Differential Equations with a Convergent Integer power Series Solution

Mark van Hoeij

April 7, 2016

Notations

Let $y \in \mathbb{Q}[[x]]$ and suppose that

1. y has a positive radius of convergence
2. There are constants $c, n \neq 0$ for which $c \cdot y(n \cdot x) \in \mathbb{Z}[[x]]$.

Then we call y a CIS (Convergent near-Integer power Series).
Also called: *globally bounded*.

If $u_0, u_1, u_2, \ldots$ an integer sequence then the *generating function*

$$y := \sum_{n=0}^{\infty} u_n x^n$$

is CIS if it converges. Case of interest: When y satisfies a differential equation of order 2.
The database oeis.org contains a huge number of examples.

Hypergeometric ${}_2F_1$ function

Let $a, b, c \in \mathbb{Q}$ with $c \notin \{0, -1, -2, \dots\}$. The hypergeometric function

$$h(x) \ := \ {}_2F_1(a, b; c|x) = 1 + \frac{a \cdot b}{c \cdot 1!}x + \frac{a(a+1) \cdot b(b+1)}{c(c+1) \cdot 2!}x^2 + \cdots$$

$h(x) \in \mathbb{Q}[[x]]$. CIS $\implies$ $\#\{\text{primes in denominators}\} < \infty$.

$$h(x) \text{ is CIS } \iff h(x) \text{ algebraic or } c \in \mathbb{N}.$$

Example:

$a, b, c = \frac{1}{12}, \frac{5}{12}, 1$ (very common) (monodromy group is arithmetic)

Then $h(x)$ is CIS (Convergent near-Integer Series).

Indeed: $h(1728\,x) \in \mathbb{Z}[[x]]$.

Hypergeometric expressions

Let $h(x) = {}_2F_1(a, b; c|x)$. A function $y(x)$ can be *expressed in terms of* $h(x)$ if it is of the form

$$y(x) = r_0 \cdot h(f) + r_1 \cdot h(f)' \quad \text{with} \quad f, r_0, r_1 \in \overline{\mathbb{Q}(x)}$$

$h(x)$ and hence $y(x)$ satisfies a differential equation of order 2.

If $h(x)$ is CIS (algebraic or $c \in \mathbb{N}$), then $y(x)$ is again CIS
$\Longrightarrow$ produces differential equations with a CIS solution $y(x)$

Conjecture: If a CIS satisfies a linear differential equation of second order, then it can be expressed in terms of a ${}_2F_1$ function.

(tested on > 100 examples from oeis.org)

An Example

- A differential operator $L = \partial^2 + a_1\partial + a_0$ corresponds to a differential equation $y'' + a_1 \cdot y' + a_0 \cdot y = 0$.
- Consider the following differential operator, which has a CIS solution:

$$L := \partial^2 + \frac{\left(27\,x^7 - 39\,x^5 + 17\,x^3 - 5\,x - 9\,x^4 - 3\right)}{3x\,(x^2-1)(x^3-x-1)(3\,x^2-1)}\partial - \frac{5\left(3\,x^2-1\right)^2}{36x\,(x^3-x-1)(x^2-1)}.$$

- Our algorithm (Kunwar, v.H. 2013) finds the solution:

$$y = \frac{{}_2F_1\left(\frac{5}{6},\frac{7}{6};1\,|\,\frac{1}{1+x-x^3}\right)}{(1+x-x^3)^{5/6}} + \frac{35}{36}\,\frac{x\left(x^2-1\right)\cdot{}_2F_1\left(\frac{11}{6},\frac{13}{6};2\,|\,\frac{1}{1+x-x^3}\right)}{(1+x-x^3)^{11/6}}.$$

- The most important step is finding the **pullback** function $f = 1/(1 + x - x^3)$.

