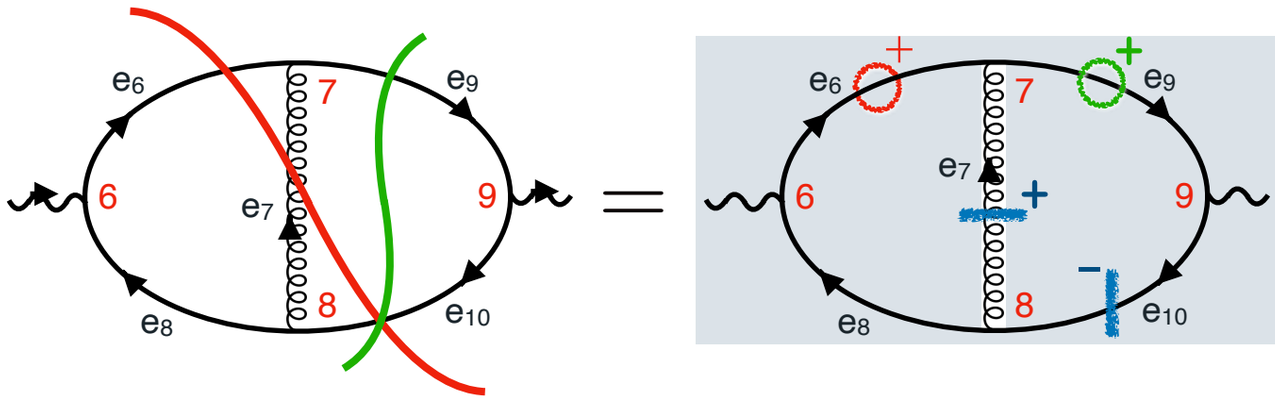

Real-emission contrib. : $\mathbf{s}_1^{\mathrm{r}}$

Evaluated on: $\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Singular at: $\eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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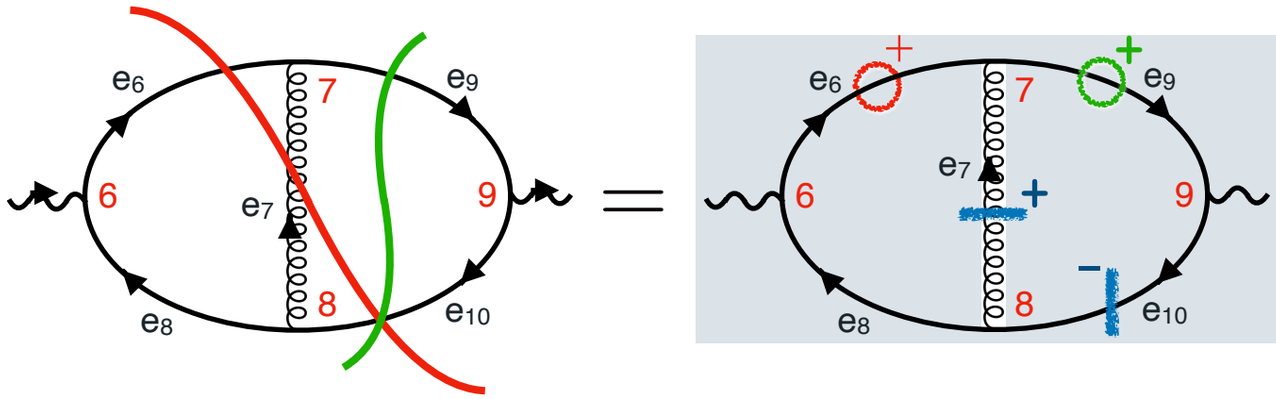
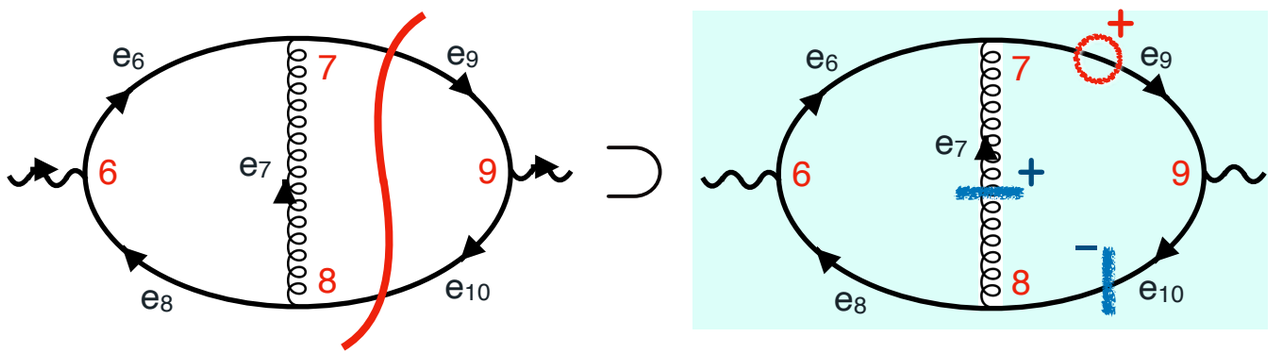


Real-emission contrib. : $\mathbf{s}_1^{\mathrm{r}}$

Evaluated on: $\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Singular at: $\eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}\,', Q^0) = 0$



Virtual contrib. : $\mathbf{s}_2^{\mathrm{v}}$

Evaluated on: $\eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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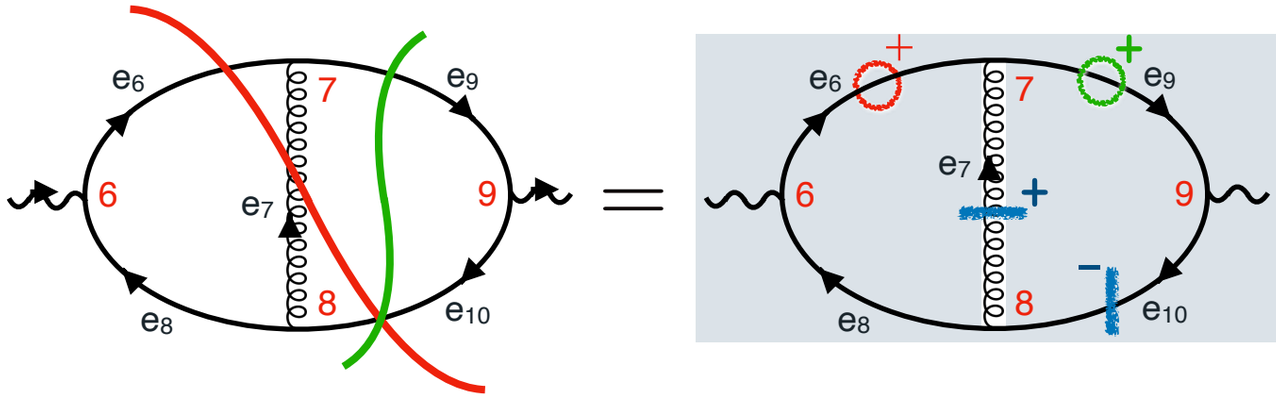
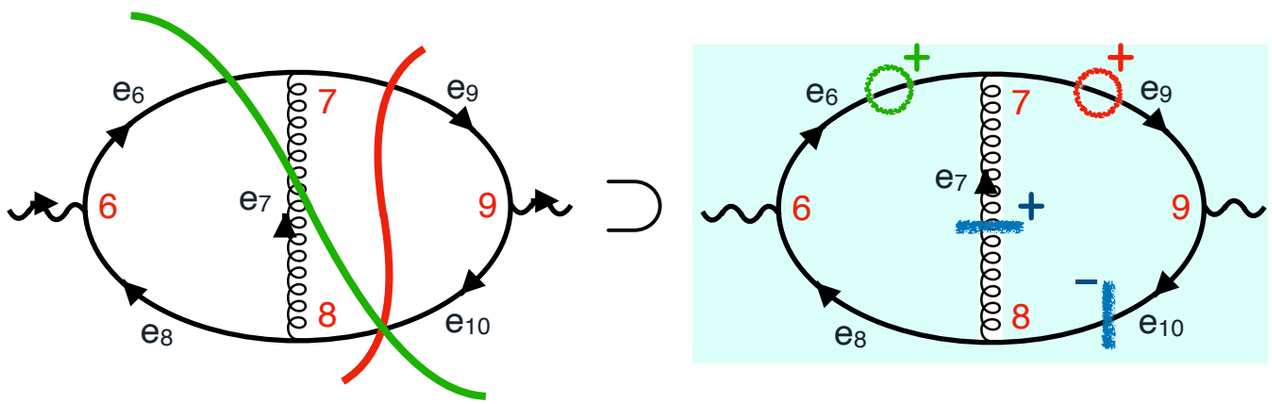


Real-emission contrib. : $\mathbf{s}_1^{\mathrm{r}}$

Evaluated on: $\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Singular at: $\eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}\,', Q^0) = 0$



Virtual contrib. : $\mathbf{s}_2^{\mathrm{v}}$

Evaluated on: $\eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Singular at: $\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

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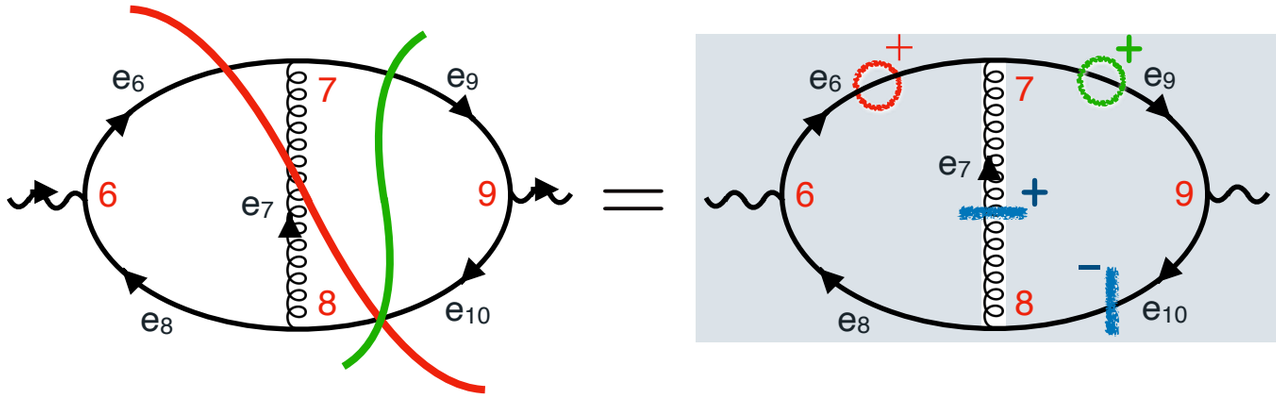
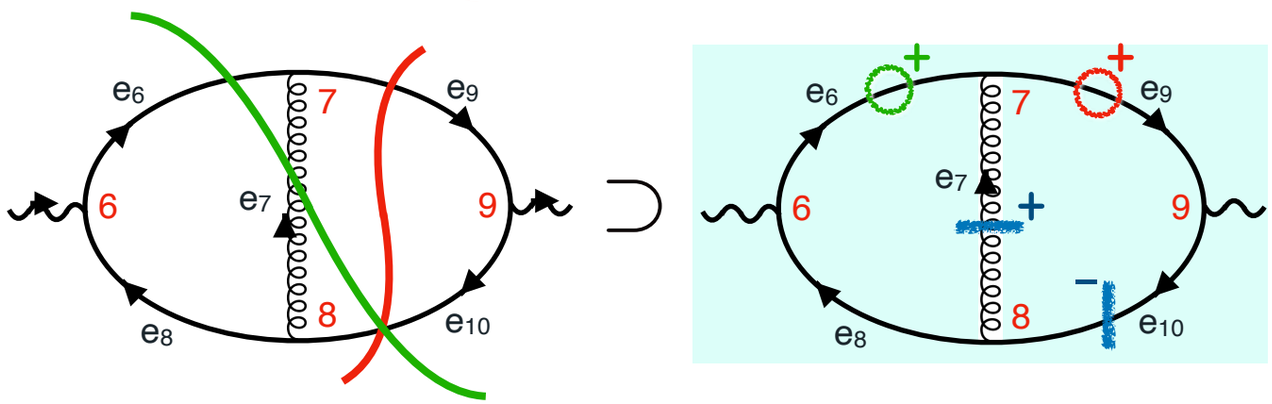


Real-emission contrib. : $\mathbf{s}_1^{\mathrm{r}}$

Evaluated on: $\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}\,', Q^0) = 0$

Singular at: $\eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}\,', Q^0) = 0$



Virtual contrib. : $\mathbf{s}_2^{\mathrm{v}}$

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This coll. singularity cancellation occurs when : $\boxed{\eta_{\mathbf{s}_1^{\mathrm{r}}}(\vec{k}', \vec{l}\,', Q^0) = \eta_{\mathbf{s}_2^{\mathrm{v}}}(\vec{k}', \vec{l}\,', Q^0) = 0}$

Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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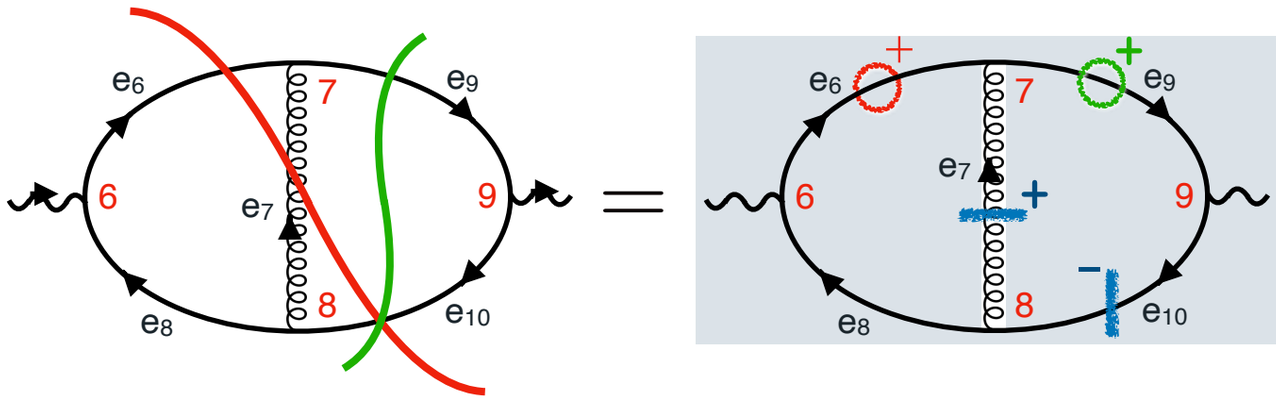

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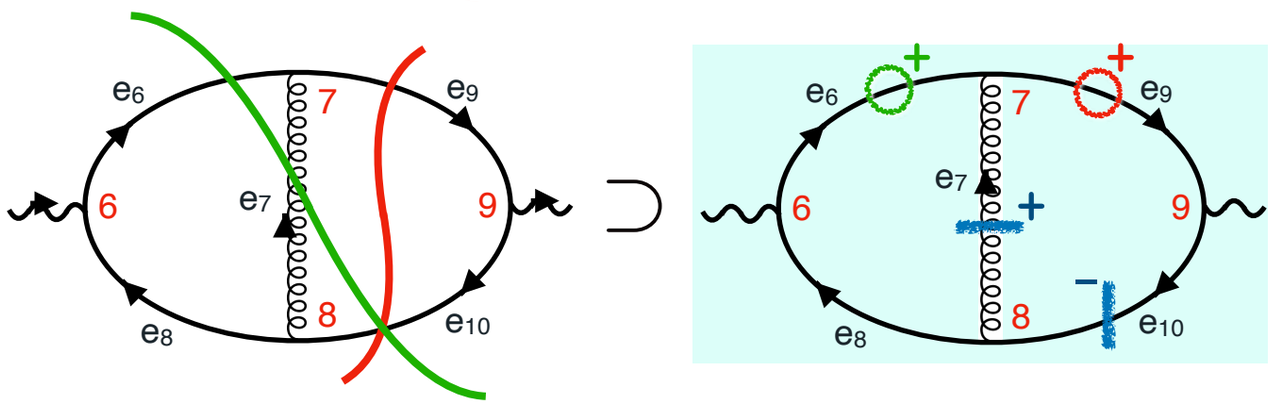

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We formally proved this cancellation at all orders by studying :

$$\int d^3\vec{k}\, d^3\vec{l} \left[I^{(DT)}\Big|_{t=t^\star_{\mathbf{s}_1^{\mathrm{r}}}} \frac{\mathcal{O}_{\mathbf{s}_1^{\mathrm{r}}}}{\left(\left|\partial_t \eta_{\mathbf{s}_1^{\mathrm{r}}}(t\vec{k}, t\vec{l}, Q^0)\right|\right)_{t=t^\star_{\mathbf{s}_1^{\mathrm{r}}}}} + I^{(DT)}\Big|_{t=t^\star_{\mathbf{s}_2^{\mathrm{v}}}} \frac{\mathcal{O}_{\mathbf{s}_2^{\mathrm{v}}}}{\left(\left|\partial_t \eta_{\mathbf{s}_2^{\mathrm{v}}}(t\vec{k}, t\vec{l}, Q^0)\right|\right)_{t=t^\star_{\mathbf{s}_2^{\mathrm{v}}}}} \right]$$

Locality Unitarity

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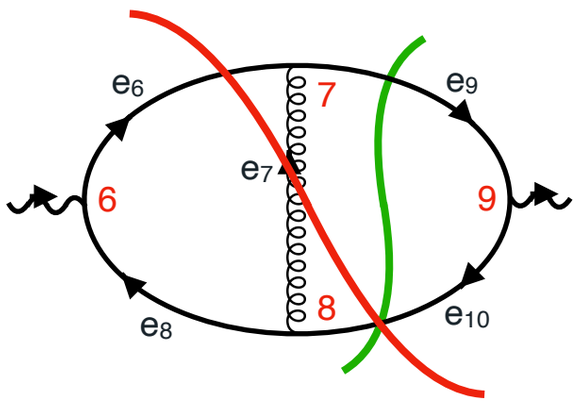
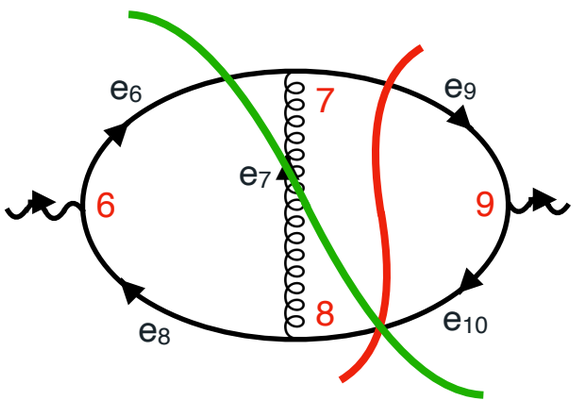
= Cutkosky cut = threshold singularity



cancels



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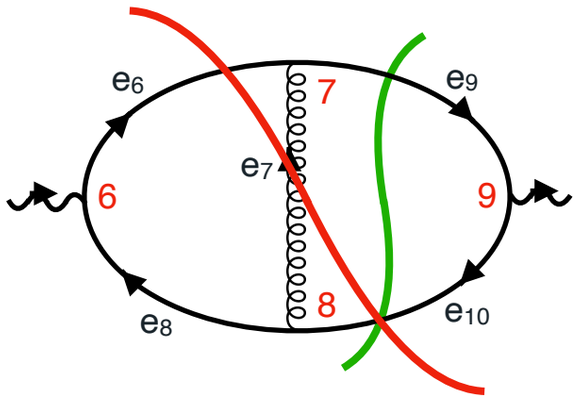

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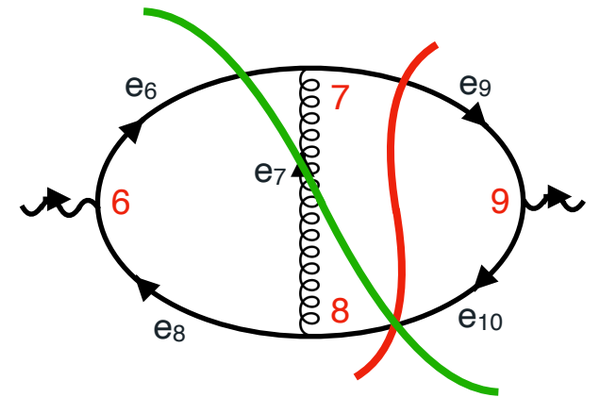

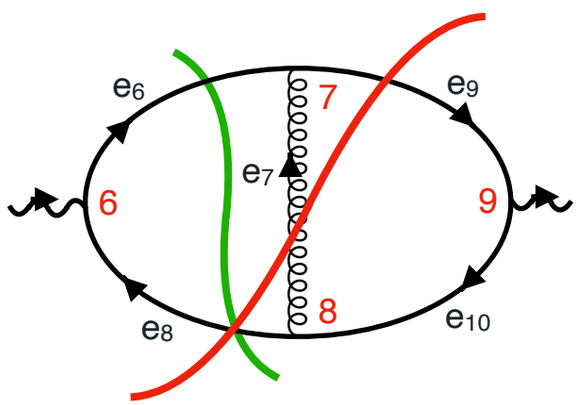

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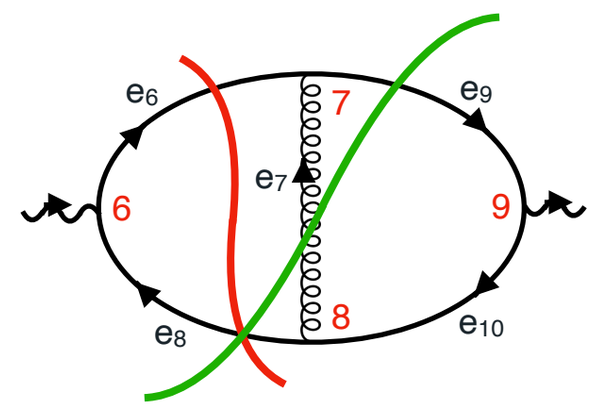

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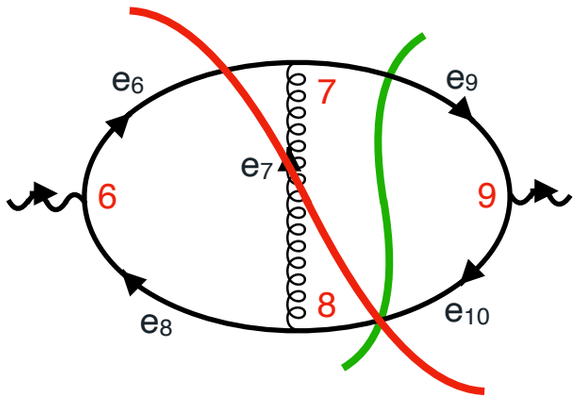

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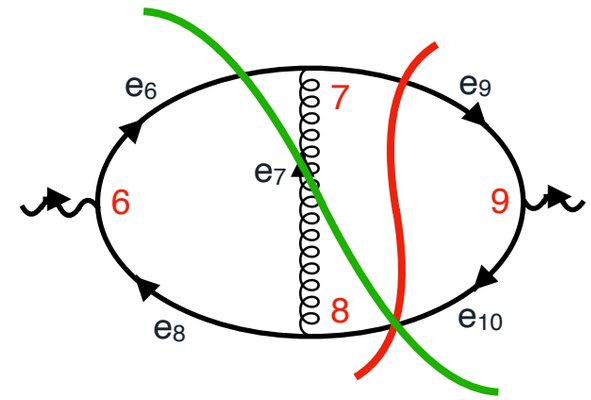

