The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

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partly based on collaboration with *E. Remiddi*

[arXiv:1311.3342], [arXiv:1509.03330]

INTRODUCTION

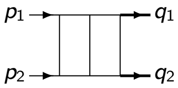
- Our understanding of **high energy physics** is (mainly) based on **perturbative calculations** in the Standard Model.

- At LHC discovery of the Higgs boson and (*for now*) no real sign of any **new physics effect**.

- High energy physics is becoming more and more **precision physics** and in order to perform **precision calculations** we need to be able to compute more and more complicated **Feynman diagrams**

Any Feynman Diagrams is (*after some tedious but elementary algebra!*) nothing but a collection of scalar **Feynman Integrals**

$\mathcal{I}(p_1, p_2, q_1) =$



with $\quad q_2 = p_1 + p_2 - q_1$

A (*possible*) representation in momentum space (*massless* case!)

$$\mathcal{I}(p_1, p_2, q_1) = \int \frac{\mathfrak{D}^d\, k\, \mathfrak{D}^d\, l}{k^2\, l^2\, (k-l)^2\, (k-p_1)^2\, (k-p_{12})^2\, (l-p_{12})^2\, (l-q_1)^2}$$

Typical **2-loop** Feynman Integral required for the computation of a $2 \to 2$ scattering process.

How do we (*tentatively!*) compute **analytically** such integrals?

1. Integrals are ill-defined in $d = 4 \rightarrow$ need a **regularization procedure**!
2. Use of dimensional regularization to regulate **UV** and **IR divergences**.
3. Dimensional regularisation turned out to be much MORE than just a **regularization scheme**!

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Dimensionally regularized Feynman integrals always converge!

This allows to derive a large number of **unexpected relations...**

- Integration by Parts, Lorentz invariance identities, Schouten Identities,...

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This large set of identities makes it *simpler* to compute **Feynman integrals** in d continuous dimensions than in $d = 4$!

A general **scalar** Feynman Integral (l-loops) can be written as

$$\mathcal{I}(\sigma_1, ..., \sigma_s; \alpha_1, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{S_1^{\sigma_1} \dots S_s^{\sigma_s}}{D_1^{\alpha_1} \dots D_n^{\alpha_n}}$$

where

$$D_n = (q_n^2 + m_n^2)\,, \qquad \text{are the } \textbf{propagators}$$

$$S_n = k_i \cdot p_j\,, \quad \text{are } \textbf{scalar products} \text{ among internal and external momenta}$$

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1. **Integration By Parts Identities** (IBPs)

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2. They generate huge systems of linear equations which relate integrals with **different powers** of numerators and denominators.

3. The integrals always belong to the same **topology**, as defined above.

The IBPs can be solved using computer algebra (Reduze2, AIR, FIRE5...)
As a result, all integrals are expressed as linear combination of a small subset of
Master Integrals

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Dimensionally regularised Feynman Integrals fulfil differential equations!

[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,...]

Let us take a **topology** of integrals which depend on two external invariants

$$s\,,\ m^2 \qquad \to x = \frac{s}{m^2}\,.$$

$$\mathcal{I}(s, m^2; \alpha_1\,, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \frac{1}{D_1^{\alpha_1} \dots D_n^{\alpha_n}}\,, \qquad \text{(same with scalar products)}$$

Assume IBPs reduce all integrals into this topology to **N Master Integrals** $m_i(s; d)$, with $i = 1, ..., N$.

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Assume IBPs reduce all integrals into this topology to **N Master Integrals** $m_i(s; d)$, with $i = 1, ..., N$.

1) All integrals depend on $x = s/m^2$ only.