Differential equations

Why the form $r_0 h(f) + r_1 h(f)'$? The hypergeometric function

$$h(x) := {}_2F_1\left(a, b; c | x\right)$$

satisfies a second order equation $L(h) = 0$ where

$$L := x(1-x)\partial^2 + (c - (a+b+1)x)\partial - ab \; \in \mathbb{Q}(x)[\partial].$$

Like CIS, the form $r_0 h(f) + r_1 h(f)'$ is closed under:

- **Change of variables:** If $f \in \mathbb{Q}(x) - \mathbb{Q}$ then $h(f)$ satisfies again some equation of order 2.
- **Gauge transformation:** If $r_0, r_1 \in \mathbb{Q}(x)$ then $r_0 h(f) + r_1 h(f)'$ satisfies again some equation of order 2.

Equivalence

Two hypergeometric functions (assume: not algebraic)

$$h_1 = {}_2F_1(a_1, b_1; c_1|x), \quad h_2 = {}_2F_1(a_2, b_2; c_2|x)$$

are called *equivalent* $h_1 \sim h_2$ when:
L can be solved in terms of h_1 iff L can be solved in terms of h_2.

This happens when h_1 can be expressed in terms of h_2
(and vice versa).

An example of equivalent ${}_2F_1$'s

$$ {}_2F_1(\frac{1}{2},\frac{1}{2};1|x) \sim {}_2F_1(\frac{1}{8},\frac{3}{8};1|x) \sim {}_2F_1(\frac{1}{12},\frac{5}{12};1|x) $$

This equivalence class contains a number of other ${}_2F_1$'s as well. If we want *some* solution, then we need to pick a member from the correct equivalence class, but it does not matter which member.

(it does matter if we care about the *size* of the solution!)

Among the many integer sequences from oeis.org whose generating function is convergent, not algebraic, and satisfies a 2nd order differential equation, *we only observe one equivalence class* (the one whose monodromy group is arithmetic)

One equivalence class

Even though oeis.org is very large, and has many integer sequences whose g.f. satisfies a 2nd order equation, for some reason we encounter only one equivalence class:

$$h(x) = {}_2F_1(\frac{1}{12}, \frac{5}{12}; 1|x).$$

Goal: Given $L \in \mathbb{Q}(x)[\partial]$, order 2, decide if it can be solved in terms of this $h(x)$, and if so, find a solution

$$r_0 h(f) + r_1 h(f)'$$

(with r_0, r_1, f algebraic functions).

Solving L

$L \in \mathbb{Q}(x)[\partial]$, $\quad h = {}_2F_1(\frac{1}{12}, \frac{5}{12}; 1|x)$, find solution:

$$y := r_0 h(f) + r_1 h(f)'$$

In the majority of cases, f is a rational function.

Why? If f is an algebraic function, f needs to be rather special (indeed, more later!) for y to satisfy a 2nd order $L \in \mathbb{Q}(x)[\partial]$.

If f is rational, we have several algorithms:

T. Fang
V. Kunwar
E. Imamoglu

$$h = {}_2F_1(\frac{1}{12}, \frac{5}{12}; 1|x)$$

There are $L \in \mathbb{Q}(x)[\partial]$, order 2, with just 5 singularities, and a solution

$$r_0 h(f) + r_1 h(f)'$$

with f a rational function of degree 18.

Reconstructing such f from just 5 points is non-trivial. To get a fast algorithm, we (V. Kunwar and v.H.) built a table of all $f \in \mathbb{C}(x)$ that can correspond to an equation with 5 singular points.

Our table has hundreds of rational functions, mostly Belyi maps and near-Belyi maps. To prove completeness, we computed combinatorial data, called (near)-dessin d'enfants.

Algebraic pullback f

We have several algorithms for finding f when $f \in \mathbb{C}(x)$. What if f is not a rational function?

$$u_n := \sum_{k=0}^{n} \binom{n}{k}^4 \qquad \text{G.f.} = \sum_{n=0}^{\infty} u_n x^n$$

This generating function satisfies a 3'rd order equation, and $\sqrt{\text{G.f.}}$ a 2'nd order equation. By the conjecture, that equation should be solvable in terms of hypergeometric functions. Indeed, we can choose a, b, c in the usual equivalence class, and write

$$\sqrt{\text{G.f.}} = r_0 \cdot {}_2F_1(a, b; c | f)$$

for some f. Now f depends on which member of the equivalence class we took, but is never a rational function.