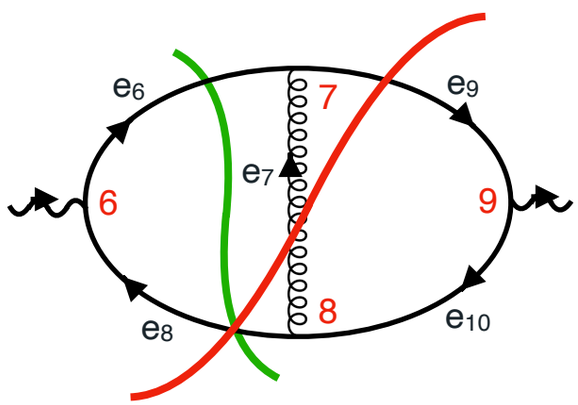

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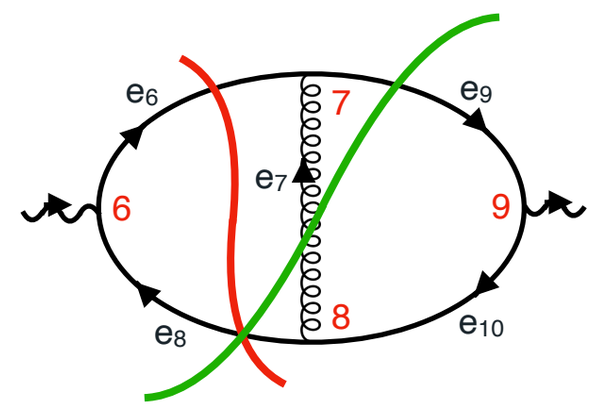
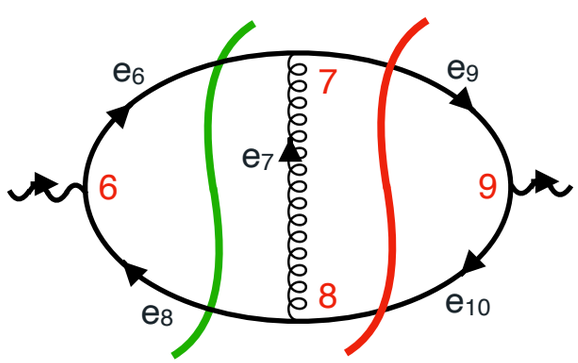


Even for **non-pinched singular threshold** ! (when $\mathcal{O}_{\mathbf{s}} \equiv 1$) :



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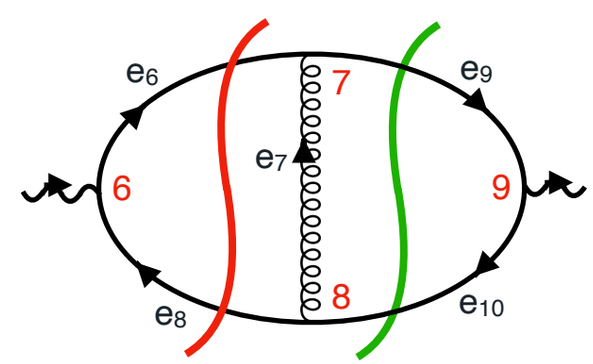

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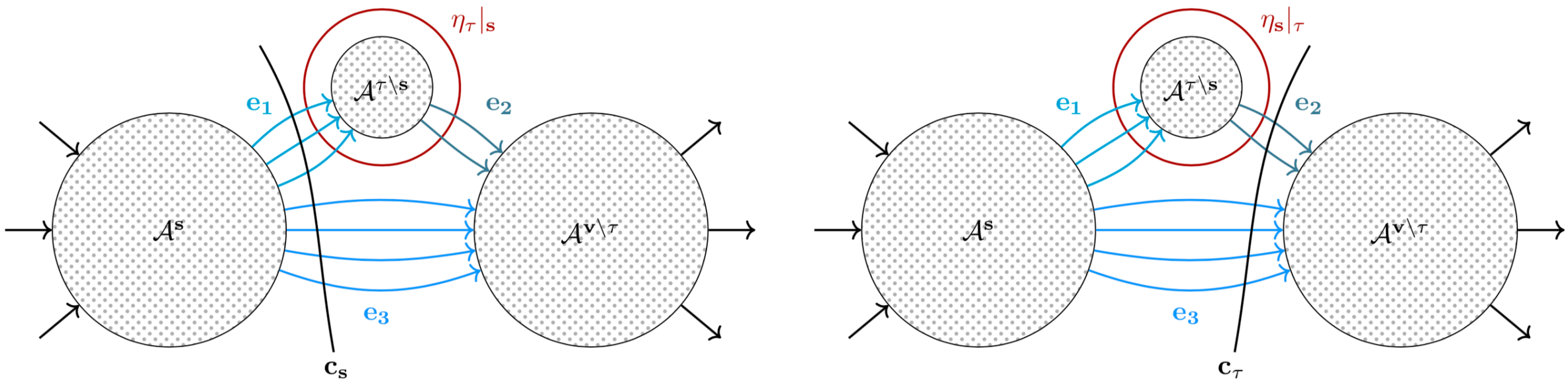

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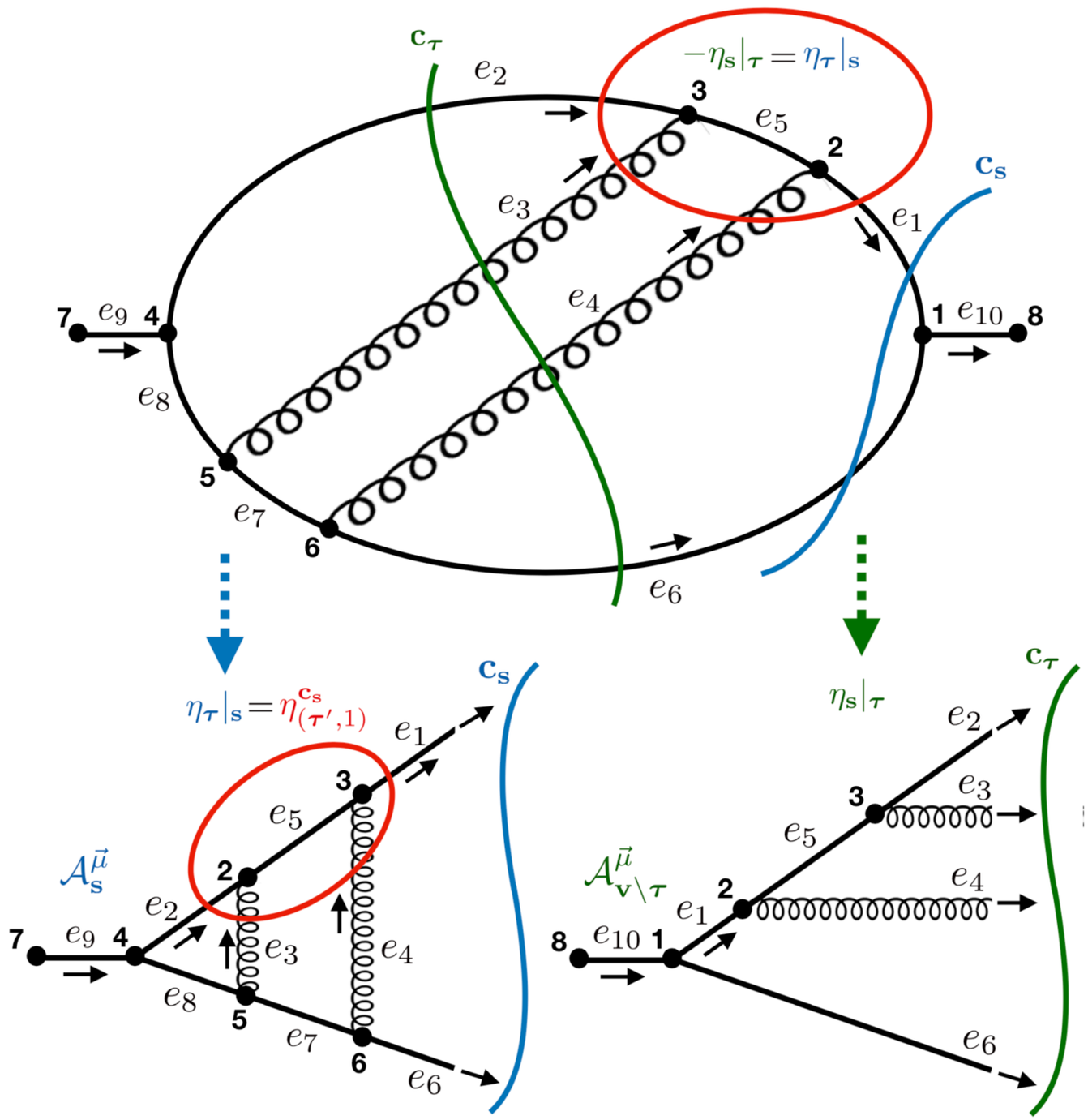

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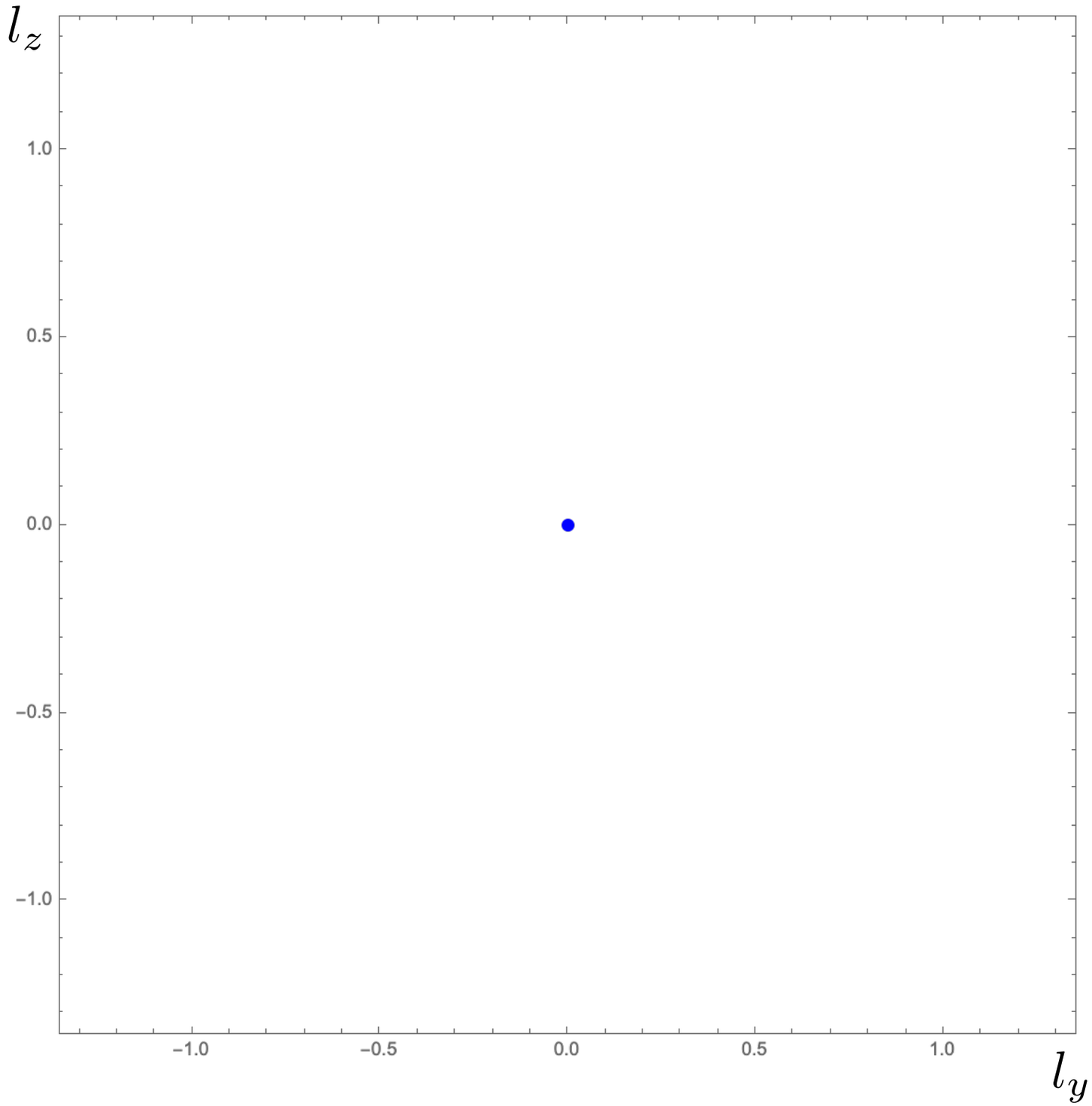

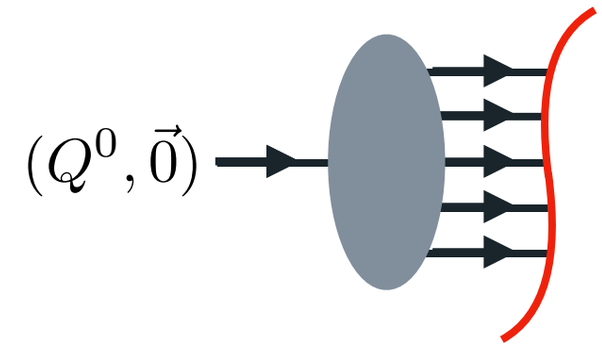

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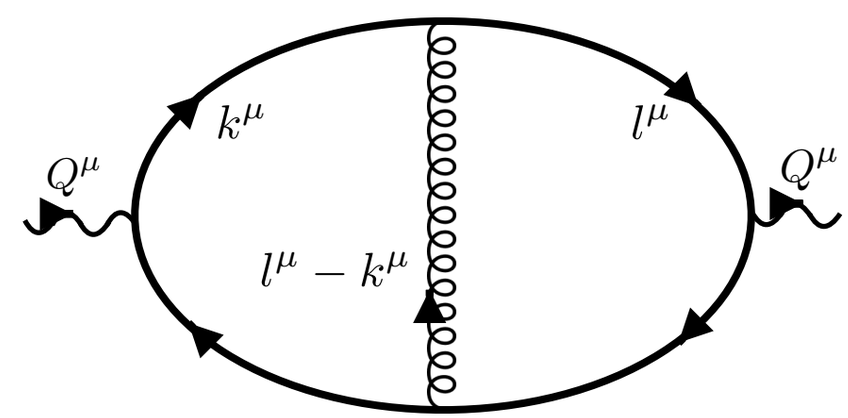

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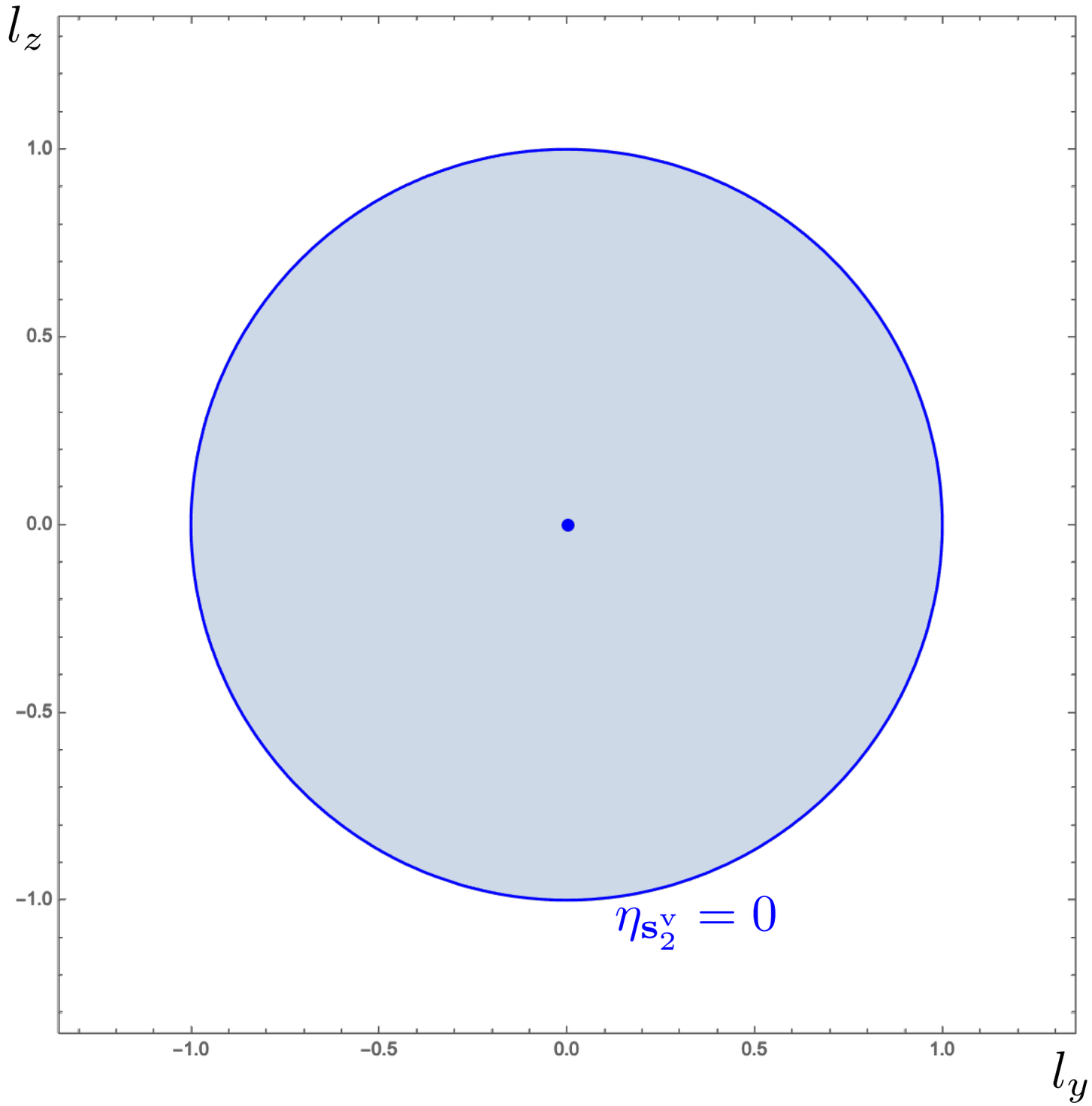

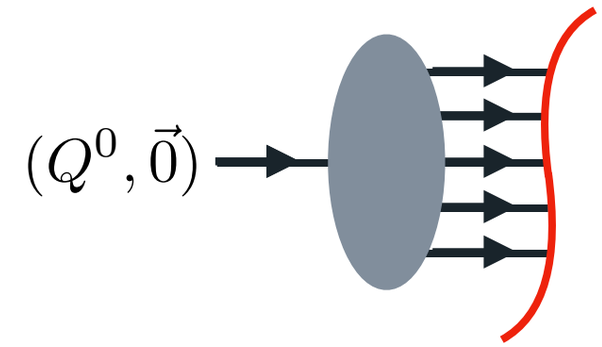

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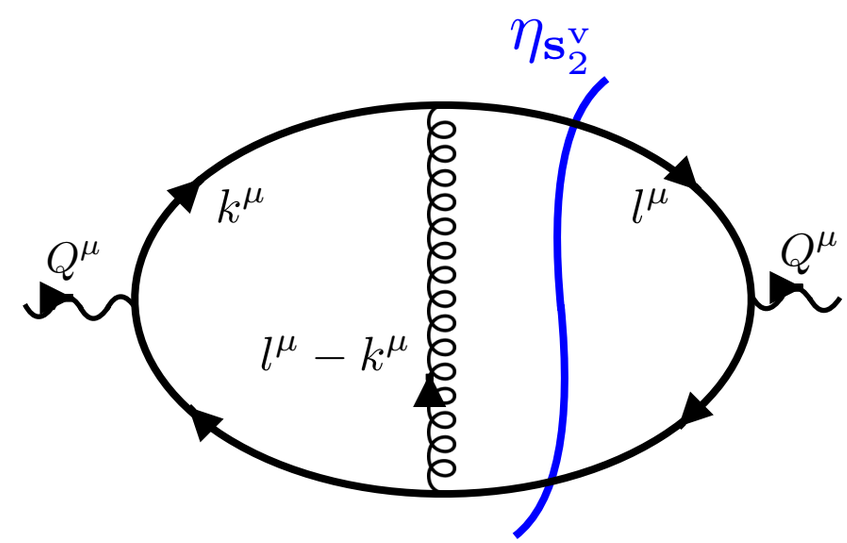

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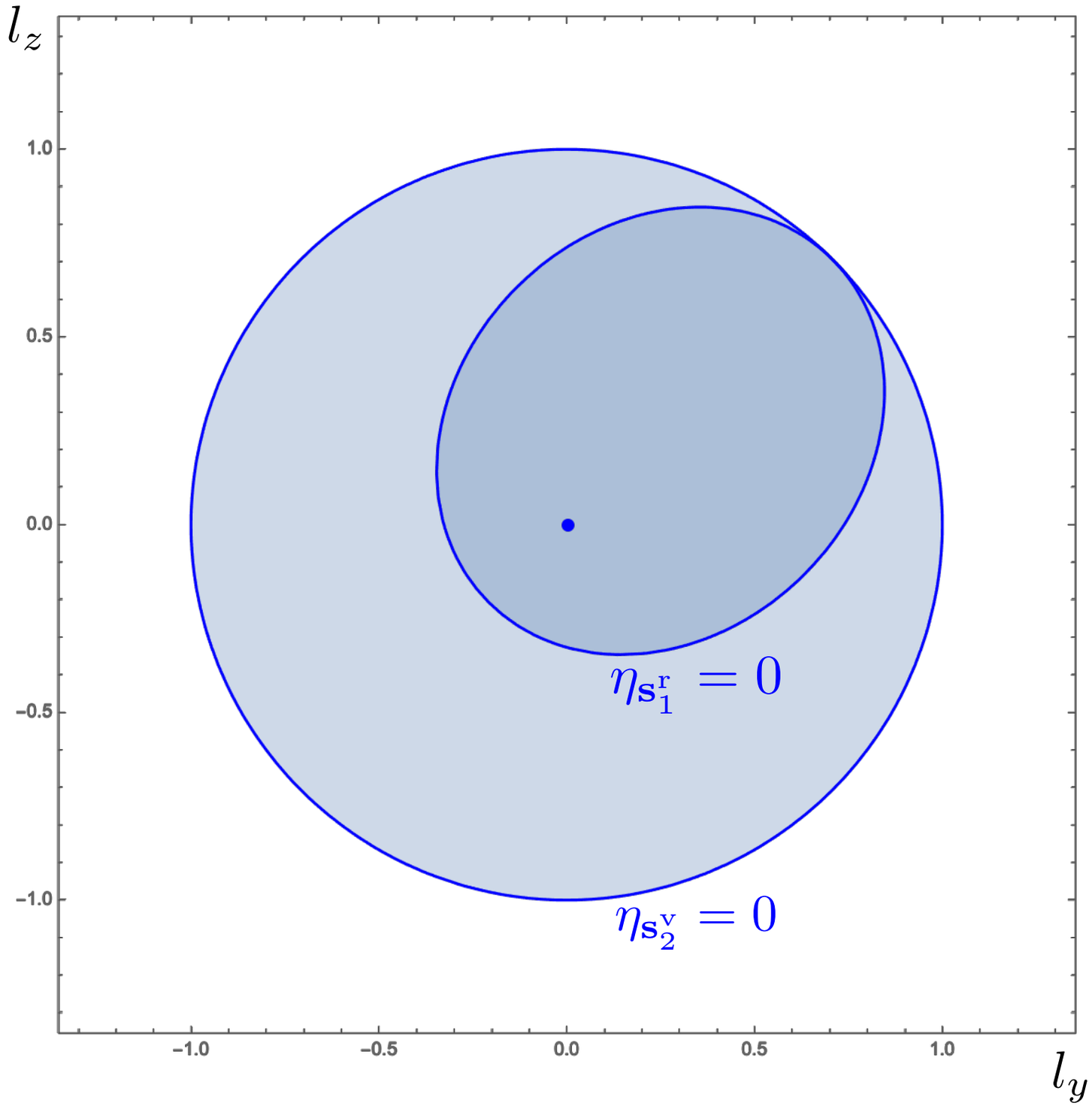