2) **Differentiation** w.r.t to an external invariant:

$$s = p_\mu \, p^\mu \qquad \rightarrow \qquad \frac{\partial}{\partial \, s} = \frac{1}{2 \, s} \left(p^\mu \frac{\partial}{\partial p^\mu} \right)$$

$$\frac{\partial}{\partial \, s} \int \prod_{j=1}^{l} \frac{d^d \, k_j}{(2\pi)^d} \frac{1}{D_1^{\alpha_1} \dots D_n^{\alpha_n}} = \int \prod_{j=1}^{l} \frac{d^d \, k_j}{(2\pi)^d} \frac{1}{2 \, s} \left(p^\mu \frac{\partial}{\partial p^\mu} \right) \frac{1}{D_1^{\alpha_1} \dots D_n^{\alpha_n}}$$

3) With IBPs integrals on r.h.s. can be **reduced** again to the MIs

$$\frac{\partial}{\partial \, s} m_i(d;s) = \sum_{j=1}^{N} c_{ij}(d;s) \, m_j(d;s) \, .$$

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How does this help in practice?

What if $\mathbf{N = 1}$? (*There is only 1 MI!*)

If there is only 1 master integral the situation is *in principle* trivial:

$$\frac{\partial}{\partial s} m(d;s) = c(d;s)\, m(d;s)$$

First order linear equation, can be solved by **quadrature**

$$m(d;s) = C_0 \exp\left(\int_0^s dt\, c(d;t) \right)$$

Note

Differential equations provide a **natural** *integral representation* for $m(d;s)$ in terms of the "correct variables", i.e. the **Mandelstam variable** s .

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What if $\mathbf{N > 1}$? (*Life is not that easy anymore!*)

If the system is coupled, it corresponds to a **N-th order** differential equation for any of the MIs. No general strategy for a solution is known.

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Observations

1. We are **free** of choosing our **basis of MIs**
2. We are interested in the expansion for $d \to 4$
3. The **physical case** $d \to 4$ can be recovered from $d \to 2\,n$ with $n \in \mathbb{N}$ [Tarasov '96; Lee '09]

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Changing the basis can **simplify** the structure of the differential equations!

A simplification for $d \to 2\,n$ **equivalent** to a simplification for $d \to 4$!!!

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The freedom of considering **all possible even numbers** of dimensions can help simplify **substantially** the problem!

Shifting differential equations of even numbers of dimensions

a) We start with a **system of deq** for N masters $m_1, ..., m_N$

$$\frac{\partial}{\partial x_{ij}} \vec{m}(d; x_{ij}) = A(d; x_{ij})\, \vec{m}(d; x_{ij})\,,$$

b) Using **Tarasov's relations**

$$\vec{m}(d-2; x_{ij}) = \Delta_{m_k}\, \vec{m}(d; x_{ij}) = C(d; x_{ij})\, \vec{m}(d; x_{ij})$$

we can define a **new basis**

$$\vec{\mathcal{I}}(d; x_{ij}) = \vec{m}(d-2; x_{ij})$$

Recall that Δ_{m_k} is a differential operator w.r.t. the internal masses. Its form depends only on the topology of the graph and can be computed algorithmically.

c) By construction, the new basis fulfils system with $d \to d-2$

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An example - shifting a **canonical basis**

a) Suppose that we can find a canonical basis in $d = 2$

$$\frac{\partial}{\partial x_{ij}} \vec{m}(d; x_{ij}) = (d-2)\, A(x_{ij})\, \vec{m}(d; x)$$

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Strong indication that **number of coupled MIs** as $d \to 4$ determines **complexity** of the problem

1) If all masters decouple differential equations become **triangular** as $d \to 4$

   a) In all these cases a **canonical basis** can be found [Kotikov '10; Henn '13]

$$\frac{\partial}{\partial x_{ij}} \vec{\mathcal{I}}(d; x_{ij}) = (d-4)\, A(x_{ij})\, \vec{\mathcal{I}}(d; x)$$

   where $A(x_{ij})$ is in **d-log form**

   b) Results in terms of **multiple polylogarithms**

$$G(0; x) = \ln x\,, \qquad G(a; x) = \ln\left(1 - \frac{x}{a}\right)$$

$$G(\vec{0}_n; x) = \frac{1}{n!}\ln^n x\,, \qquad G(a, \vec{n}; x) = \int_0^x \frac{dt}{t-a} G(\vec{n}; t)$$

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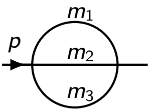
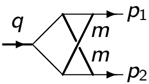
2) If all but two masters decouple $\rightarrow$ a 2×2 block **remains coupled**.