$u_n = \sum_{k=0}^{n} \text{binomial}(n,k)^4$

I inserted the following G.f. in oeis.org

$$(\cdots)\cdot {}_2F_1\left(\frac{1}{8},\frac{3}{8};1|\frac{4(\sqrt{1+4x}+\sqrt{1-16x})(\sqrt{1+4x}-\sqrt{1-16x})^5}{5(2\sqrt{1-16x}+3\sqrt{1+4x})^4}\right)^2$$

James Wan's award-winning poster at ISSAC'2013 used this to prove a formula for $1/\pi$. Now

$$f \in \mathbb{Q}(\sqrt{\frac{1+4x}{1-16x}})$$

and yet $L \in \mathbb{Q}(x)[\partial]$. So pullback f is *not unique*: $\sqrt{\ } \mapsto -\sqrt{\ }$.

Main idea: Classifying non-uniqueness $\Longrightarrow$ all non-rational f's.

Classifying non-uniqueness

Suppose f_1, f_2 are algebraic functions, and

$$ {}_2F_1(\frac{1}{12}, \frac{5}{12}; 1|f_1) = r_0 \cdot {}_2F_1(\frac{1}{12}, \frac{5}{12}; 1|f_2) \; + \; r_1 \cdot (\cdots)' $$

with f, r_0, r_1 algebraic functions. Can prove: $r_1 = 0$.

Question: What is the relation between f_1, f_2? (classification of non-unique f's).

Answer: The above relation holds for some algebraic r_0 iff f_1, f_2 satisfy

$$ \phi_N(1728/f_1, 1728/f_2) = 0 $$

for some N, where ϕ_N is the equation of the modular curve $X_0(N)$.

(ϕ_N is the algebraic relation between $j(\tau)$ and $j(N\tau)$)

Constructing $L \in \mathbb{Q}(x)[\partial]$ with f algebraic

Strategy: For $N = 2, 3, \ldots$ do

1. Search for algebraic f_1, f_2, conjugated over $\mathbb{Q}(x)$, with $\phi_N(1728/f_1, 1728/f_2) = 0$.
2. Compute $L = \partial^2 + a_1\partial + a_0 \in \mathbb{Q}(x, f_1)[\partial]$ with solution $h(f_1)$.
3. Clear the ∂^1 term by multiplying solutions by $\exp(\int a_1/2)$.
4. Store the resulting $\partial^2 + r$ in a table if r is in $\mathbb{Q}(x)$.

The solver can use the table to handle algebraic f's. How to make the algorithm complete? First, suppose $[\mathbb{Q}(x, f) : \mathbb{Q}(x)] = 2$.

f in a quadratic extension over $\mathbb{Q}(x)$

In this case, f has only two conjugates, say f_1, f_2.

$\mathbb{Q}(f_1, f_2)$ is isomorphic to the function field of $X_0(N)$ for some N. To end up with a 2nd differential equation over $\mathbb{Q}(x)$, the field $\mathbb{Q}(f_1, f_2)$ should have a genus-0 subfield of index 2.

So $X_0(N)$ must be rational, elliptic, or hyper-elliptic. Such N's are classified (Ogg 1974), so it should be possible to make the differential solver complete when f is quadratic over the rational functions.

Remaining issues

1. Higher degree algebraic functions.
2. Proofs.
3. Implementations.
4. What about hypergeometric functions from other equivalence classes?

   Why do they not occur in oeis.org?
   How to classify their non-rational f's?
5. How to tackle the conjecture? (that if $\sum u_n x^n$ is CIS and satisfies a 2'nd order differential equation, then it can be expressed in terms of hypergeometric functions).