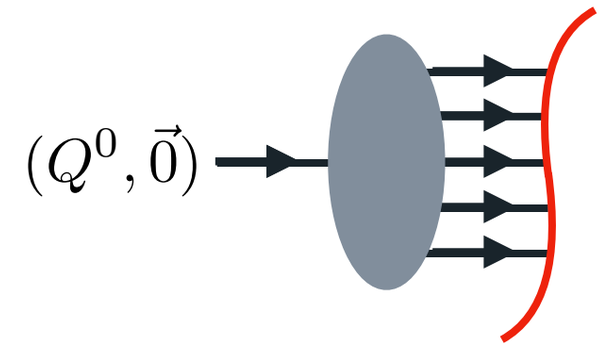

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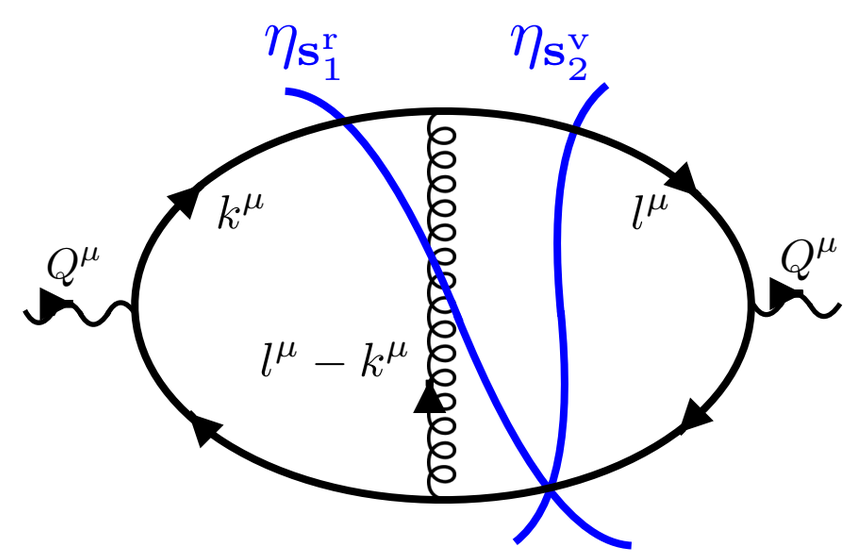

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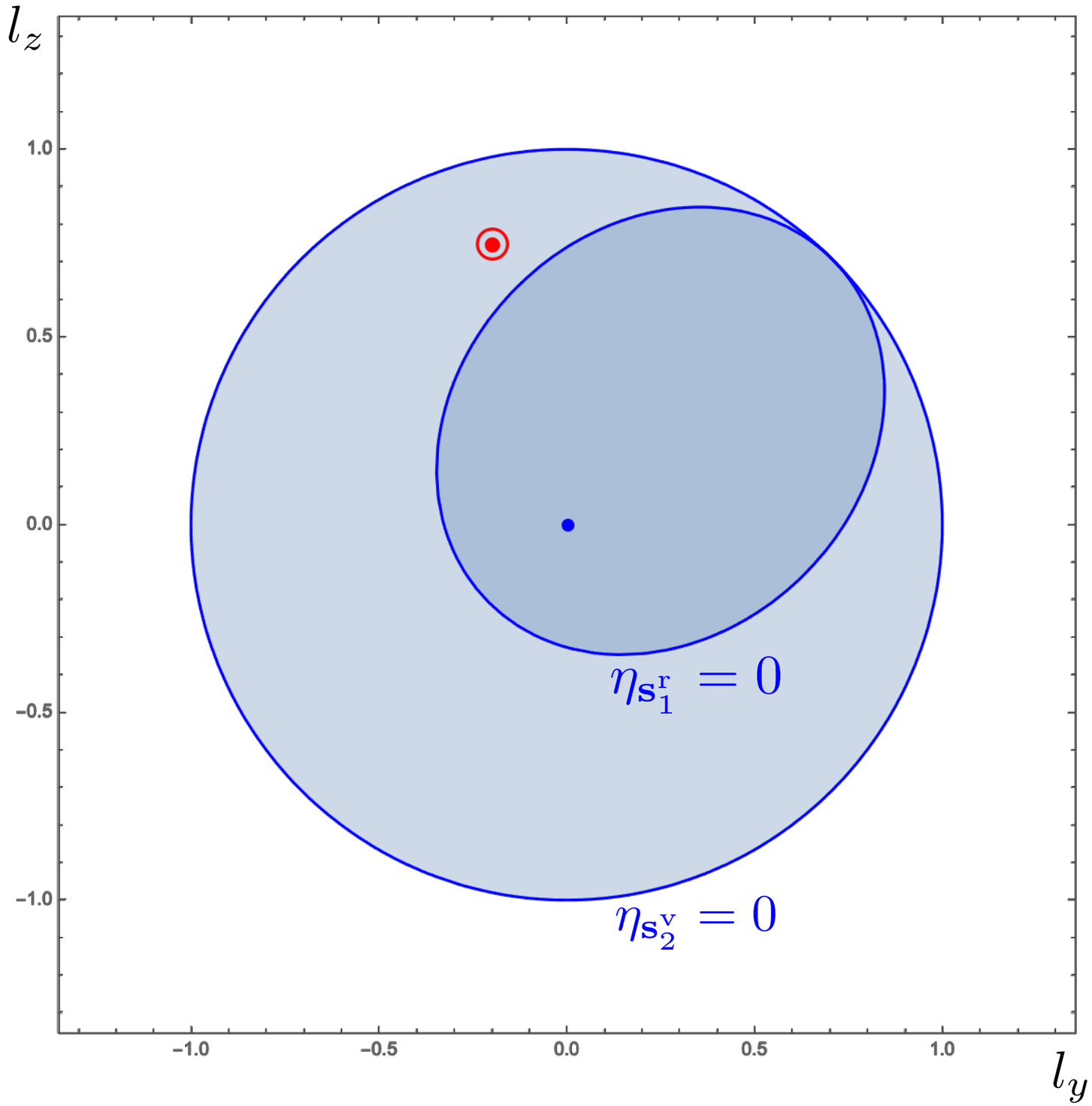

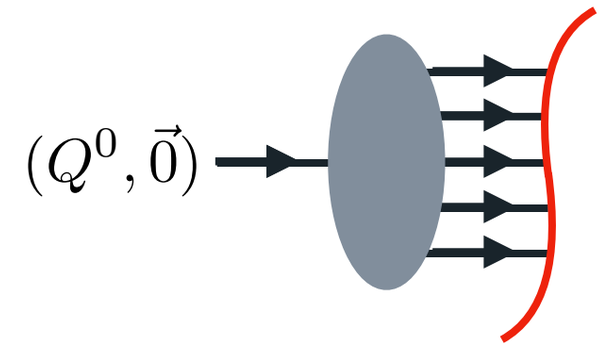

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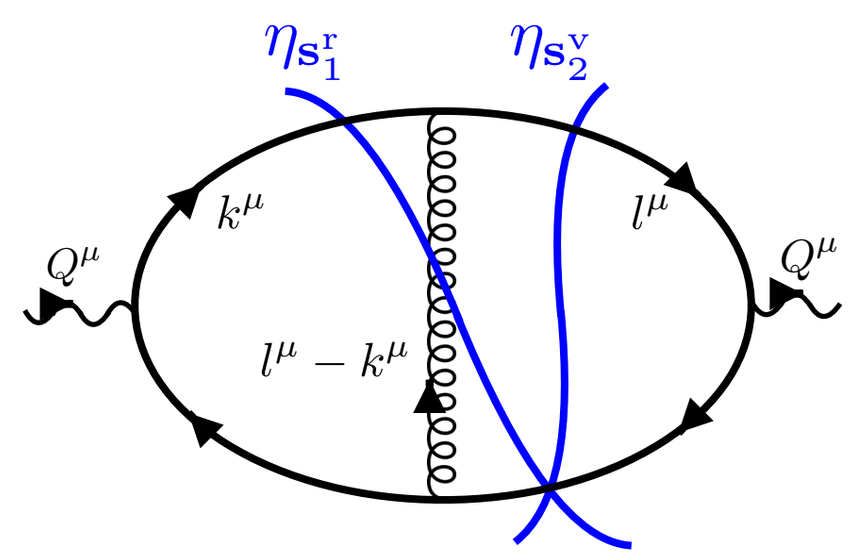

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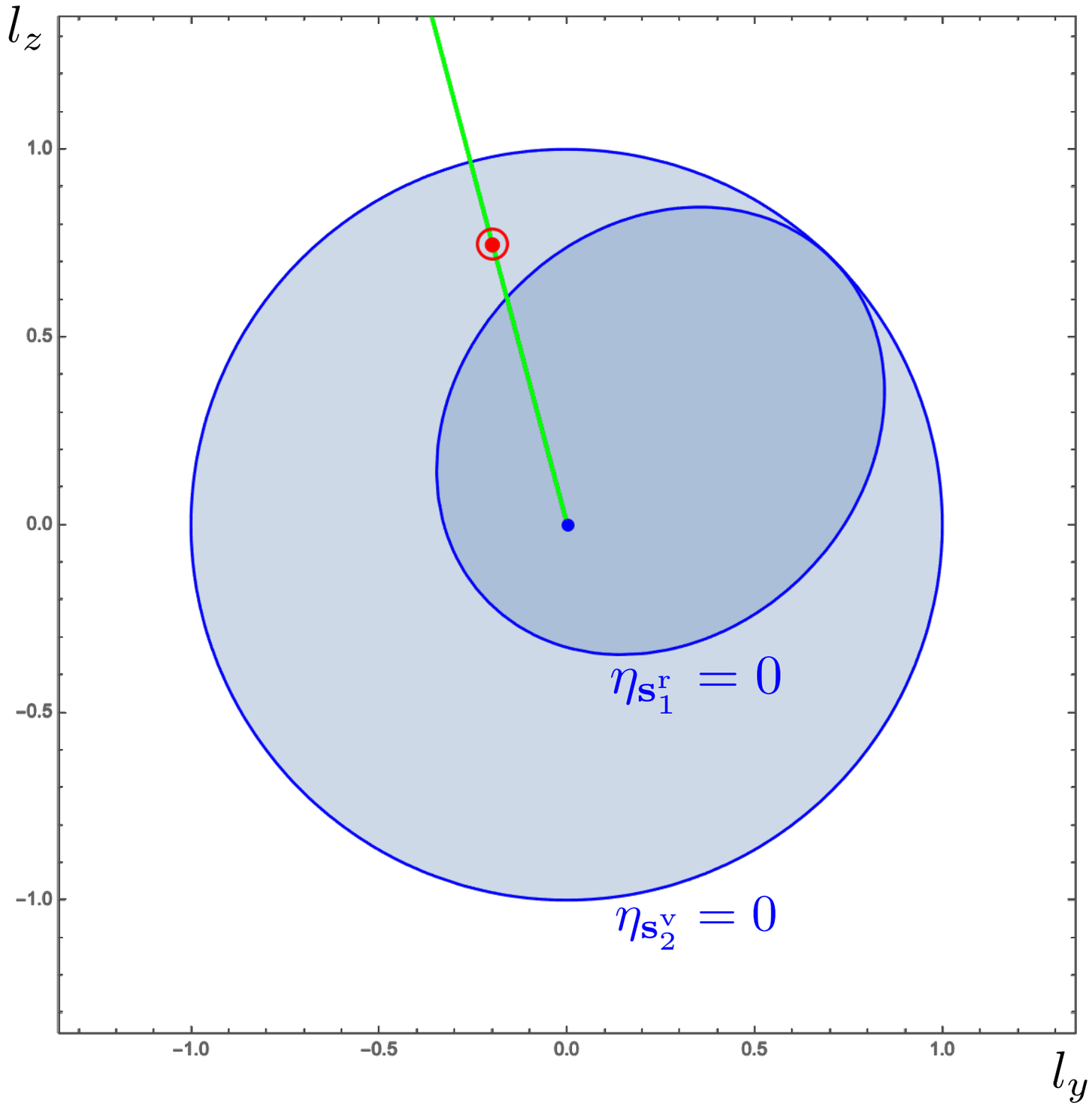

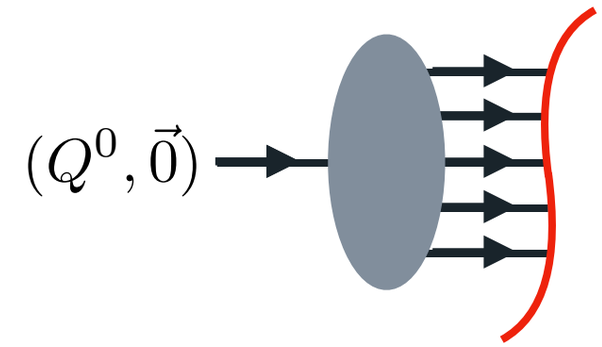

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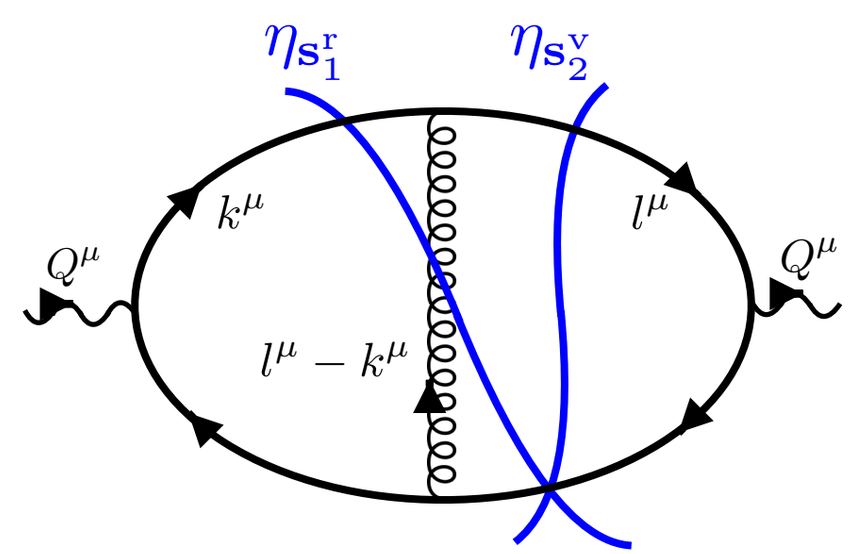

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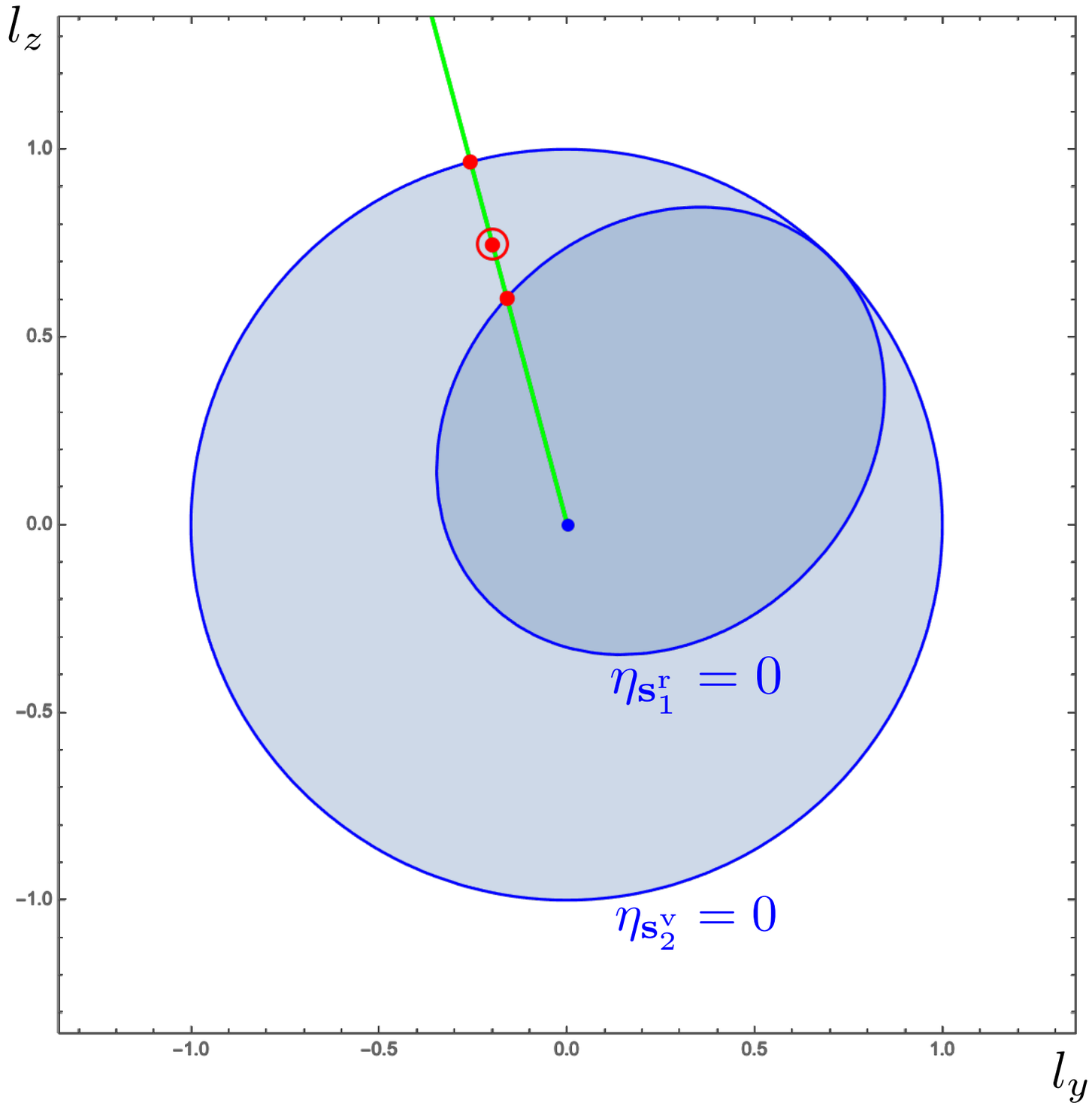

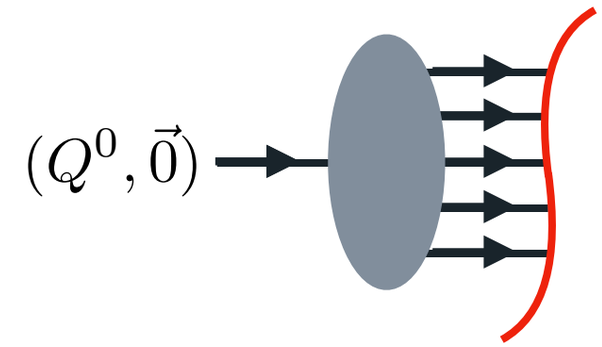

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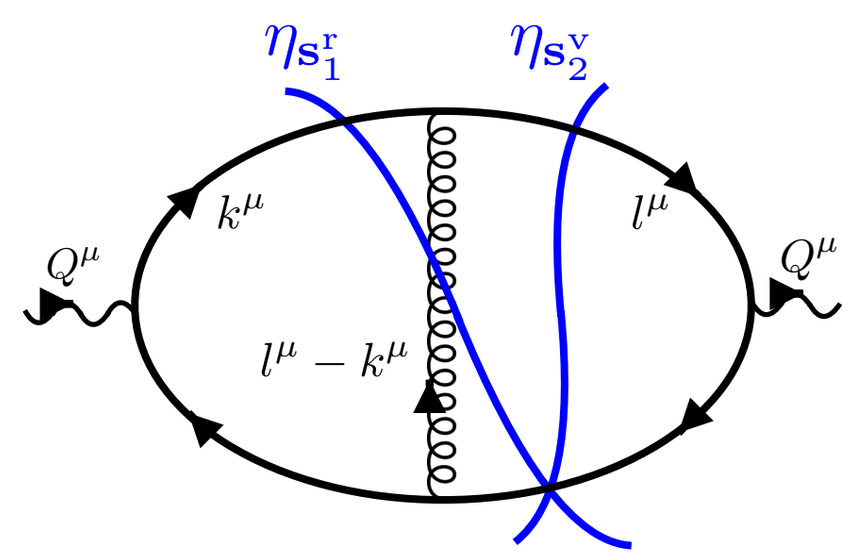

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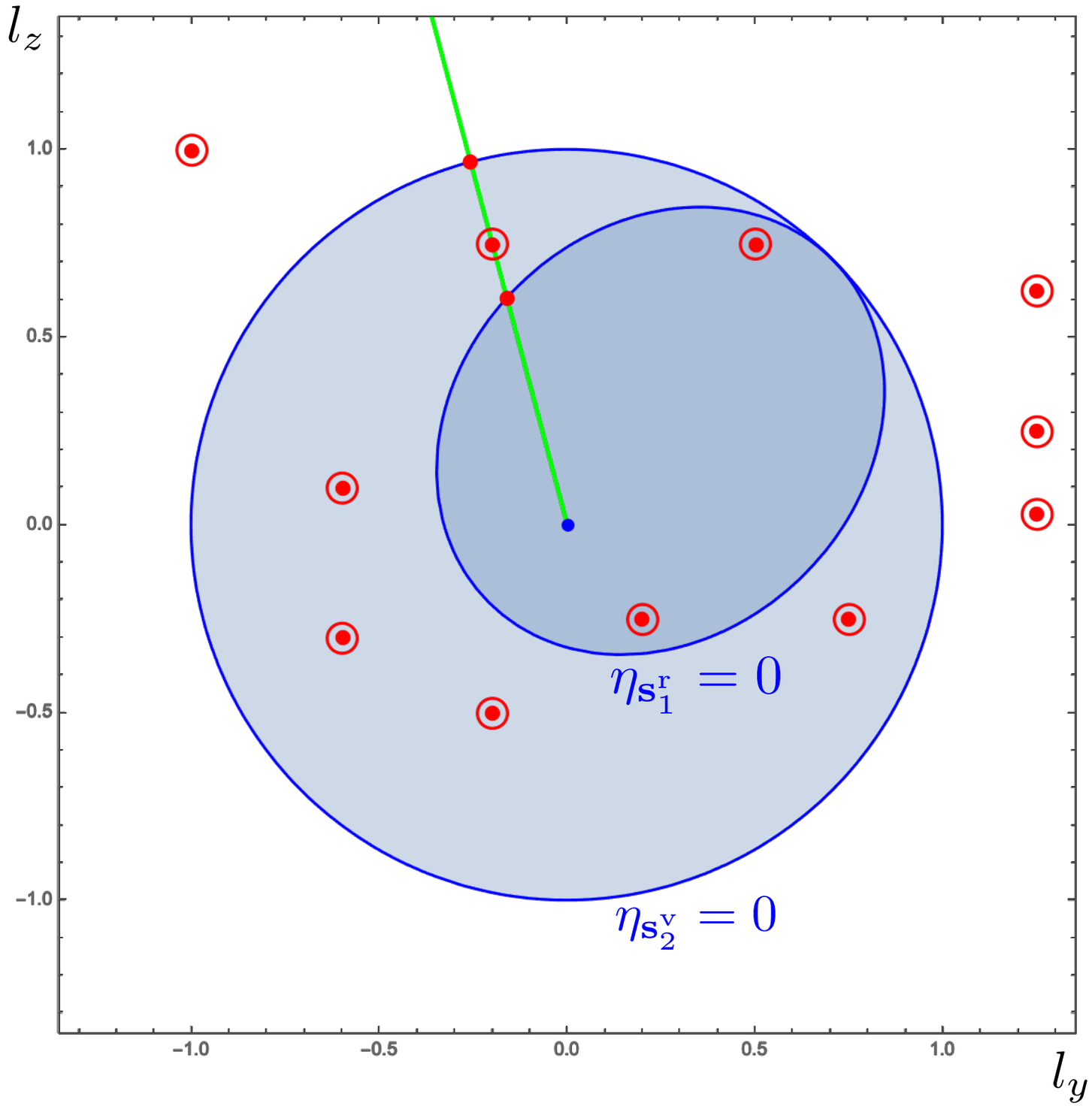