a) Not many "classified" examples. The most famous ones

$S(d; p^2) =$ 

, $T(d; q^2) =$ 

b) $S(d; p^2)$ is reduced to **4 master integrals**
$T(d; q^2)$ is reduced to **3 master integrals**

c) In both cases **2** masters cannot be decoupled in the limit $d \rightarrow 2\,n$.
Both are known to contain **elliptic functions**

This stimulated current attempt to generalise **multiple polylogarithms** to **elliptic** (multiple?) **polylogarithms**

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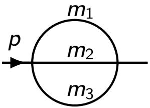
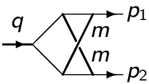
$S(d; p^2) =$ 

, $T(d; q^2) =$ 

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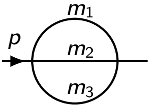
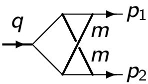
$S(d;p^2) =$ 

, $T(d;q^2) =$ 

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This stimulated current attempt to generalise **multiple polylogarithms** to
elliptic (multiple?) **polylogarithms**

Of course there are more complicated example where **three** or more master integrals cannot be decoupled in the limit $d \to 2\,n$, but the latter are at the moment still poorly studied and it is not clear what kind of **new functions** one could expect to appear in such calculations.

To me *one* **central question** becomes:

How many MIs remain coupled in $d \to 2\,n$?

- Can we imagine a criterion to determine whether some MIs can be decoupled in the limit $d \to 2\,n$?
- Given a graph with N MIs, they fulfil N coupled differential equations in d dimensions because the N MIs are **linearly independent in** d!

Conjecture

The differential equations can be decoupled in $d = 2\,n$ if the IBPs **degenerate** in the limit $d \to 2\,n$, allowing some of the MIs to become **linearly dependent**.

If this happens we would expect them to bring **no new information** as $d \to 2\,n$ and therefore it should be possible to decouple them from the differential equations.

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In order to verify this we need to **read the IBPs** in $d = 2\,n$ dimensions

1) Start with a topology with 2 masters $\mathcal{I}_1(d;x)$, $\mathcal{I}_2(d;x)$ in d dimensions and which satisfy

$$\begin{cases} \frac{\partial}{\partial x}\mathcal{I}_1(d;x) = c_{11}(d;x)\,\mathcal{I}_1(d;x) + c_{12}(d;x)\,\mathcal{I}_2(d;x) \\ \\ \frac{\partial}{\partial x}\mathcal{I}_2(d;x) = c_{21}(d;x)\,\mathcal{I}_1(d;x) + c_{22}(d;x)\,\mathcal{I}_2(d;x) \end{cases}$$

2) Now generate IBPs in d dimensions, then fix $d = 2\,n$, and finally try to solve them

NB

The MIs are in general divergent BUT in the IBPs there can be NO factors $1/(d-2\,n)$! One can more formally expand IBPs in Laurent series in $(d-2\,n)$, and solve the chained systems of IBPs order by order. The **homogeneous part** is always the same. This is what determines the **degeneracy** on the MIs in $d = 2\,n$.

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3) Assume that by solving the $d = 2\,n$ IBPs we find

$$\mathcal{I}_2(2\,n; x) = b(x)\,\mathcal{I}_1(2\,n; x)\,,$$

i.e. in $d = 2\,n$ there is *only 1 master integral*. It must fulfil a *first order differential equation* in the limit $d \to 2\,n$.

4) Where does such a relation come from?
If in the d-dimensional IBPs there is something like

$$K(d, x) = \frac{1}{d - 2\,n}\left(b_1(d; x)\,\mathcal{I}_1(d; x) + b_2(d; x)\,\mathcal{I}_2(d; x)\right)\,,$$

5) This implies that, *as far as the IBPs are concerned*,

$$b_1(d; x)\,\mathcal{I}_1(d; x) + b_2(d; x)\,\mathcal{I}_2(d; x) = \mathcal{O}(d - 2\,n),$$

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To make the **decoupling manifest** we perform the rotation

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The system of deq under this rotation becomes

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We expect the diff. equation for the newly introduced master to **decouple**, i.e.