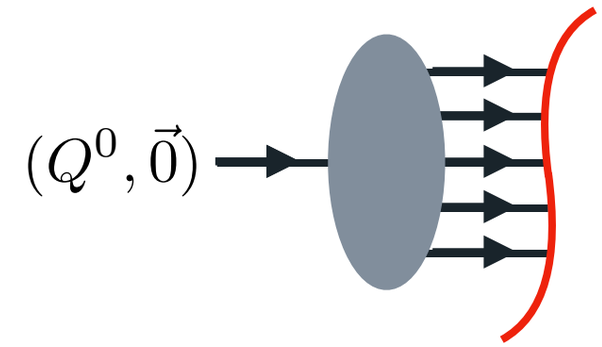

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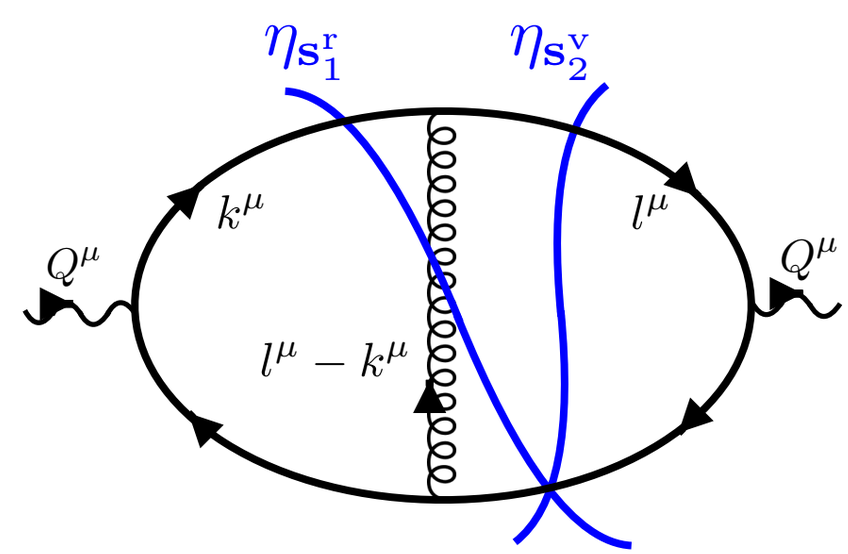

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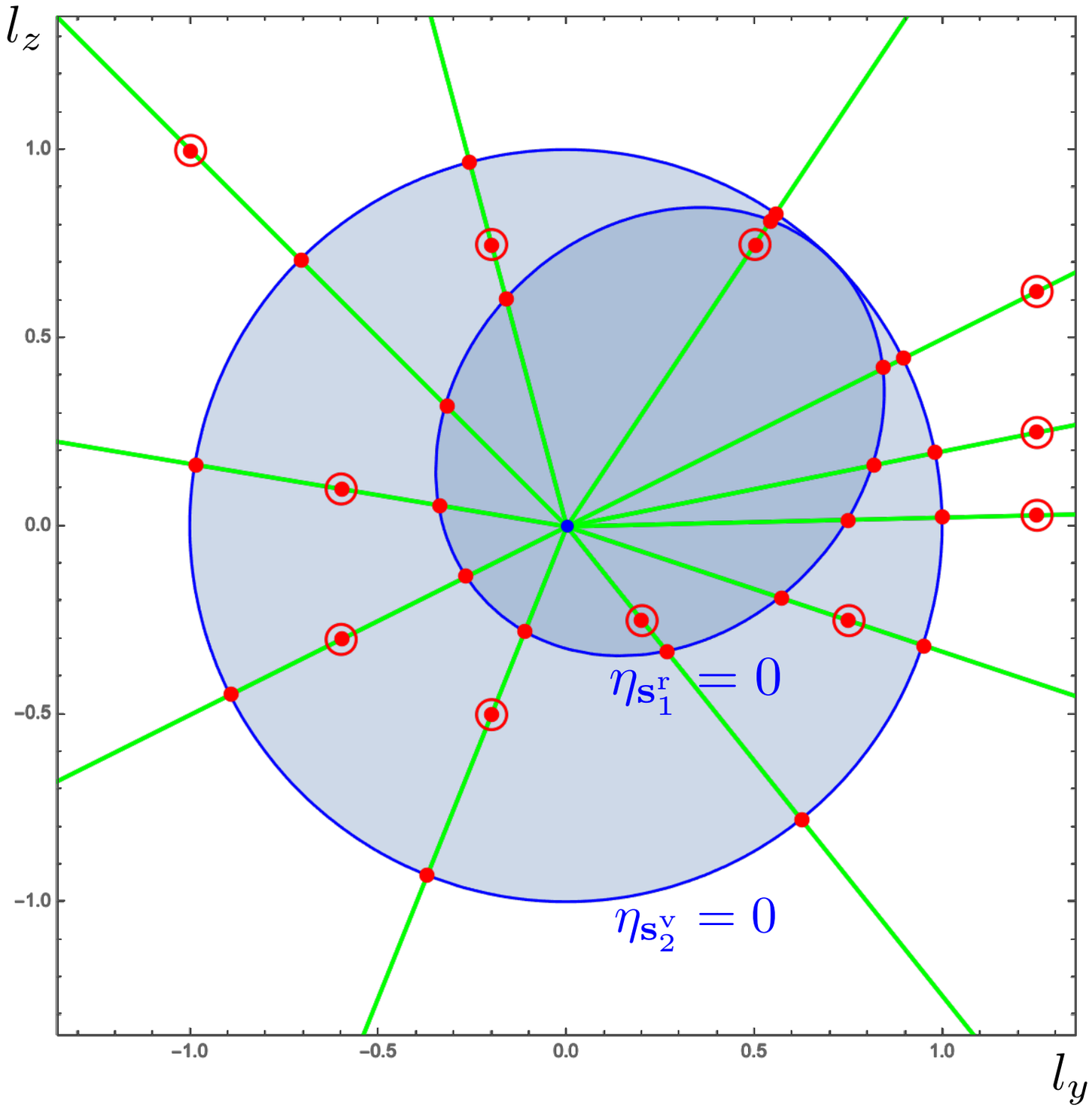

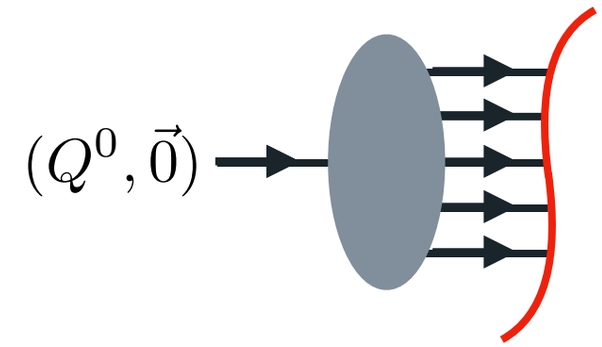

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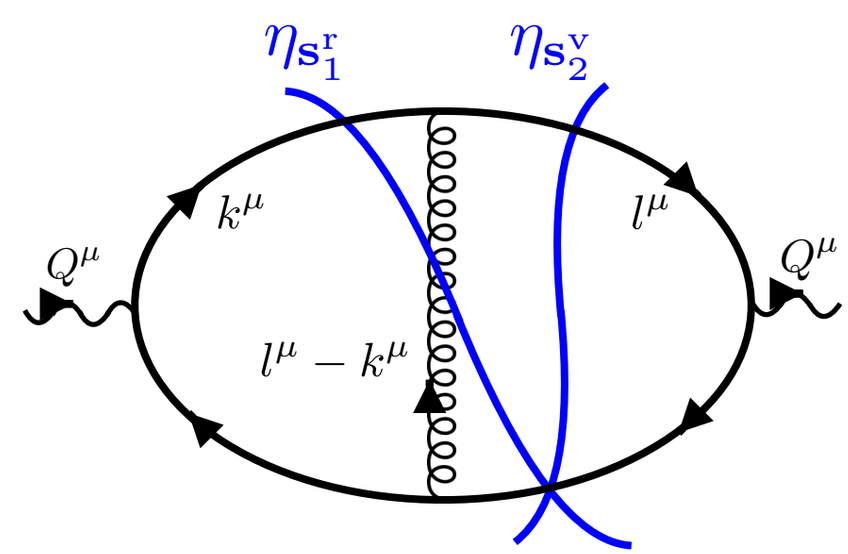

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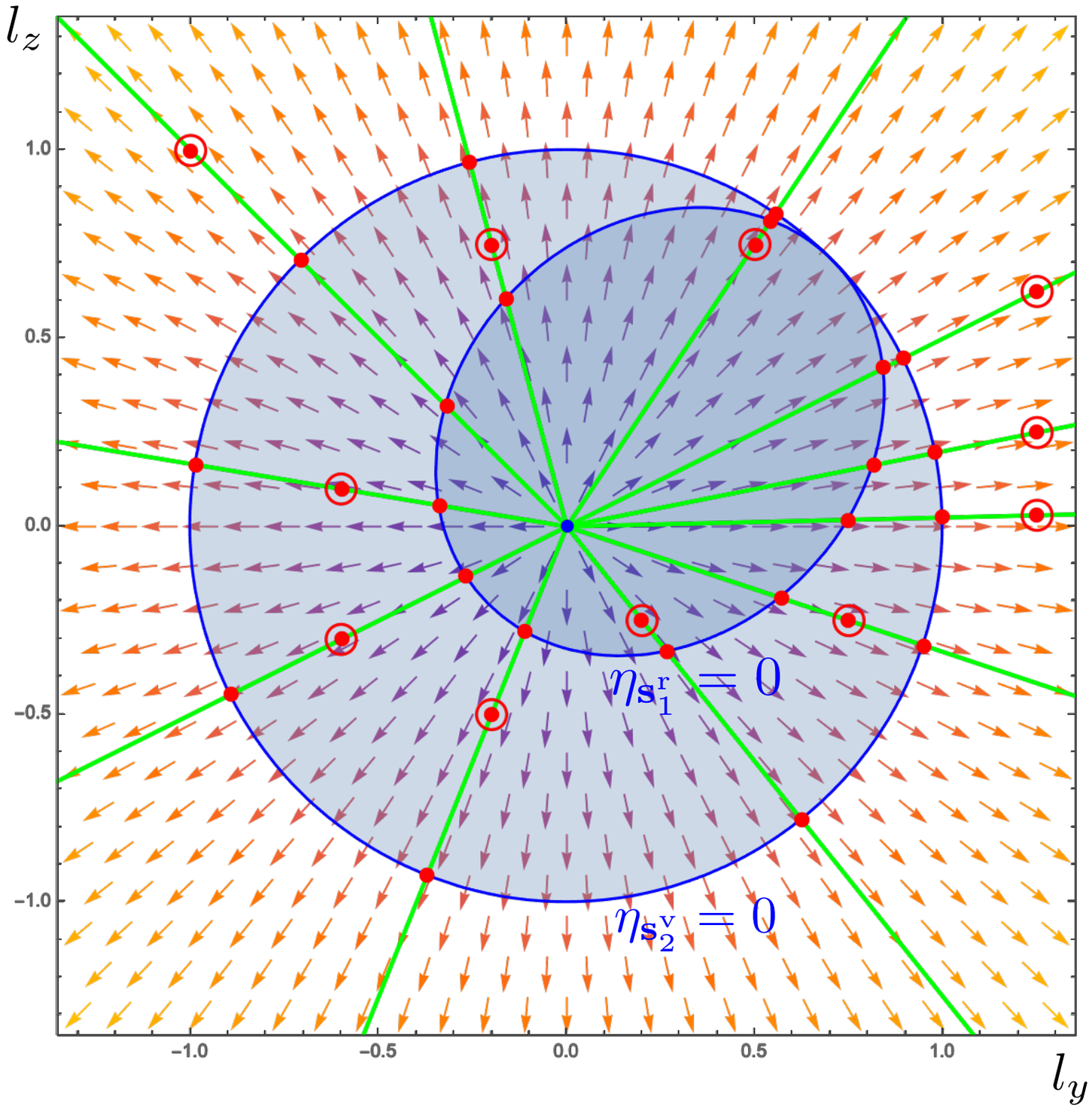

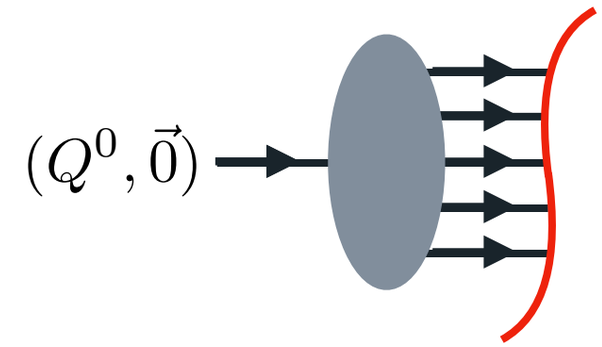

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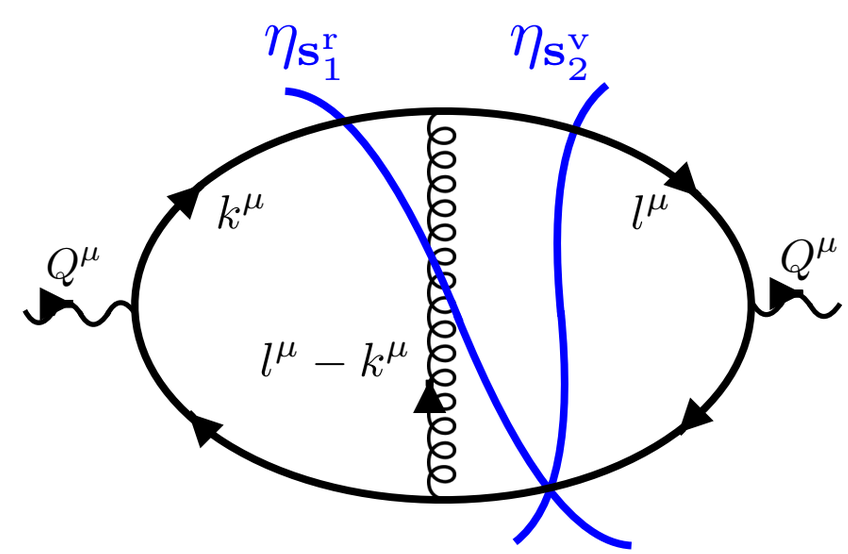

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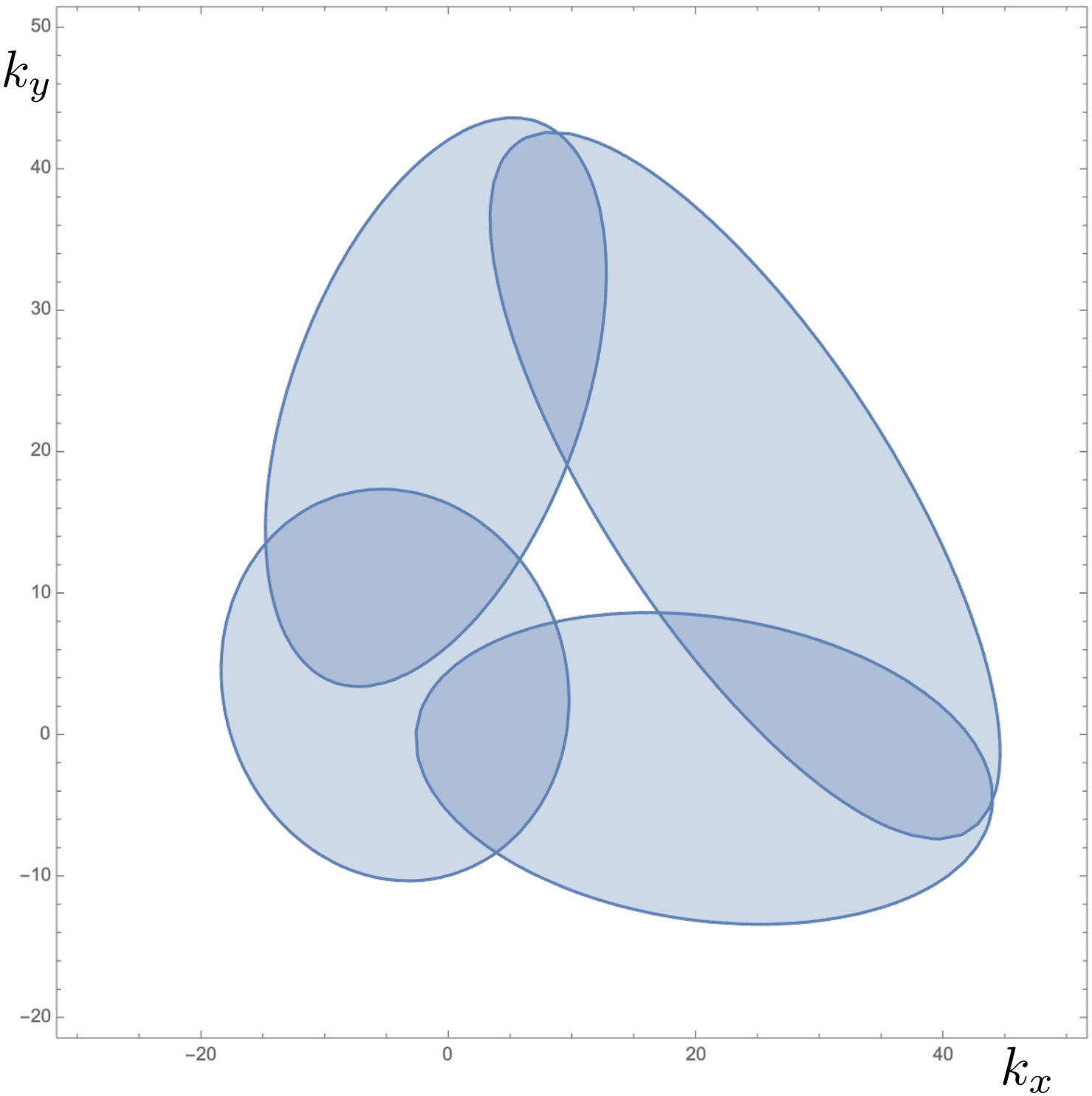

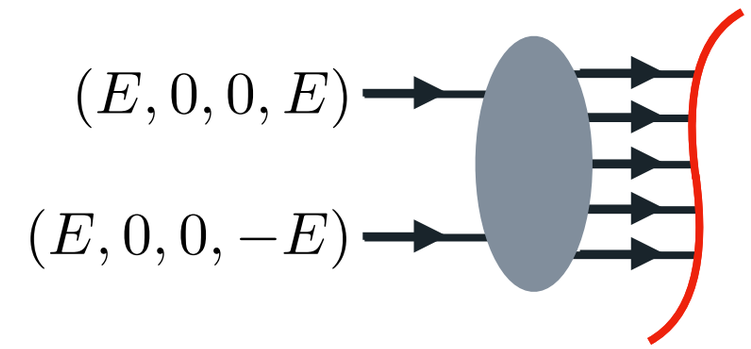

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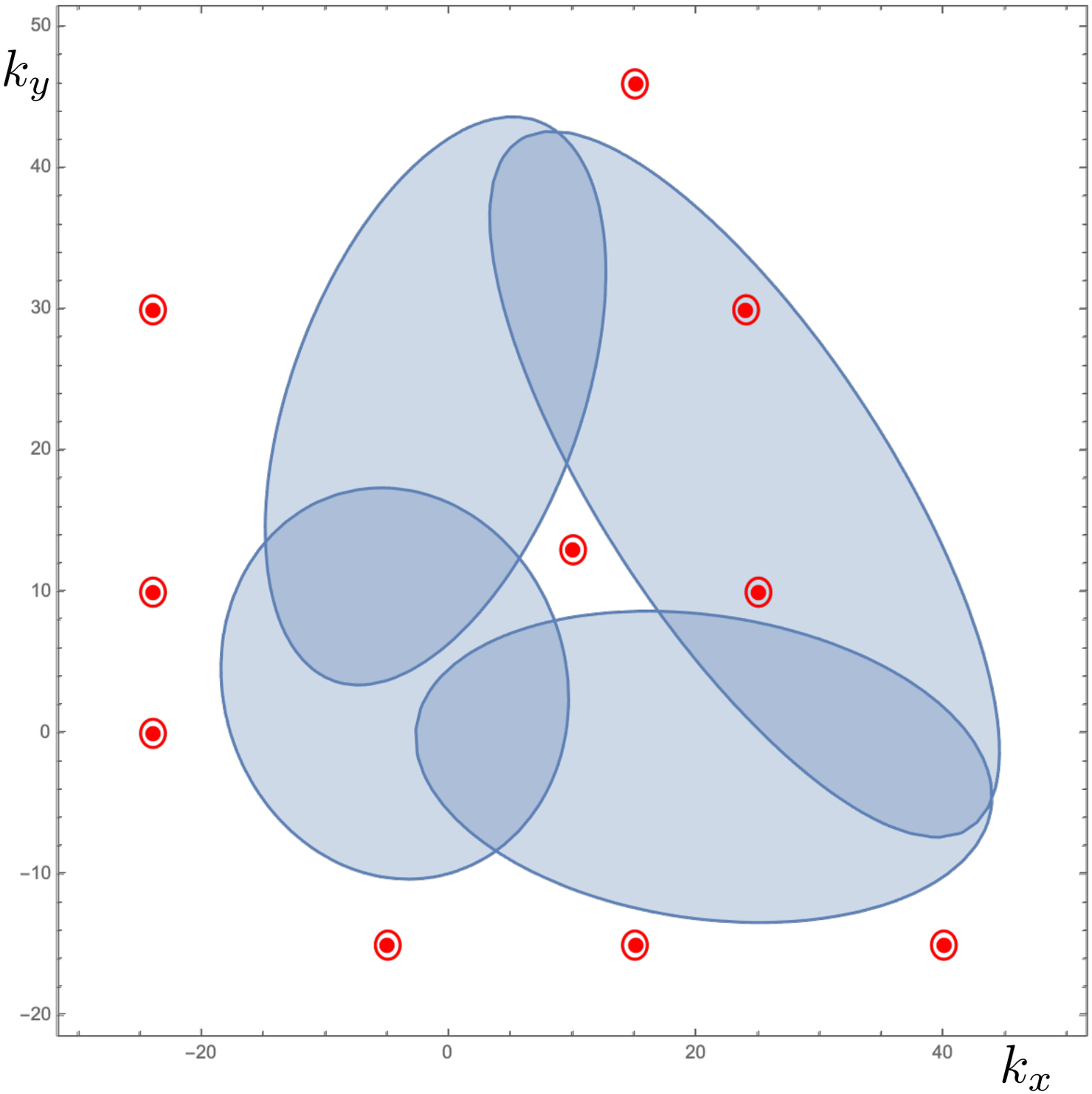