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No mathematical proof. Decoupling expected due to *linear dependence* of the two master integrals in the limit $d \to 2\,n$. The rotation introduced above makes **manifest** the *first order differential equation* satisfied by one of the two original masters (in this case $\mathcal{J}_2 = \mathcal{I}_2$).

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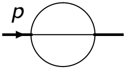
Explicit example, an **easy** two-loop sunrise



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1. Using **IBPs** we can reduce all these integrals to only **2 Master Integrals**.

$$S_1 = \mathcal{I}(s, m^2; 1, 1, 1, 0, 0), \qquad S_2 = \mathcal{I}(s, m^2; 1, 1, 2, 0, 0).$$

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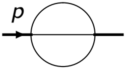

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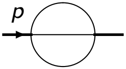

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Decoupling of the diff. equations in $d = 4$

- Following argument above, let us study the IBPs in $d = 4$. First of all note that both masters have a (UV) double pole

$$S_1(d;s) = \frac{1}{(d-4)^2}\, S_1^{(-2)}(4;s) + \frac{1}{(d-4)}\, S_1^{(-1)}(4;s) + \mathcal{O}(1)$$
$$S_2(d;s) = \frac{1}{(d-4)^2}\, S_2^{(-2)}(4;s) + \frac{1}{(d-4)}\, S_2^{(-1)}(4;s) + \mathcal{O}(1)\,.$$

  This is also *highest pole* for any other integral in this topology as $d \to 4$!

- Expand IBPs in $(d-4)$ and solve them order by order. As expected first order (double pole) gives degeneracy

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Note that

In this case solving the IBPs in $d = 4$ provides a real relation among the highest poles of the two master integrals! The graph does not contain any sub-topology and we have not neglected anything.

An (easy) explicit calculation gives

$$S_1(d;s) = \frac{1}{(d-4)^2}\left(\frac{m^2}{2}\right) + \mathcal{O}\left(\frac{1}{(d-4)}\right)$$
$$S_2(d;s) = \frac{1}{(d-4)^2}\left(\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{(d-4)}\right),$$

Confirming the relation above

$$S_2^{(-2)}(4;s) = \frac{1}{m^2} S_1^{(-2)}(4;s).$$

which comes for free if we *properly* read the IBPs in $d = 4$.

As we discussed, this degeneracy can be found in the d-dimensional IBPs

a) Scan the result of the IBPs reduction in d dimensional, look for relations which contain an overall $1/(d-4)$. We find, for example

$$\mathcal{I}(s, m^2; 2, 1, 1, 0, 0) = \left(\frac{1}{d-4}\right) \frac{(d-3)}{s-m^2} \left(4\, m^2\, S_2(d;s) - (3d-8) S_1(d;s)\right)$$

And many other similar relations, all **equivalent** in the limit $d \to 4$!

b) Taking the $d \to 4$ limit of these relation we find of course

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We can now finally simplify the differential equations.

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Decoupling of the diff. equations in $d = 2$

1) Let us check what happens in $d = 2$. In $d = 2$ the two masters have a double pole of IR origin.

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$$\mathcal{I}(s, m^2; 1, 1, 1, 1, 0) = \left(\frac{1}{d-2}\right) \frac{m^2}{3} \left(\left[(2-d)\frac{s}{m^2} + (3-d)\right] S_1(d; s) - (s - m^2) S_2(d; s)\right)$$

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- The differential equations for $\mathcal{J}_1$ and $\mathcal{J}_2$ **decouple**

$$\frac{d\mathcal{J}_1}{ds} = (d-2)\left[\frac{2}{s-m^2} - \frac{3}{2\,s}\right]\mathcal{J}_1 + (d-2)\left[\frac{3(d-2)}{2\,s} - \frac{2}{s-m^2}\right]\mathcal{J}_2$$
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Writing

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Question

Is it always the case? Is it enough to find a relation between the highest poles of the two master integrals in order to decouple the differential equations?