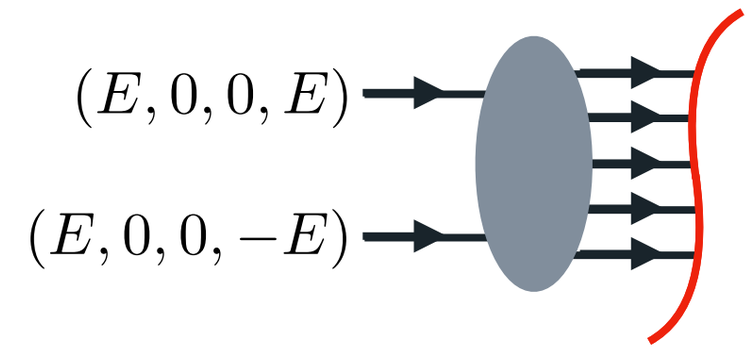

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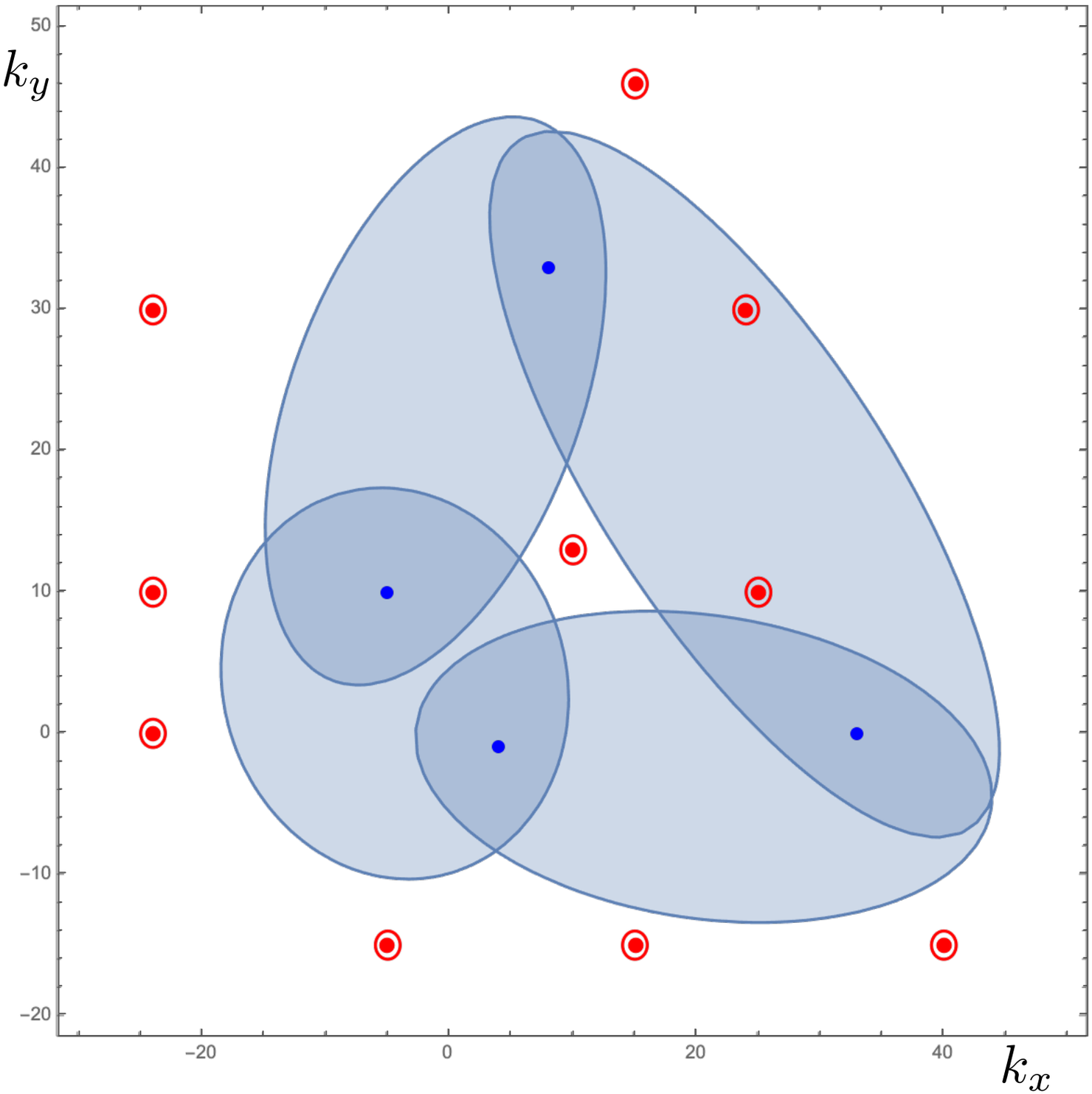

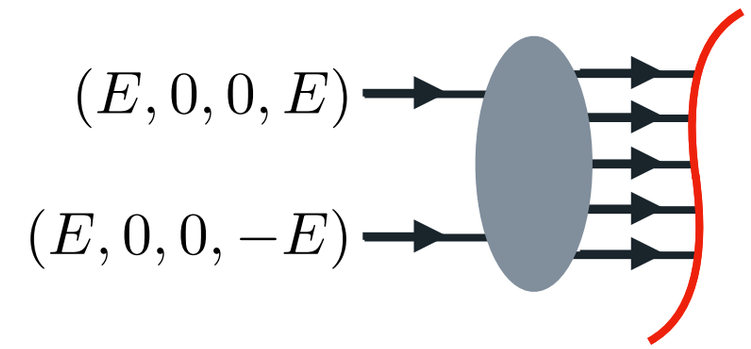

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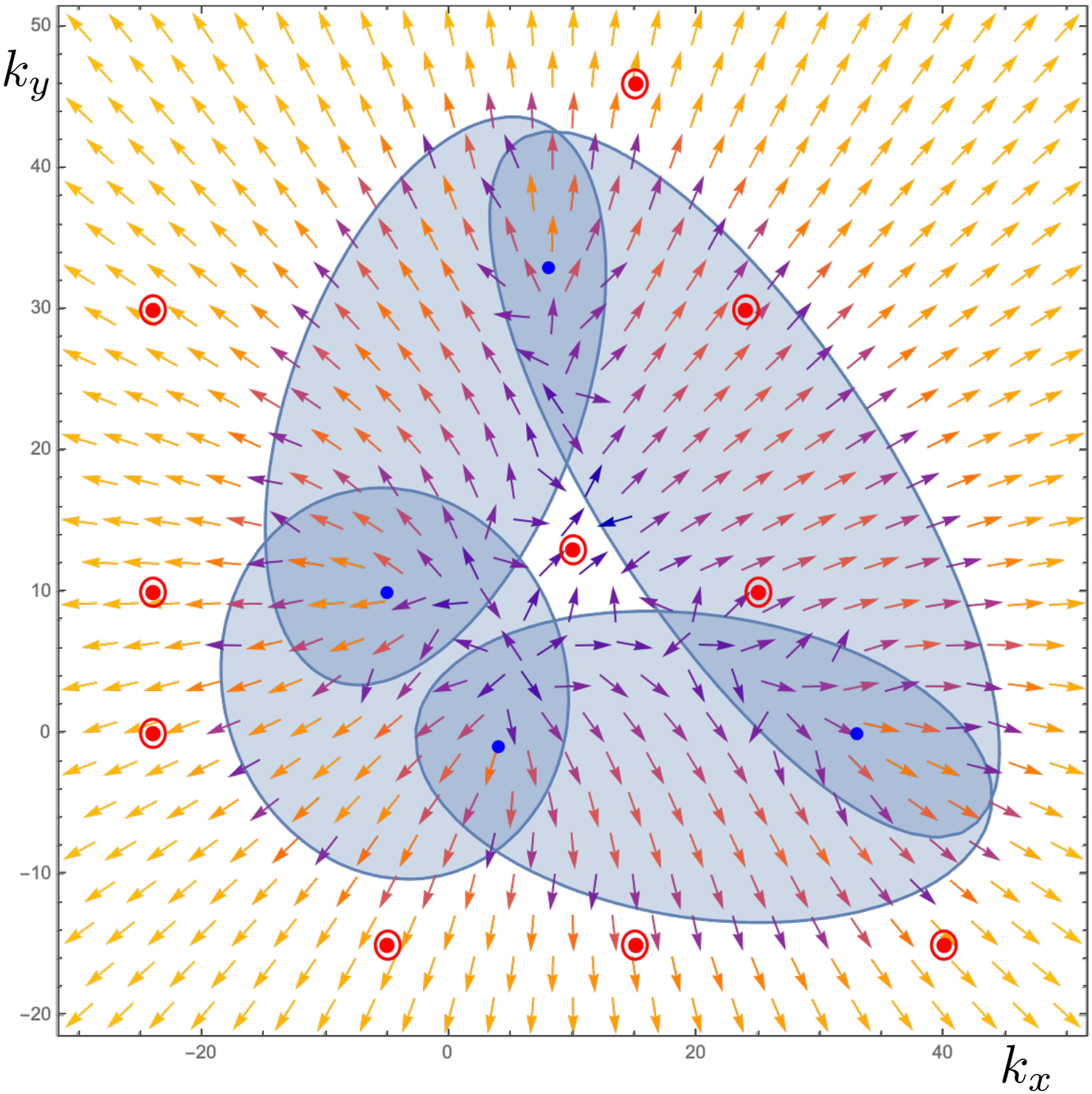

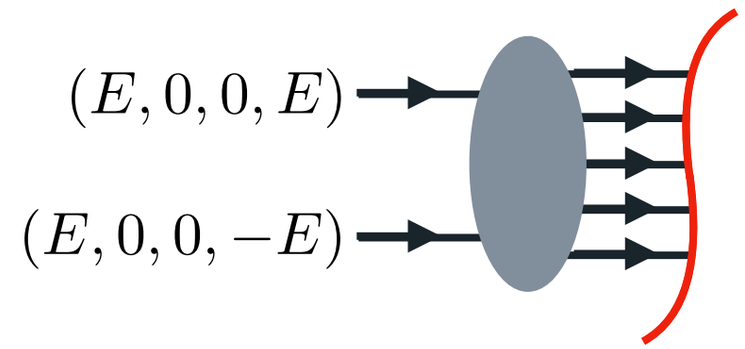

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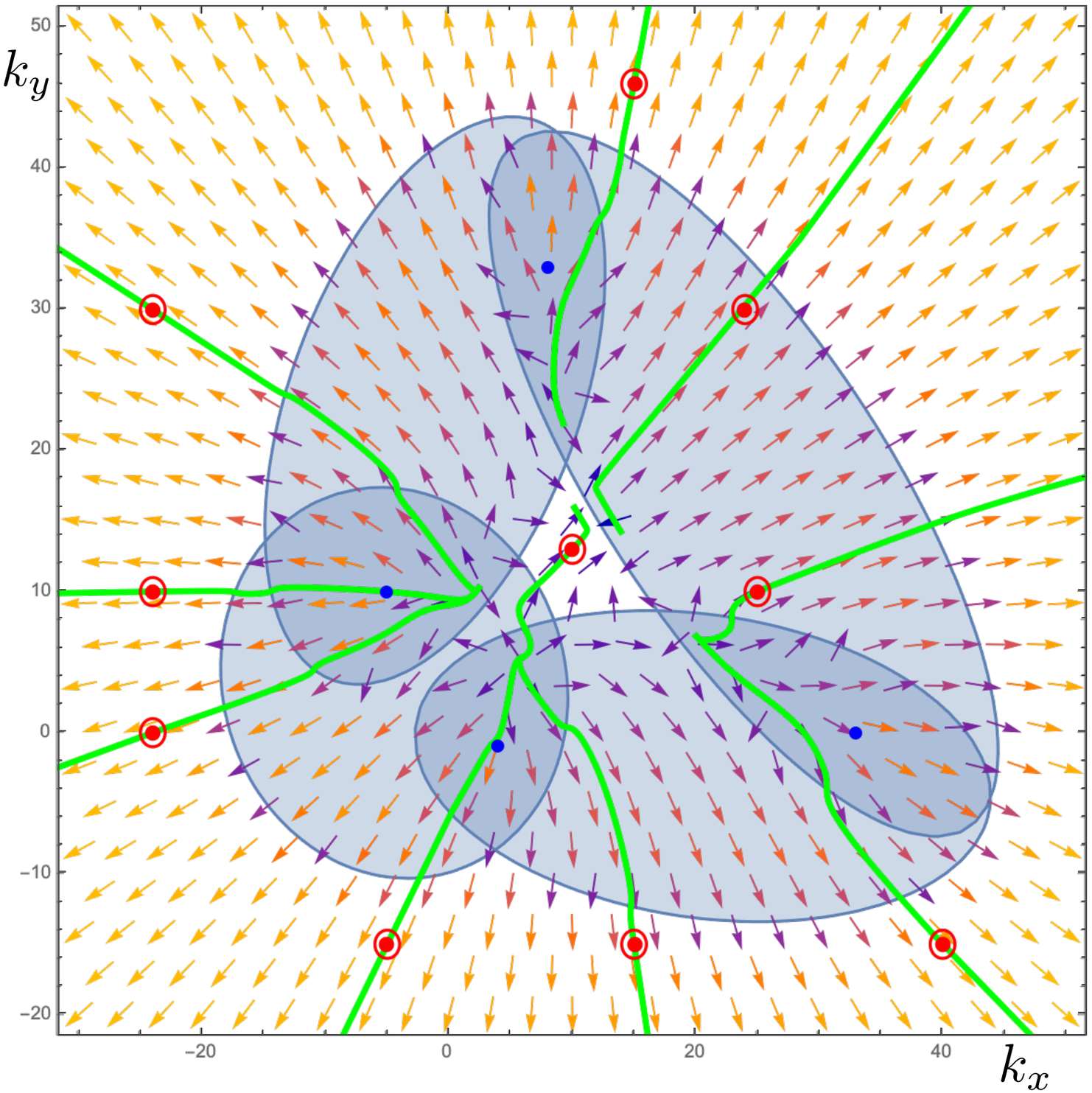

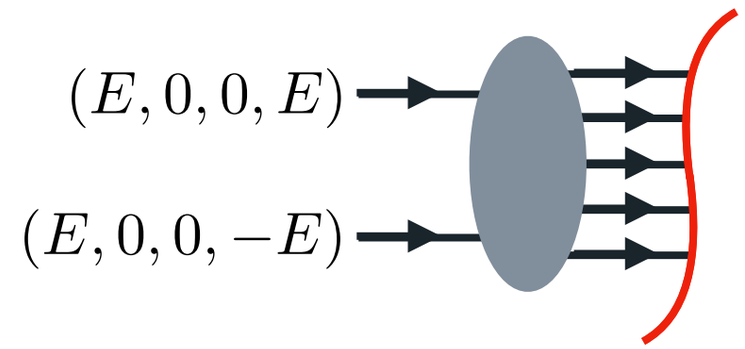

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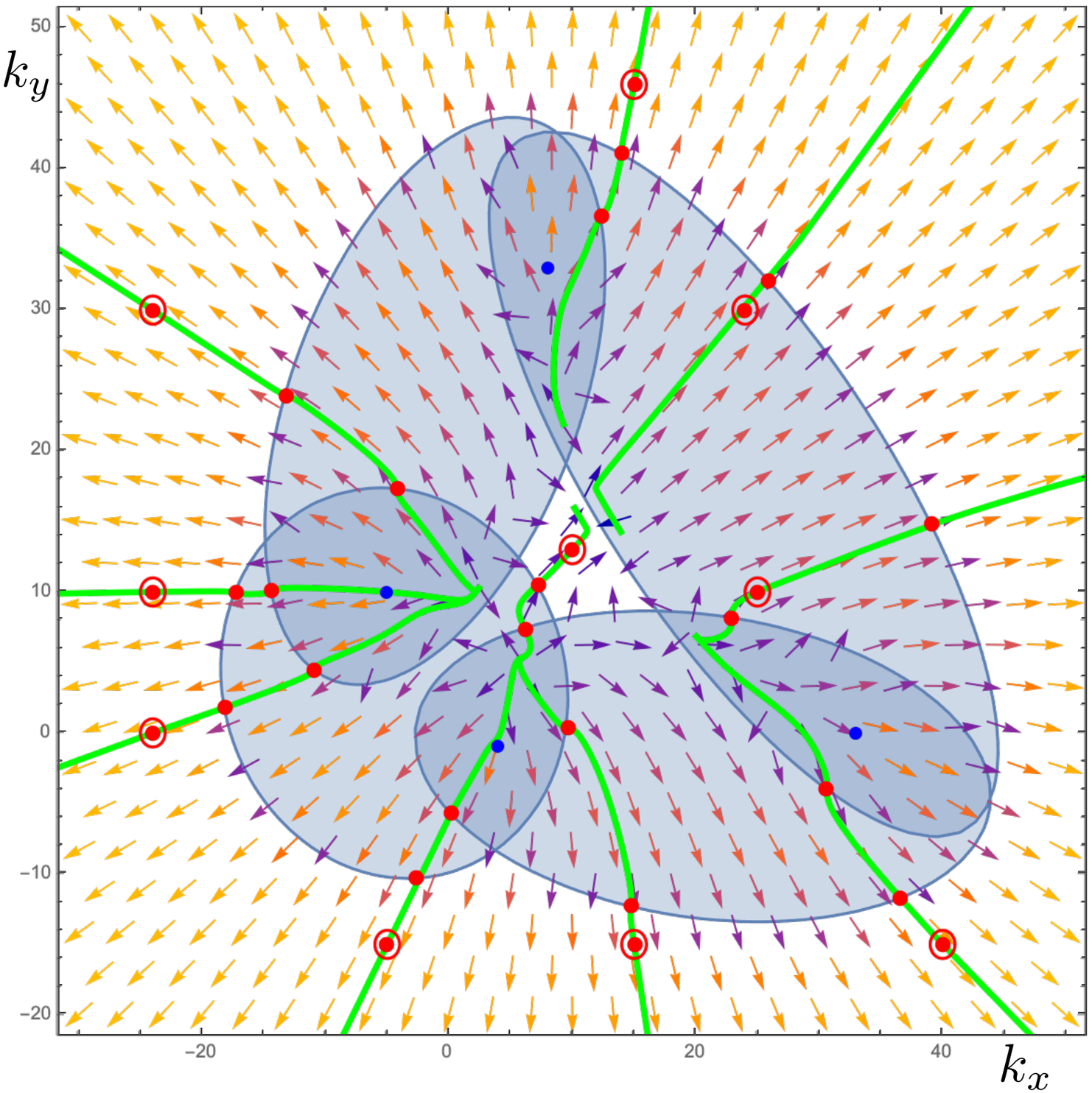

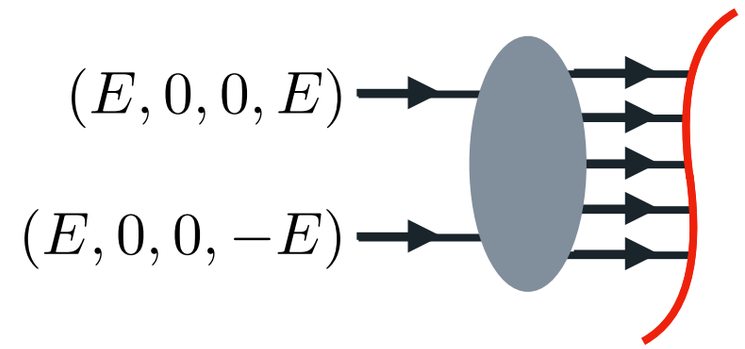

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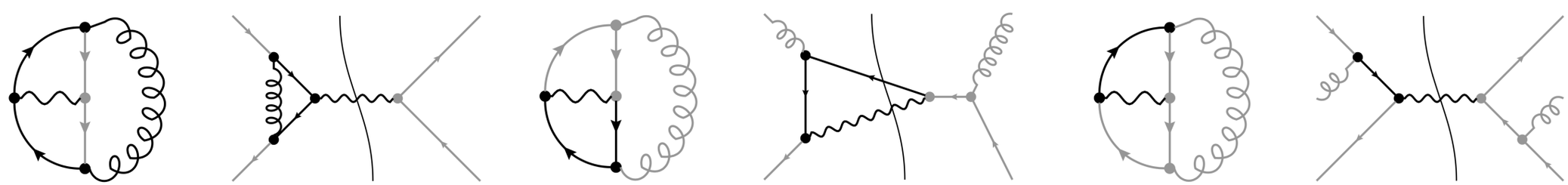
➡ 

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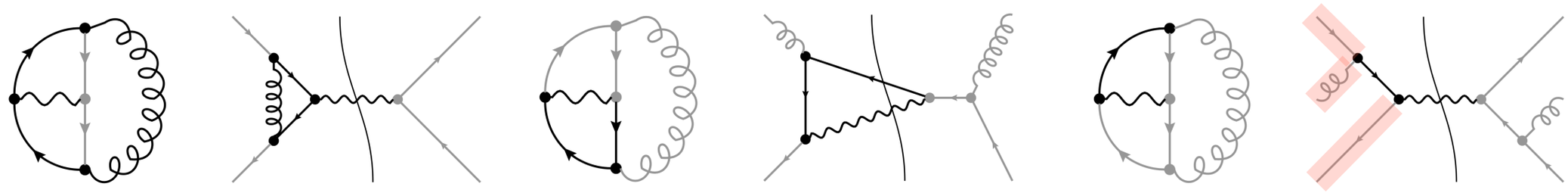
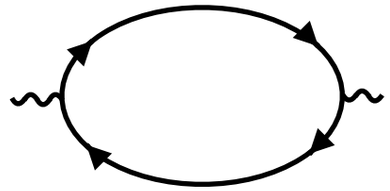
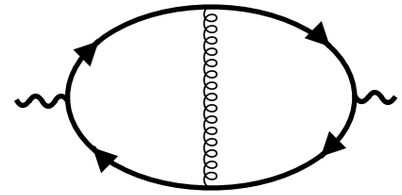
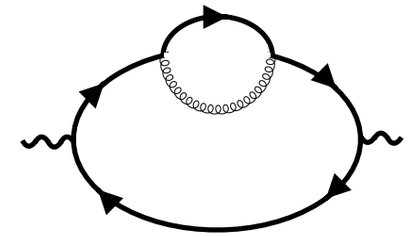
➡ 

Plan

- Challenges in pushing fixed-order accuracy
- Local Unitarity : general concepts
- (Multi-)Loop Tree Duality : formulation
- (Multi-)Loop Tree Duality : singularities / deformation
- Local Unitarity : introduction by example
- Local Unitarity : numerical results
- Conclusion

Numerical results for $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+\, 2\times$  $\Big]$

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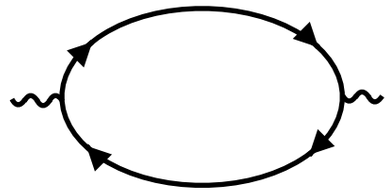
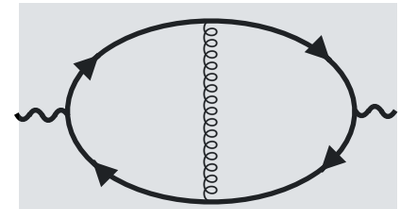
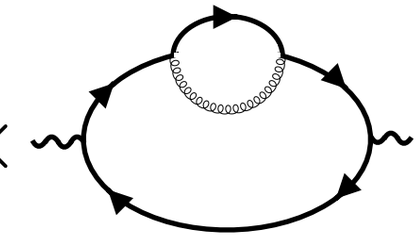
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+ 2\times$  $\Big]$

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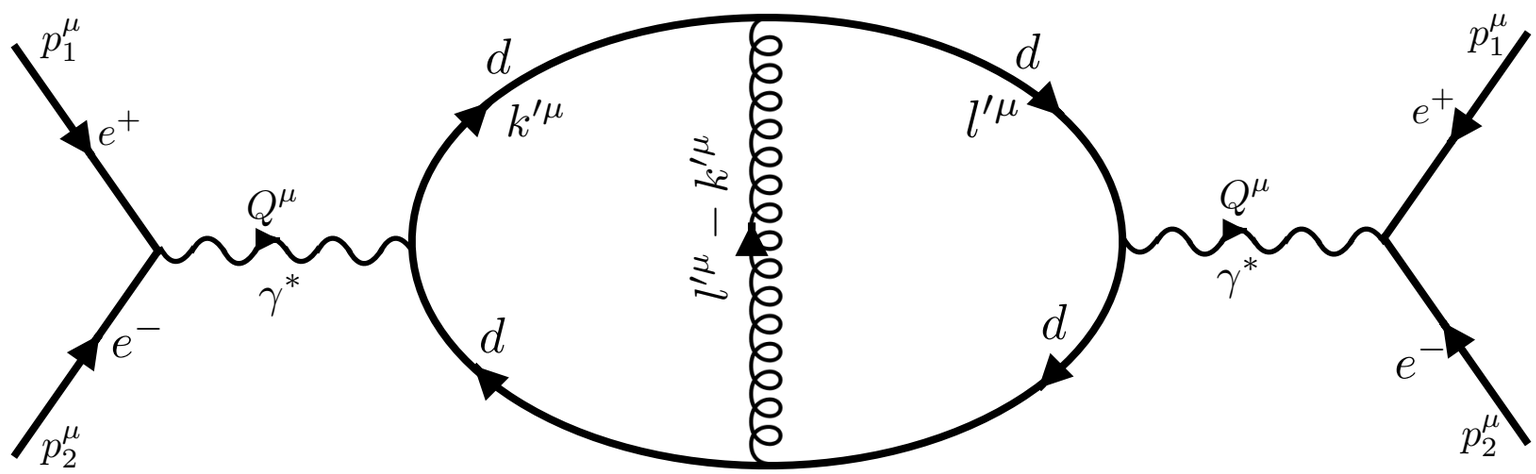

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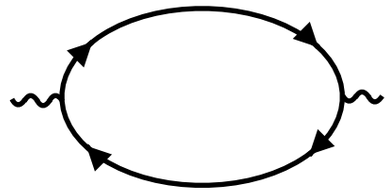
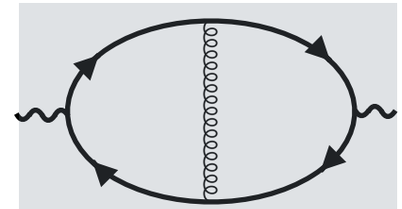
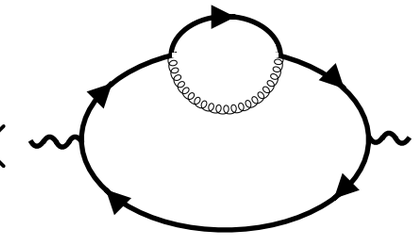
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+\, 2\times$  $\Big]$

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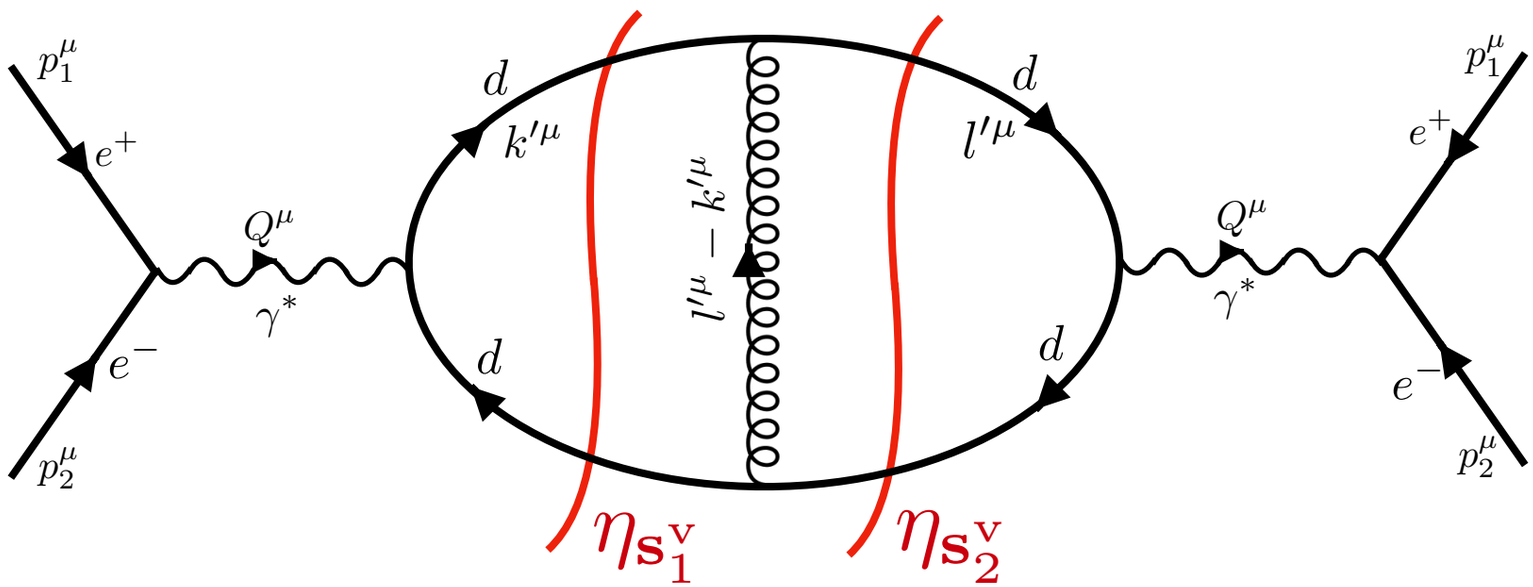

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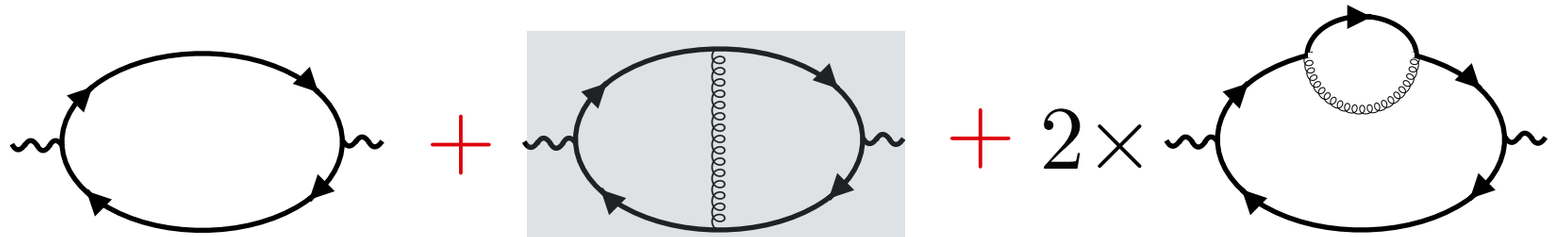
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big]$

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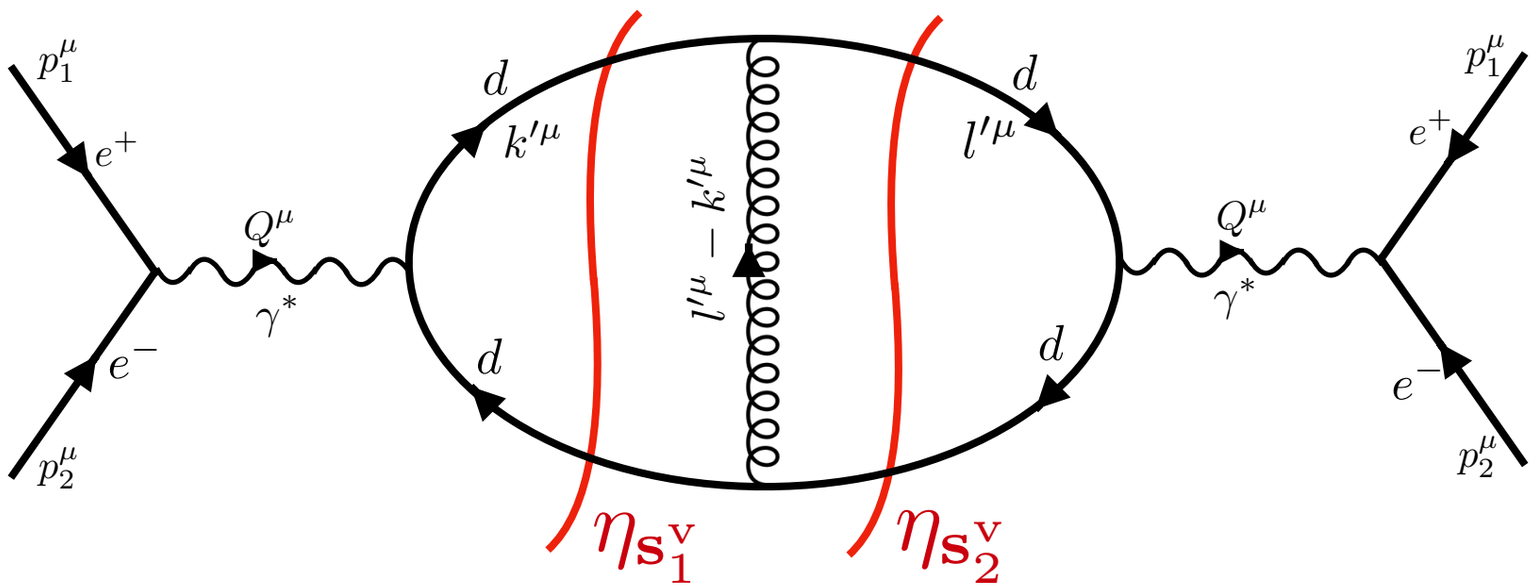

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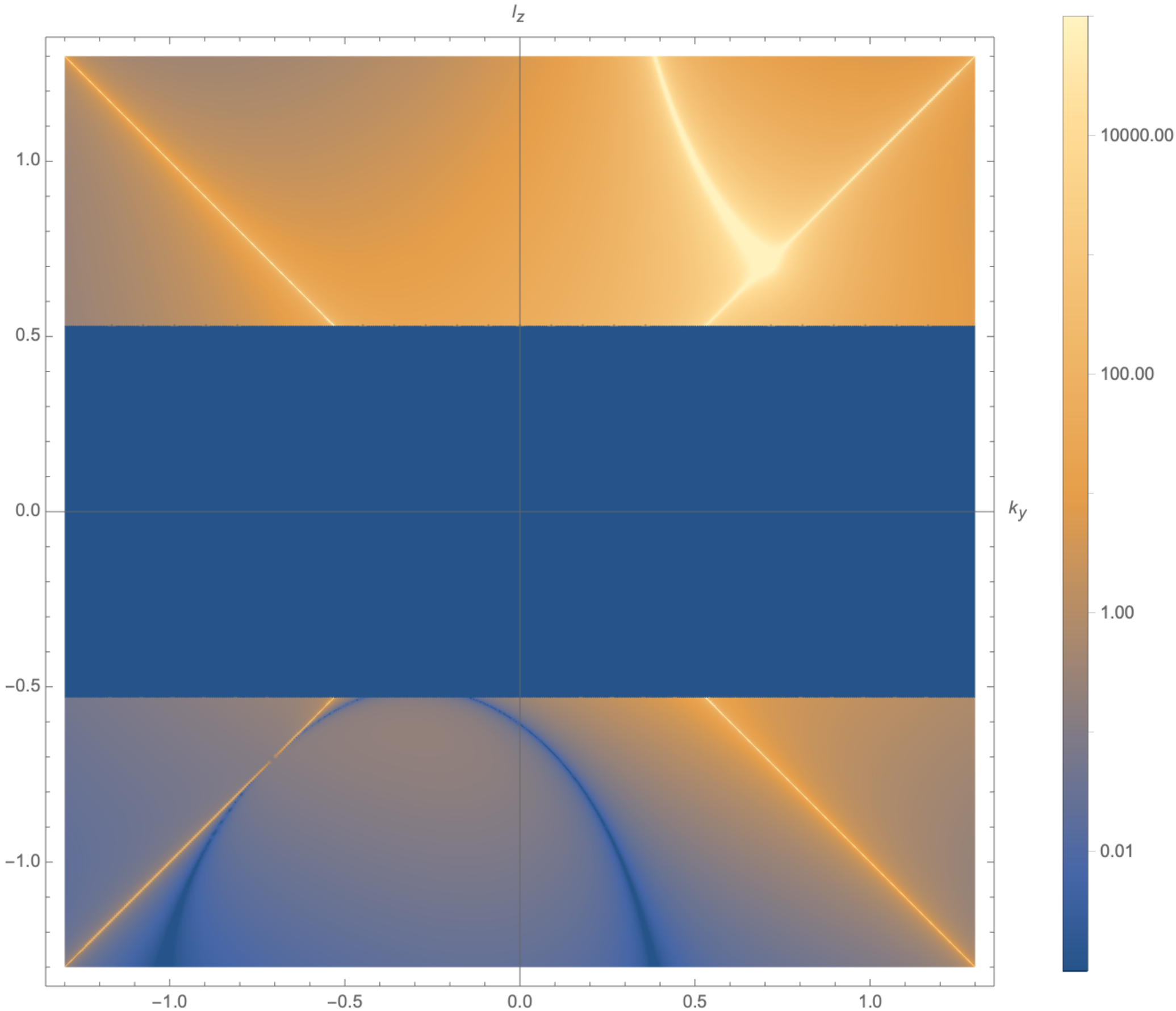

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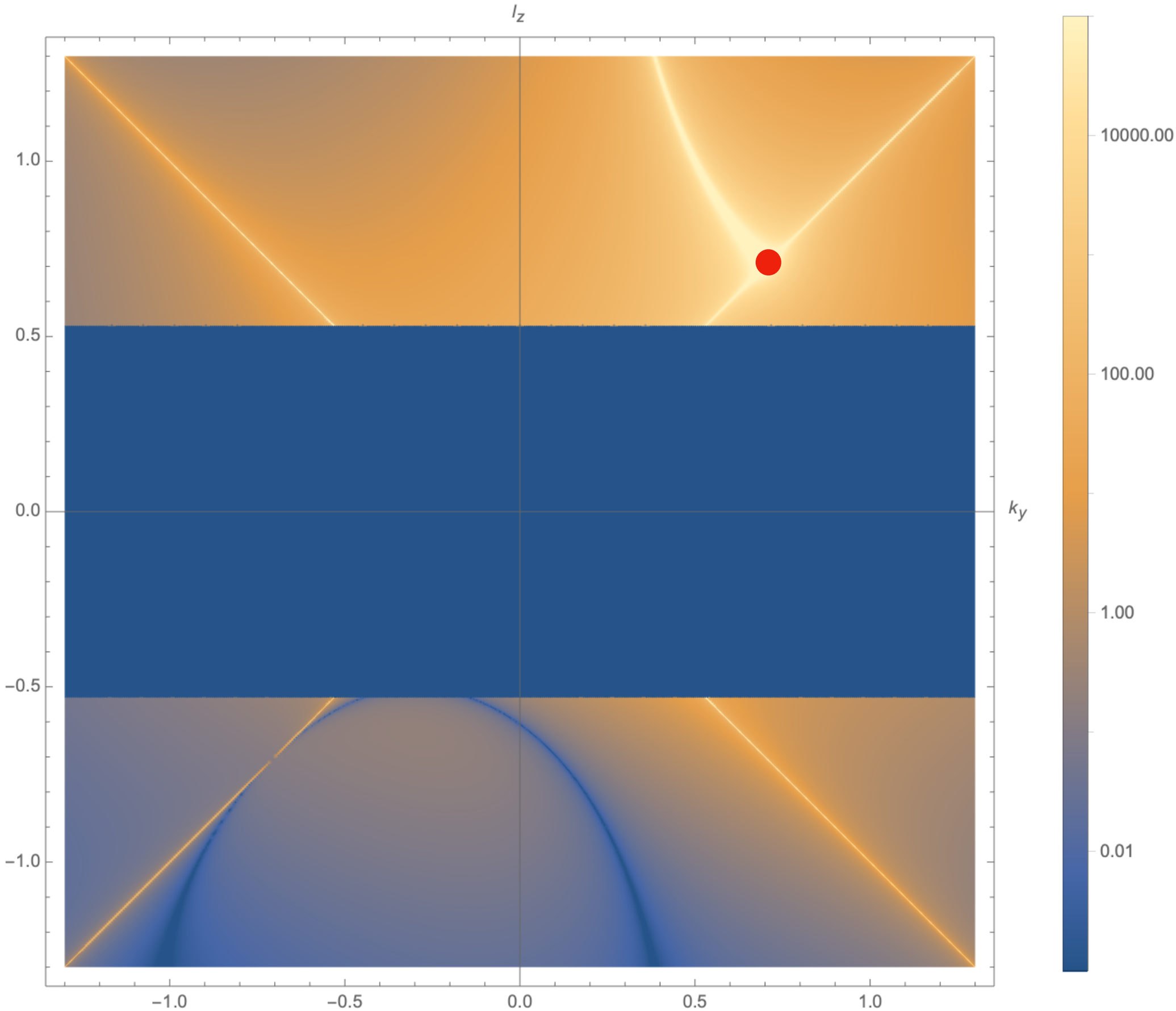

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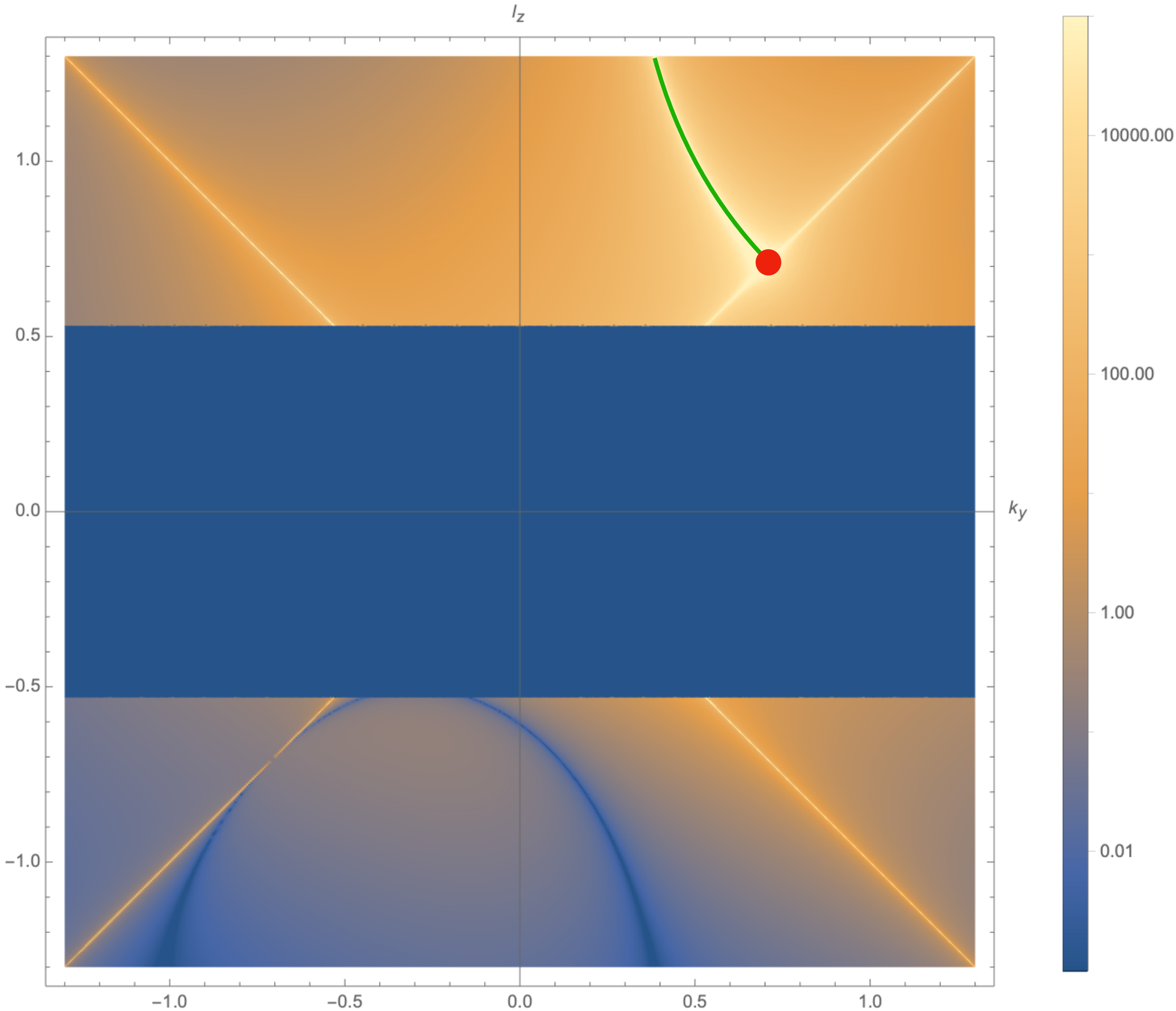