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The answer is, of course **NO** . Example **two-loop massive sunrise**

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1. Using **IBPs** we we find again only **2 Master Integrals**

$$S_1 = \mathcal{I}(s, m^2; 1, 1, 1, 0, 0), \qquad S_2 = \mathcal{I}(s, m^2; 1, 1, 2, 0, 0).$$

2. If we neglect sub-topologies and solve IBPs in $d = 2$ or $d = 4$ we do not find any further relations. Equations cannot be decoupled! **Elliptic polylogaritms**!

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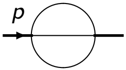

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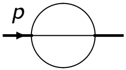

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a) This time, **do not neglect** sub-topologies and expand all IBPs in Laurent series starting from the highest pole $1/(d-4)^2$

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$$S_1^{(-2)}(d \to 4; s) = \frac{3}{2\,m^2}\, T^{(-2)}(d \to 4)$$
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Poles of MIs are fake! In other words, there exists a completely **finite** basis in $d = 2, 4, 6, \ldots$ etc, such that all poles are entirely determined by sub-topologies!

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Method can be easily generalised to more complicated topologies

a) In **different numbers of dimensions** $d = n \in \mathbb{N}$

b) For any **other topology** irrespective of number of loops or masses!

As a (*natural*) extension **the three-loop banana graph**

$I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) =$

$$= \int \mathfrak{D}^d k_1 \mathfrak{D}^d k_2 \mathfrak{D}^d k_3 \frac{(k_1 \cdot p)^{n_5} (k_2 \cdot p)^{n_6} (k_3 \cdot p)^{n_7} (k_1 \cdot k_2)^{n_8} (k_1 \cdot k_3)^{n_9}}{(k_1^2 - m_1^2)^{n_1} (k_2^2 - m_2^2)^{n_2} (k_3^2 - m_3^2)^{n_3} ((k_1 + k_2 + k_3 - p)^2 - m_4^2)^{n_4}}$$

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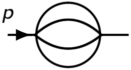
$I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) =$ 

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The case of **all equal masses**

All masses equal $m_4 = m_3 = m_2 = m_1 = m \to 3$ MIs in d dimensions

$$\mathcal{I}_1(d;s) = I_1(d;1,1,1,1,0,0,0,0,0)\,, \quad \mathcal{I}_2(d;s) = I_1(d;2,1,1,1,0,0,0,0,0)\,,$$
$$\mathcal{I}_3(d;s) = I_1(d;3,1,1,1,0,0,0,0,0)\,.$$

They remain independent if $d = 2 \to$ The scalar master integrals fulfil a **third-order differential equation**!

$$D_d^{(3)}\,\mathcal{I}_1(d;s) = 0\,,$$

$$\begin{aligned} D_d^{(3)} =& \frac{d^3}{d\,s^3} + \frac{3\left(64m^4 + 10(d-5)m^2 s - (d-4)s^2\right)}{s(s-4m^2)(s-16m^2)}\,\frac{d^2}{d\,s^2} \\ &+ \frac{(d-4)(11d-36)s^2 - 64(d-4)d\,m^4 - 4\left(216 + d(7d-88)\right)m^2\,s}{4\,s^2(s-4m^2)(s-16m^2)}\,\frac{d}{d\,s} \\ &+ \frac{(3-d)(3d-8)\left(2(d+2)m^2 + (d-4)s\right)}{4\,s^2\,(s-4m^2)(s-16m^2)} \end{aligned}$$

The case of **two different masses** - two different configurations

$$I_2^A(d;n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9) = I_4(d;n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9)\Big|_{m_3=m_2=m_1=m_a,m_4=m_b}$$

$$I_2^B(d;n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9) = I_4(d;n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9)\Big|_{m_2=m_1=m_a,m_4=m_3=m_b}$$

A) In configuration A 5 independent MIs in d dimensions

$$\mathcal{I}_1^A(d;s) = I_2^A(d;1,1,1,1,0,0,0,0,0)\,,\quad \mathcal{I}_2^A(d;s) = I_2^A(d;2,1,1,1,0,0,0,0,0)\,,$$
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Simplifying the system of differential equations in $d = 2$

A) Solving IBPs in $d = 2$ we find that 4 MIs remain independent

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\begin{aligned}
m_a^2(s - 5m_a^2 + m_b^2)\,\mathcal{I}_5^A(2;s) = \\
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B) Similarly we find here two relations such that again only 4 MIs remain independent