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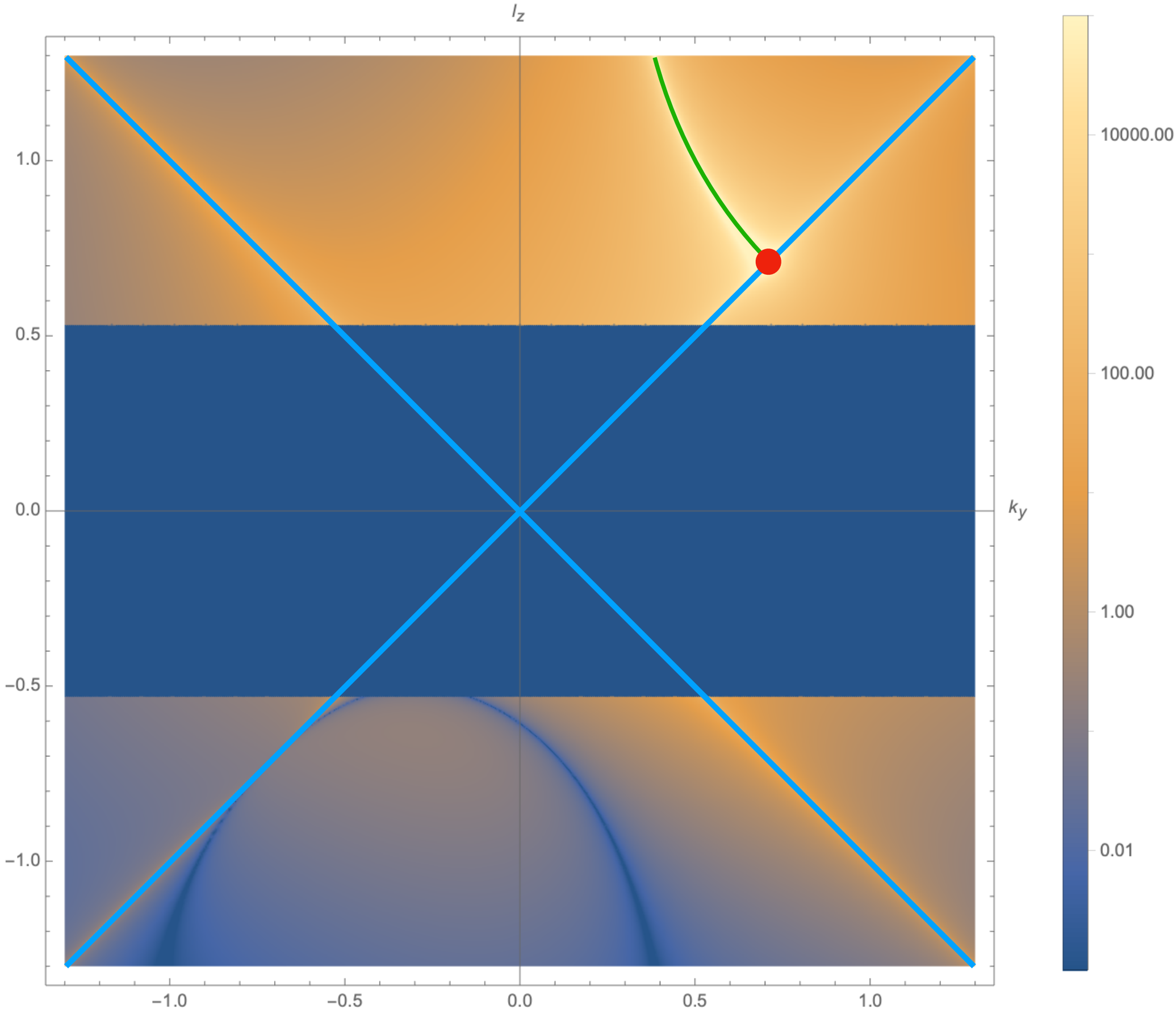
collinear

non-pinch cancellation



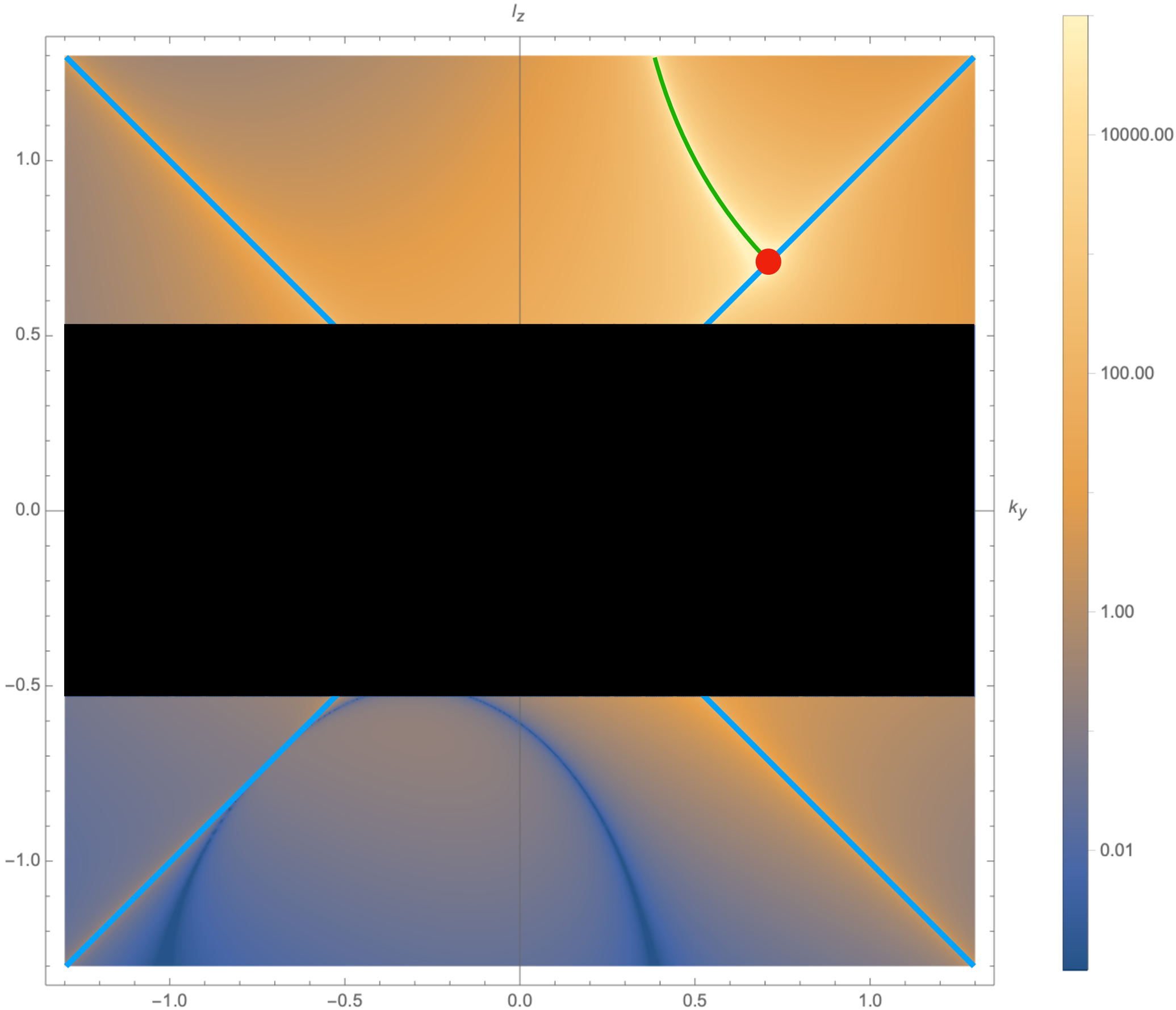
$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

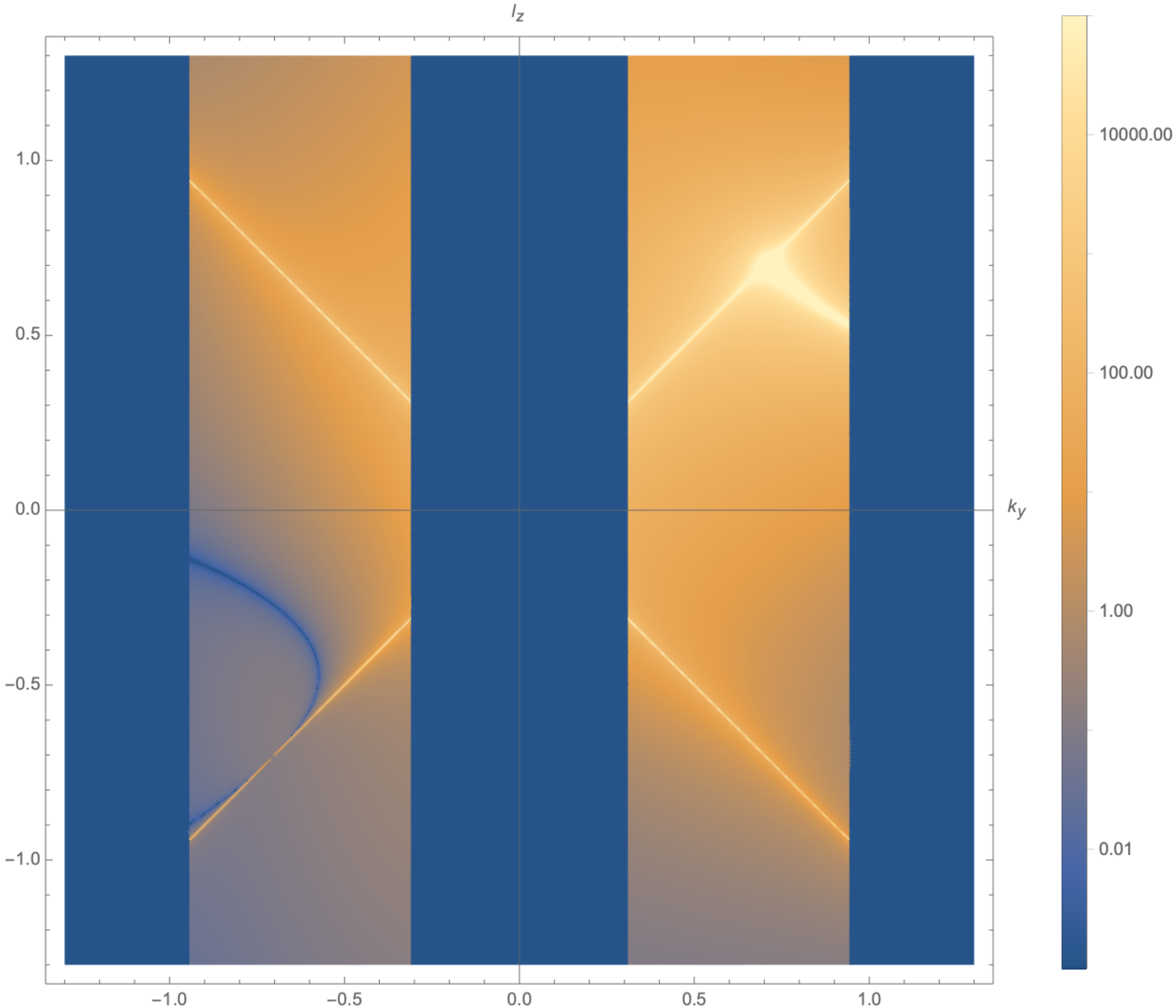
collinear

non-pinch cancellation

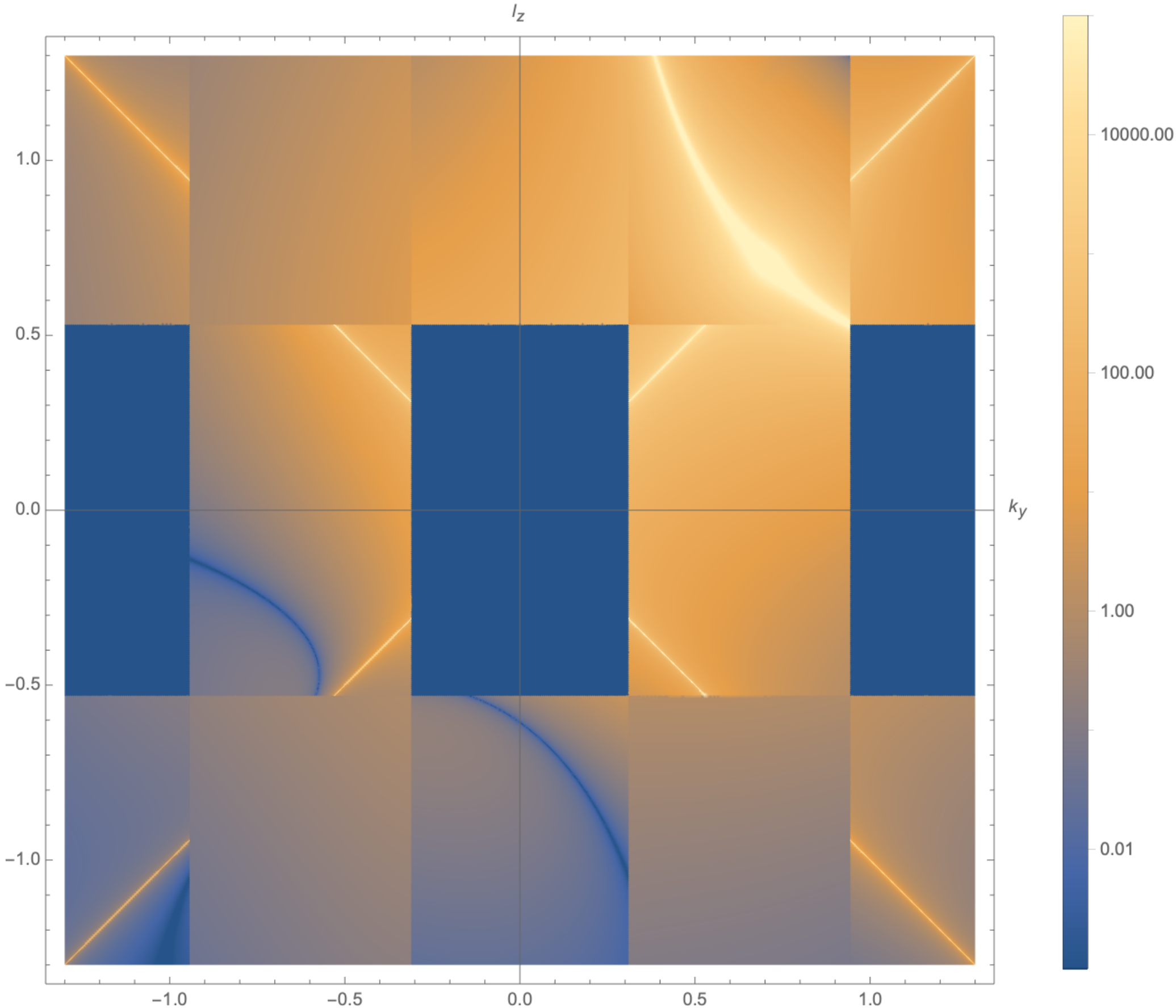
observable cuts



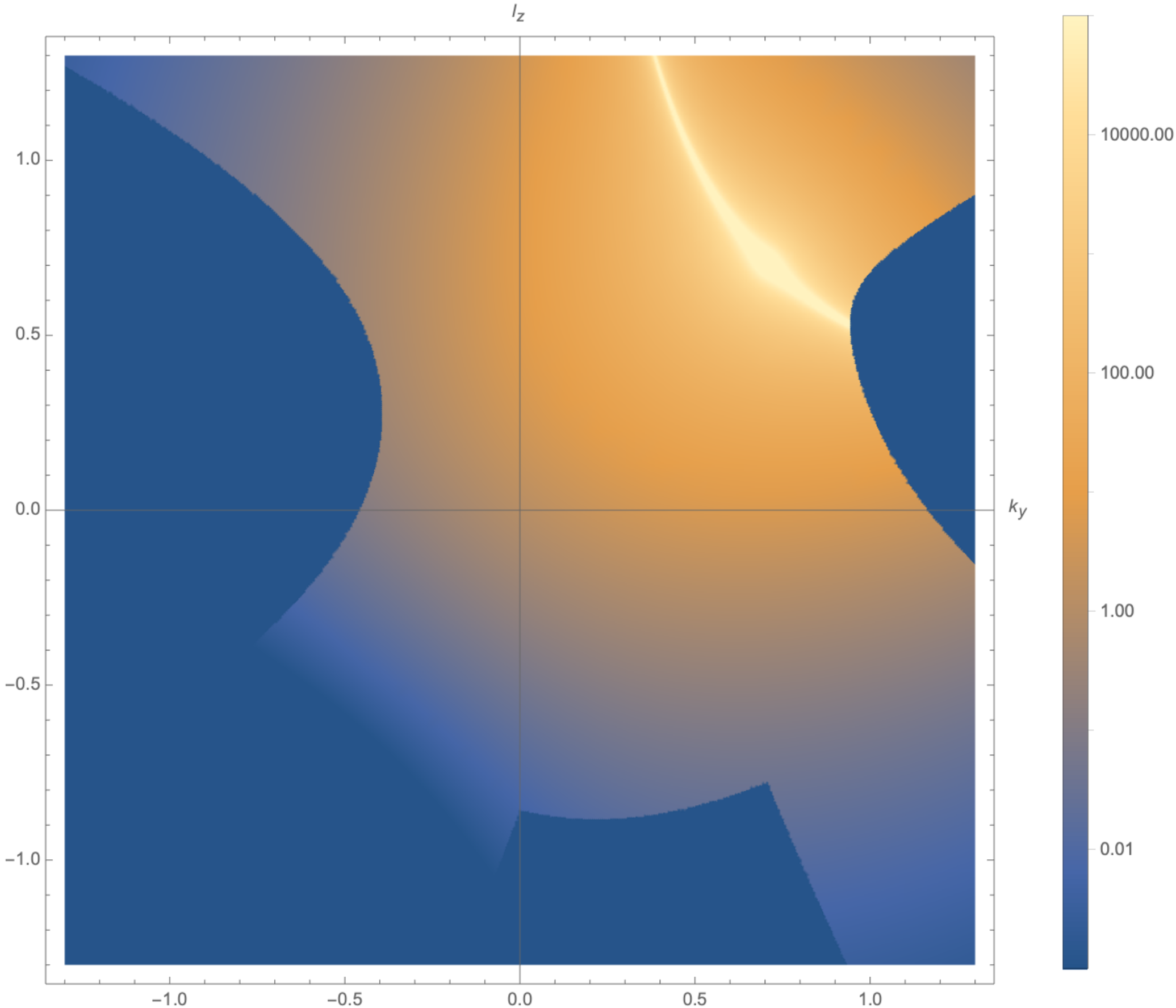
$I_{\mathbf{s}_2^{\mathrm{v}}}$



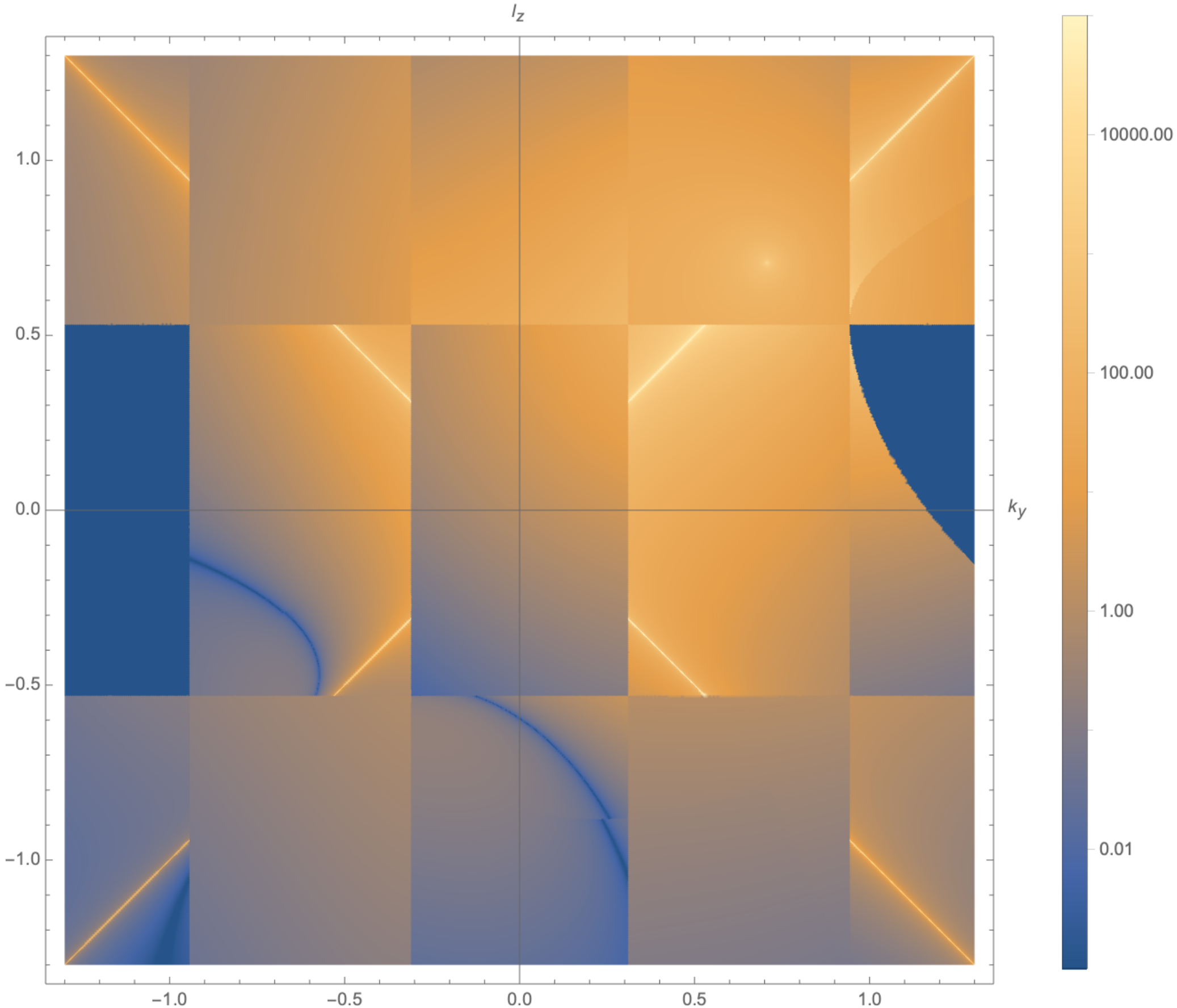
$$I_{\mathbf{s}_1^{\mathrm{v}}} + I_{\mathbf{s}_2^{\mathrm{v}}}$$