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Simplifying the system of differential equations in $d = 2$

A) Solving IBPs in $d = 2$ we find that 4 MIs remain independent

$$m_a^2(s - 5m_a^2 + m_b^2)\,\mathcal{I}_5^A(2;s) = + \frac{3m_a^2 + m_b^2 - s}{12\, m_a^2}\,\mathcal{I}_1^A(2;s) + \frac{51m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + 2s)}{12\, m_a^2}\,\mathcal{I}_2^A(2;s) + \frac{m_b^2(m_b^2 - s)}{6\, m_a^2}\,\mathcal{I}_3^A(2;s) + \frac{21m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + s)}{6}\,\mathcal{I}_4^A(2;s)\,.$$

B) Similarly we find here two relations such that again only 4 MIs remain independent

$$\mathcal{I}_5^B(s;2) = a_1\,\mathcal{I}_1(2;s) + a_2\,\mathcal{I}_2(2,s) + a_3\,\mathcal{I}_3(2,s) + a_4\,\mathcal{I}_4(2,s)$$
$$\mathcal{I}_6^B(s;2) = b_1\,\mathcal{I}_1(2;s) + b_2\,\mathcal{I}_2(2,s) + b_3\,\mathcal{I}_3(2,s) + b_4\,\mathcal{I}_4(2,s)$$

As for the two-loop sunrise, these relations can be used to decouple 1 (2) MIs from the system of differential equations.
4 differential equations remain **coupled**, corresponding to a **fourth-order differential equation** for the scalar amplitude.

The **general case** with 4 different masses

In the most general case one finds 11 different master integrals in d dimensions

$$\mathcal{I}_1(d;s) = I_4(d;1,1,1,1,0,0,0,0,0)\,,\quad \mathcal{I}_2(d;s) = I_4(d;2,1,1,1,0,0,0,0,0)\,,$$
$$\mathcal{I}_3(d;s) = I_4(d;1,2,1,1,0,0,0,0,0)\,,\quad \mathcal{I}_4(d;s) = I_4(d;1,1,2,1,0,0,0,0,0)\,,$$
$$\mathcal{I}_5(d;s) = I_4(d;1,1,1,2,0,0,0,0,0)\,,\quad \mathcal{I}_6(d;s) = I_4(d;3,1,1,1,0,0,0,0,0)$$
$$\mathcal{I}_7(d;s) = I_4(d;2,2,1,1,0,0,0,0,0)\,,\quad \mathcal{I}_8(d;s) = I_4(d;2,1,2,1,0,0,0,0,0)$$
$$\mathcal{I}_9(d;s) = I_4(d;2,1,1,2,0,0,0,0,0)\,,\quad \mathcal{I}_{10}(d;s) = I_4(d;1,2,2,1,0,0,0,0,0)$$
$$\mathcal{I}_{11}(d;s) = I_4(d;1,2,1,2,0,0,0,0,0)\,.$$

Repeating the exercise of solving the IBPs in $d=2$ we find 5 independent relations such that **6 master integrals** remain independent!

$\rightarrow$ corresponds to a *sixth-order differential equation for the scalar amplitude*

OPEN QUESTIONS

1. Is the criterion above necessary, together with sufficient?

2. Looking for relations in different even numbers of dimensions can give *apparently different* and *complementary* results.
   In what do these relations differ?

3. How do we know when we should stop looking? $d = 2, 4, 6, ...$

CONCLUSIONS

a) **Differential equations** are a very powerful tool for evaluating complicated *multi-loop and multi-scale* Feynman integrals.

b) If system of differential equations can be **decoupled** in the limit $d \to 4$ their solution as **Laurent series** in $(d-4)$ becomes much easier.

c) We shouldn't limit to study the system in the limit $d \to 4$.
A simplification in $d \to 2\,n$ *is equivalent!*

d) I showed that **relations** useful to **decouple** systems of differential equations in the limit $d \to n \in \mathbb{N}$ can be found easily by studying the IBPs in the considered limit.

THANKS!