$$I_{\mathbf{s}_1^{\mathrm{r}}} + I_{\mathbf{s}_2^{\mathrm{r}}}$$

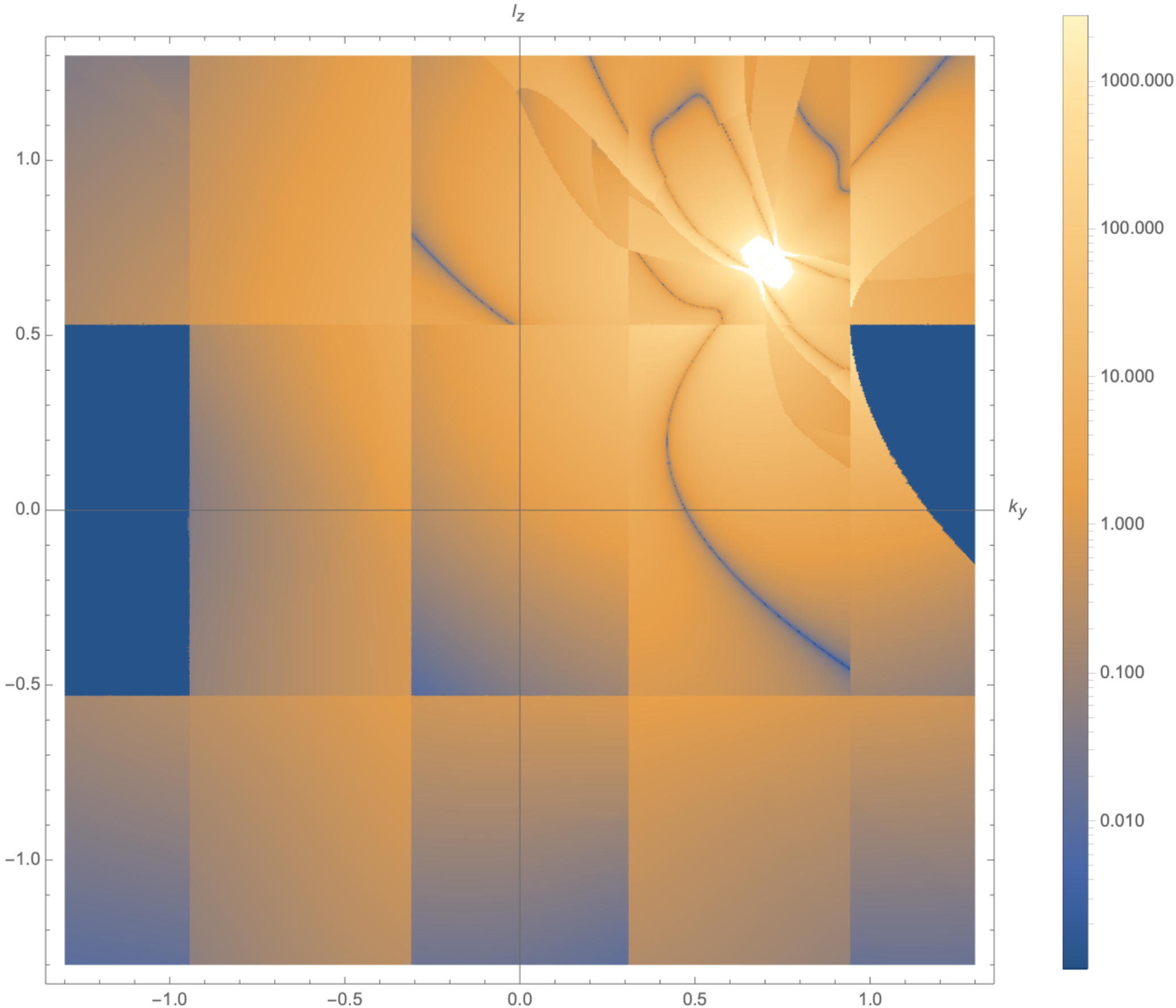


I_Σ

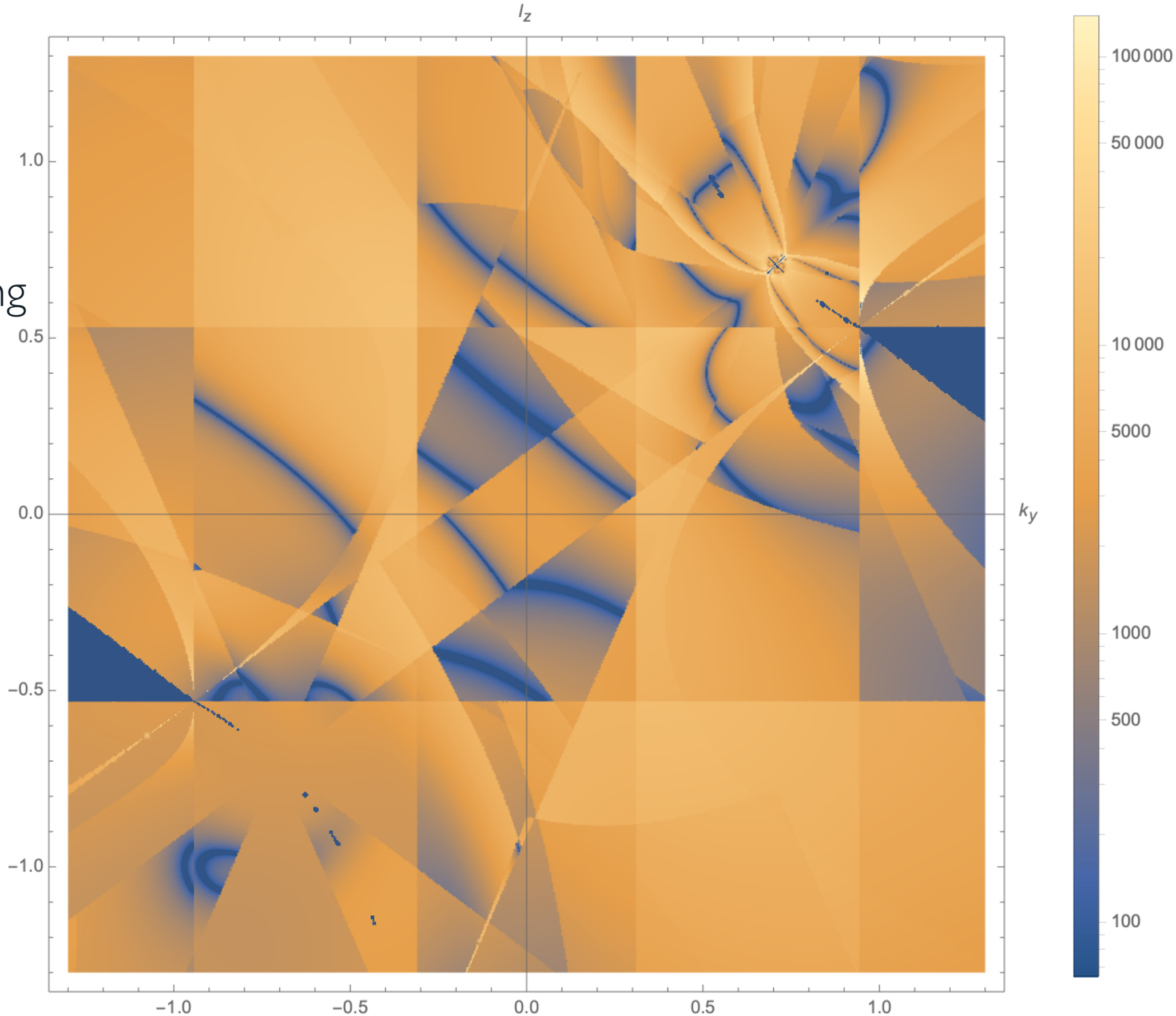


$\mathrm{Re}\left[I_\Sigma\right]$

with deformation



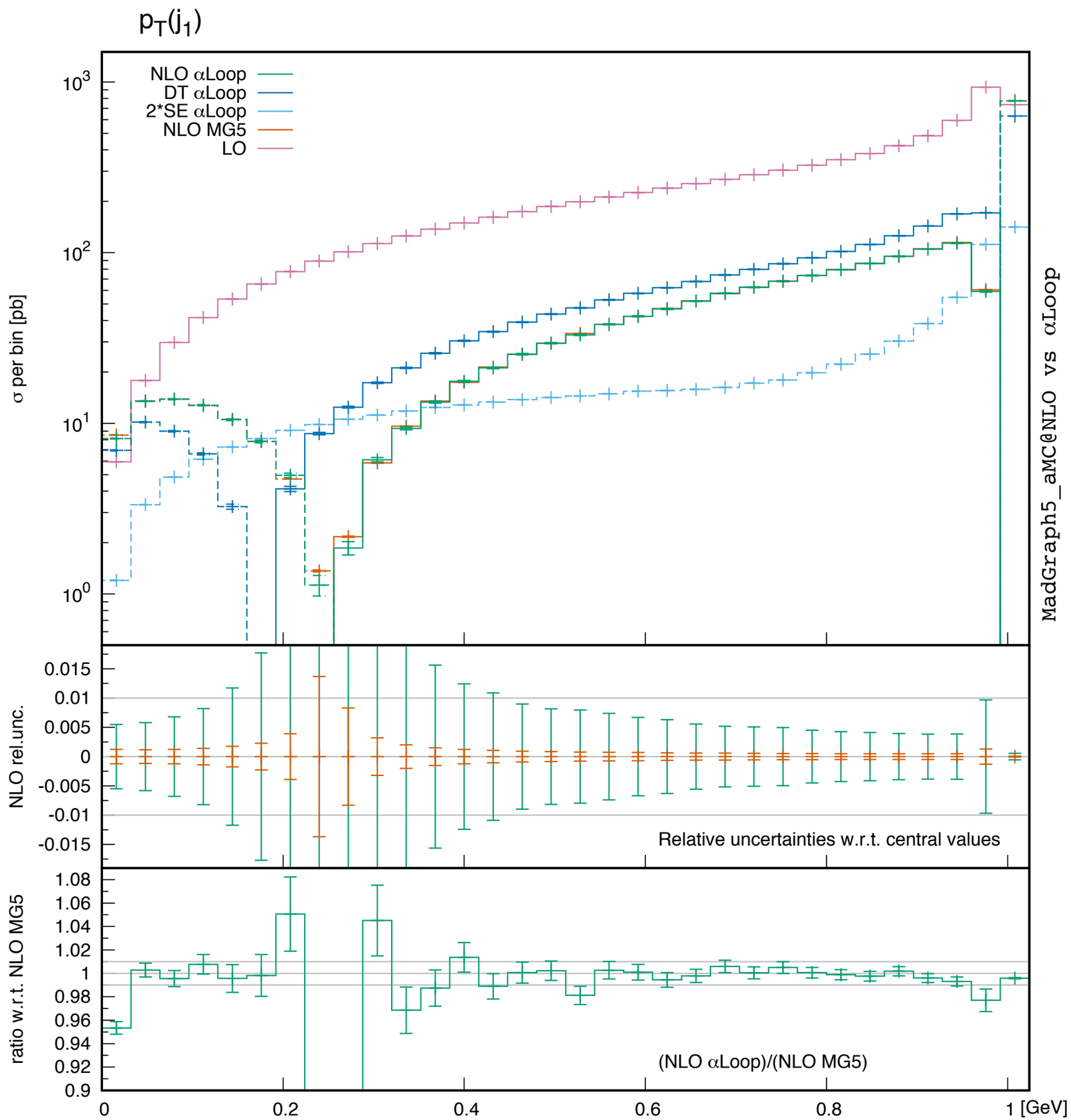
$\mathrm{Re}\left[I_\Sigma\right]$
with
deformation
and
multichanneling



Numerical results from αLoop

Differential results for

$$e^+e^- \to \gamma^\star \to d\bar{d} \text{ @ NLO}$$



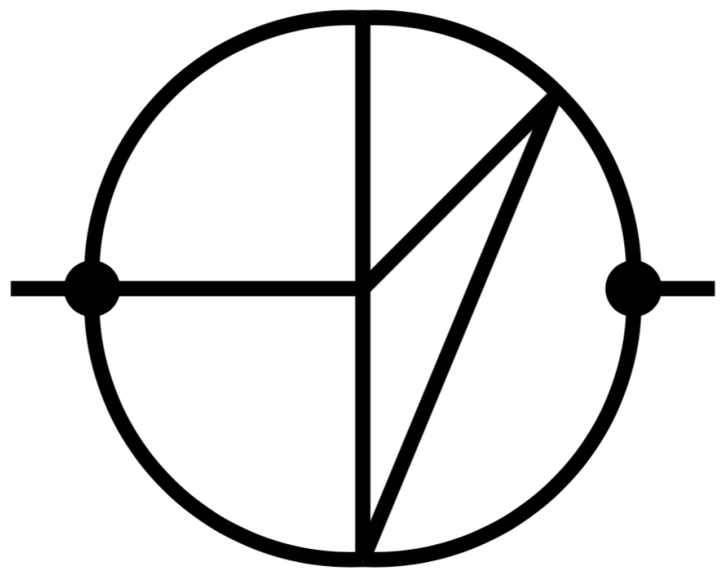
DT = Double-Triangle

SE = Self-Energy

Numerical results from αLoop

Piece of the "N⁴LO" correction to $\phi^\star \to \phi\phi$ within $\{\phi^3, \phi^4\}$-theory

One individual **5-loop supergraph** :

LU [] $= \frac{5\pi}{(16\pi)^5}\frac{441}{40}\zeta(7)$

[**FORCER**
B. Ruijl, T. Ueda, J. Vermaseren
arxiv: 1704.06650]

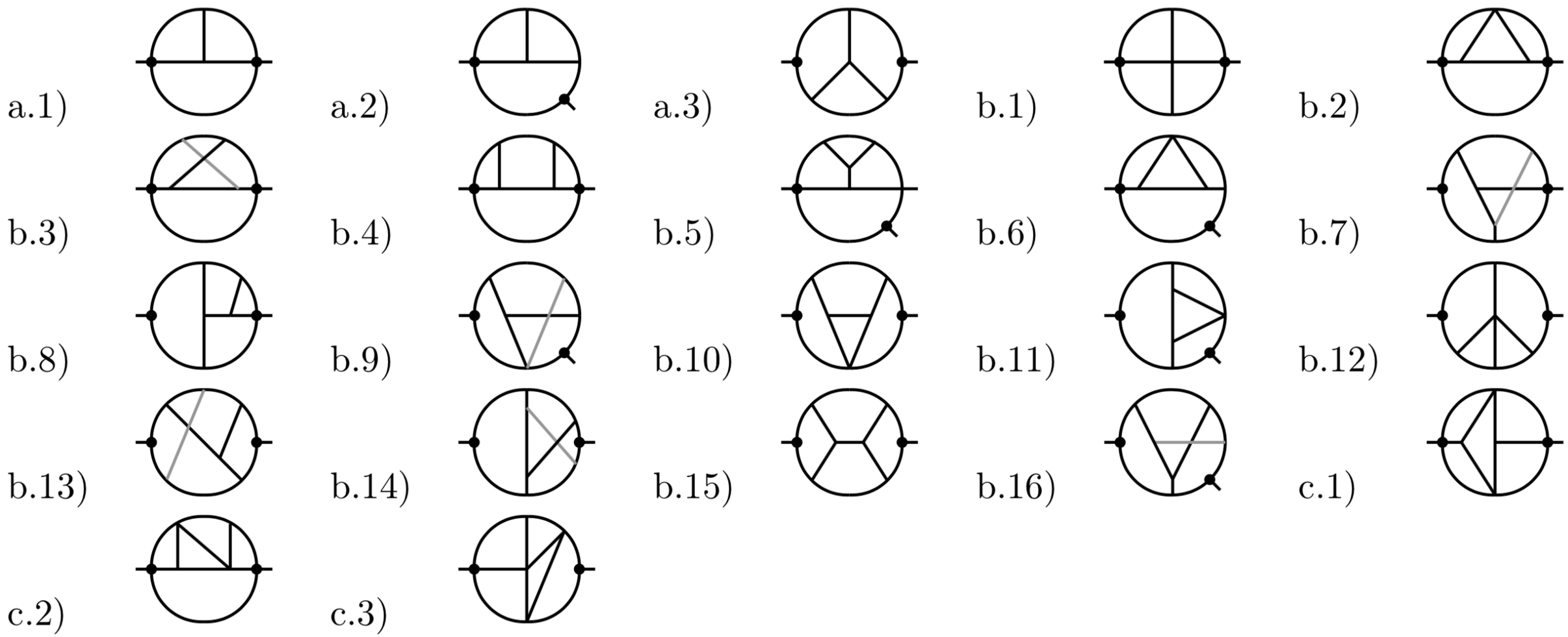
[**R***
F. Herzog, B. Ruijl
arxiv: 1703.03776]

without **contour deformation** (doing it differentially would have required one)

Γ	N_p $[10^6]$	t/p $[\mu s]$ min	t/p $[\mu s]$ avg	N_{ch}	FORCER $[GeV^2]$	αLoop $[GeV^2]$	exp.	Δ $[\sigma]$	Δ $[\%]$
c.3	1	1600	69000	130	1.77832	1.7797(33)	-9	0.42	0.00077

Numerical results from αLoop

We also tested **all** possible *finite* **3-loop** and **4-loop** supergraph topologies :



Plan

- Challenges in pushing fixed-order accuracy
- Local Unitarity : general concepts
- (Multi-)Loop Tree Duality : formulation
- (Multi-)Loop Tree Duality : singularities / deformation
- Local Unitarity : introduction by example
- Local Unitarity : numerical results
- Conclusion

Summary : a wishlist

Wishes	Local Unitarity
Fully local	✓
No restriction on number of scales	✓
No reduction to master integrals	✓
No special functions	✓
No dimensional regularisation	✓
No IR phase-space counterterms	✓
No IR loop counterterms	✓
Contour deformation only when needed	✓
Applicable to all processes and pert. orders	✓

Summary : Local Unitarity

Formalism :

- **Local** cancellation of all **final-state IR singularities**
- Completely **generic** (masses, kinematics, topologies, observables…)
- New **theoretical perspectives** on perturbative expansions

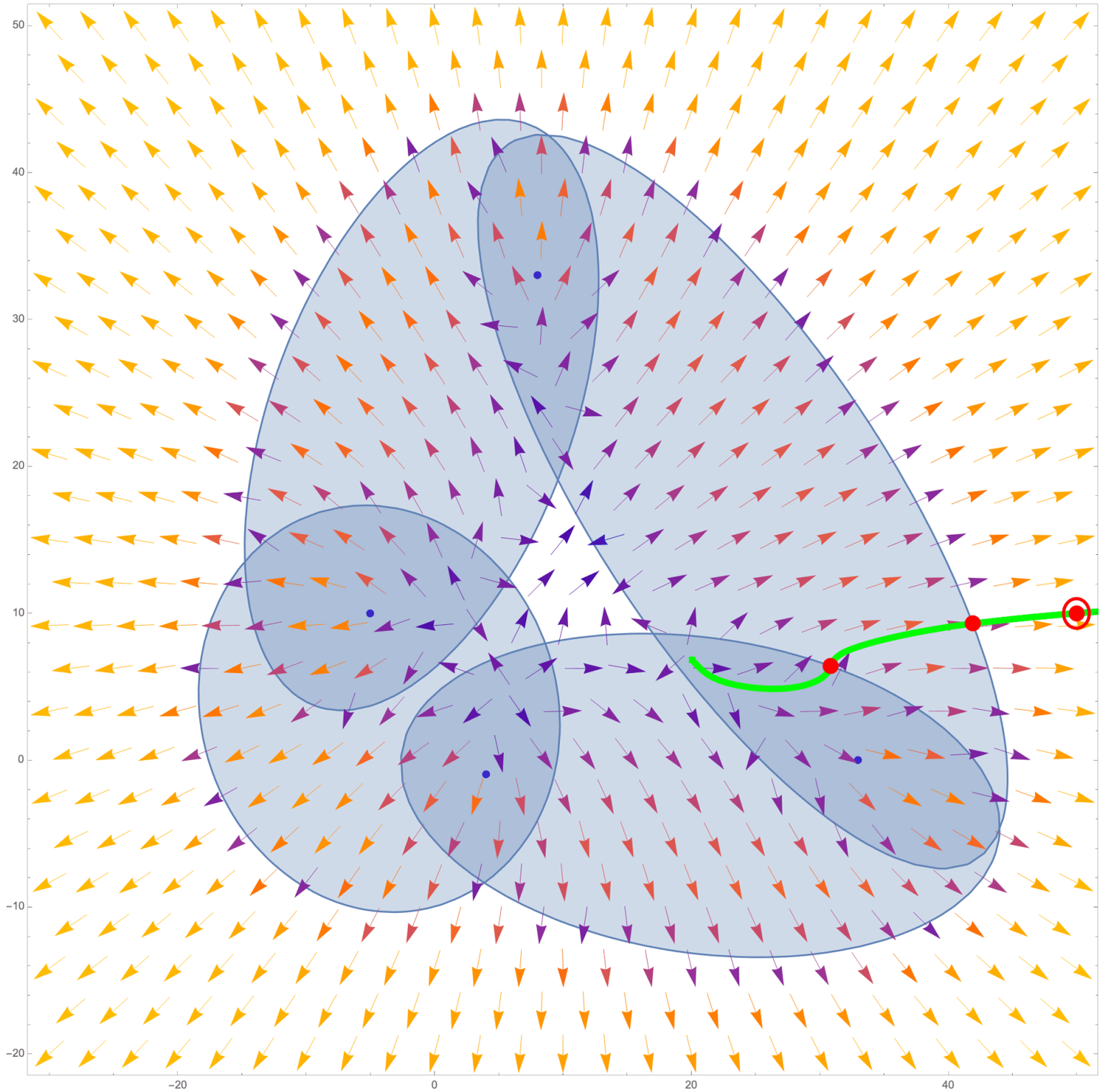
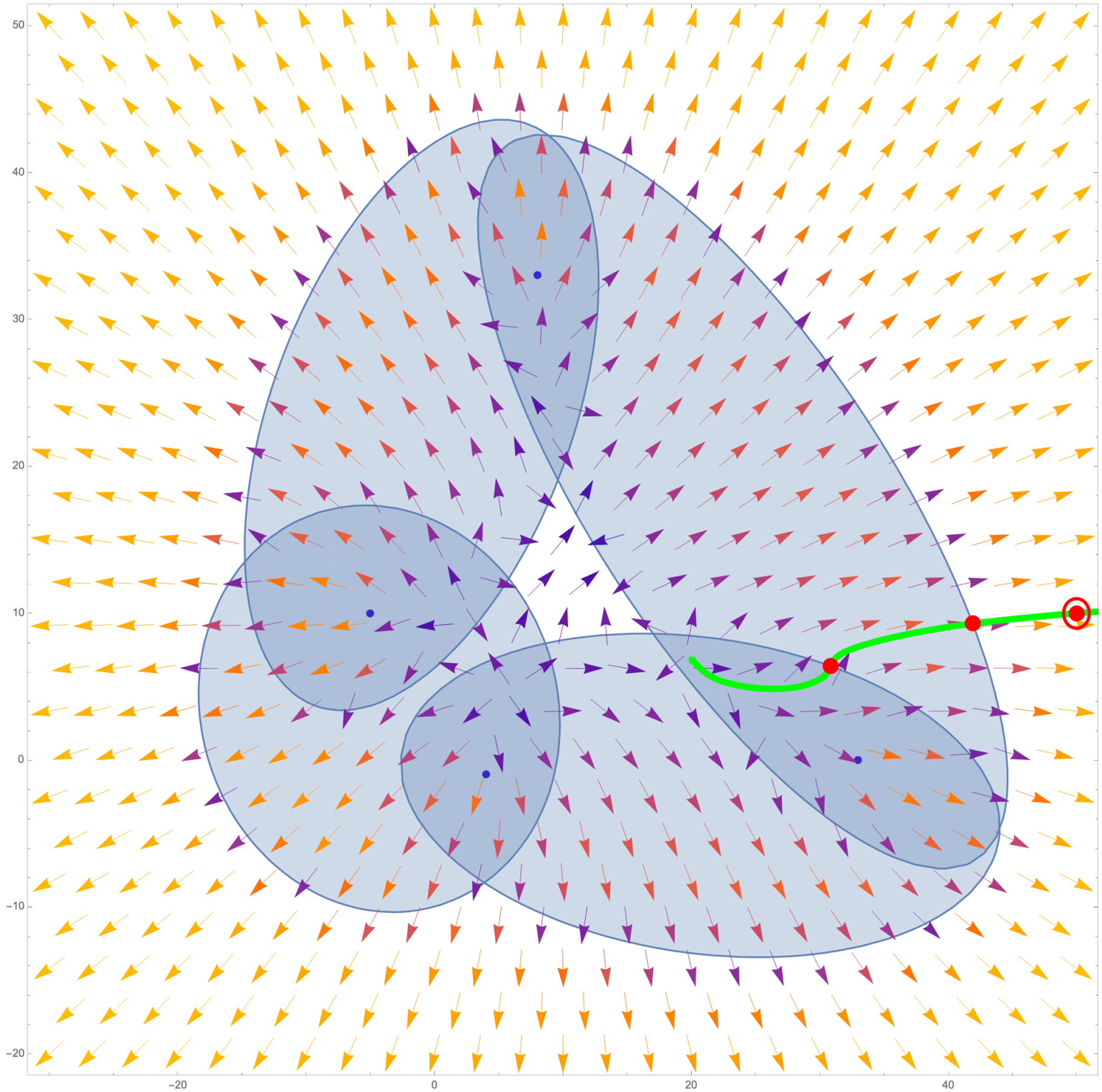
Numerical application :

- **Proof-of-concept** with fully differential $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO
- Tested **complicated IR cancellations** with scalar 5-loop supergraphs
- Full **automation** already underway : **αLoop**

Outlook :

- Apply **LU** to **new unknown corrections** : $H > jj$ @ NNLO with b, c, t
- Generalise **LU** so as to apply of **initial-state singularities** as well

Questions?



Questions?