A quasi-finite basis for multi-loop Feynman integrals

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PRISMA

based on arXiv:1411.7392 with Robert Schabinger and Erik Panzer

Theory Seminar, DESY Zeuthen
19. March 2015

Multi-loop Feynman integrals

consider **L loop Euclidean Feynman integrals**:

$$I = \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \cdots \frac{\mathrm{d}^d k_L}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \cdots D_N^{a_N}}$$

where $a_i \in \mathbb{Z}$ and e.g. $D_1 = k_1^2 - m_1^2$ etc.

linear dependencies:

- integration-by-parts (IBP) identities [Tkachov, Chetyrkin '81]
- systematic reduction to small number of master integrals [Laporta '00]
- think of it as linear vector space with some arbitrary basis (master integrals)

expansion in $\epsilon = (4-d)/2$

- typically sufficient for phenomenological applications
- Laurent coefficients are simpler integrals

solving methods typically based on

1. direct integration of Feynman (Schwinger) parameter integrals
2. differential equations

An improved basis for differential equations

method of differential equations [Kotikov '91]:

- relies on IBP reduction
- considers system of differential equations for basis integrals wrt external invariants
- for special choice of basis [Kotikov '10; Henn '13]:

$$\frac{\partial}{\partial s_i} I_j(d, s_m) = (d-4) A_{jk}^{(i)}(s_m) I_k(d, s_m)$$

- full decoupling after expansion in $\epsilon = (4-d)/2$
- clear view on structure of solution

obtain **pure functions** due to particular choice of basis:

- pure functions $\Leftrightarrow$ every term of ϵ expansion has uniform weight
- avoids spurious denominators in amplitude
- avoids spurious higher order ϵ terms
- also useful for exact d dependent single scale integrals [Li, AvM, Schabinger, Zhu '14]

however: differential equations fail to determine various single scale integrals

An improved basis for Feynman parameters

consider **Feynamn parameter representation** of multi-loop integral

$$I = \frac{\Gamma\left(\nu - \frac{Ld}{2}\right)(-1)^{\nu}}{\prod_{i=1}^{N} \Gamma(\nu_i)} \left[\prod_{j=1}^{N} \int_0^{\infty} \mathrm{d}x_j \right] \delta(1 - x_N)\, \mathcal{U}^{\nu-(L+1)d/2} \mathcal{F}^{-\nu+Ld/2} \prod_{k=1}^{N} x_k^{\nu_k - 1}$$

where $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity

presence of **subdivergencies** (= divergencies from Feynman parameter integrations) implies:

- can't directly expand in ϵ
- can't directly evaluate numerically

generic approaches to **singularity resolution**:

1. sector decomposition [Binoth, Heinrich '00]
2. regularising dimension shifts [Panzer '14]
3. New **basis of quasi-finite Feynman integrals**
   (quasi-finite = free of subdivergencies)

Sector decomposition: shortcomings

calculate to $\mathcal{O}(\epsilon)$:

$$I(\epsilon) = \int_0^1 \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$:

$$I_1(\epsilon) = \int_0^{1/2} \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$
$$I_2(\epsilon) = \int_{1/2}^{1} \mathrm{d}t \; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)\,.$$

rescale, expand in plus distributions, evaluate:

$$I_1(\epsilon) = -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3}\pi^2 - 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$
$$I_2(\epsilon) = -\frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3}\pi^2 + 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

result:

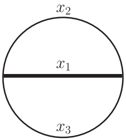
$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3}\pi^2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

note:

- split up of domain introduces **spurious terms** $\ln(2)$
- spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility & prevents **analytical integration** a la [Brown '08; Panzer '14]

An example for subdivergencies

$$T_{111}(m^2, 4-2\epsilon) =$$



$$= \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \int \frac{\mathrm{d}^d k_2}{i\pi^{d/2}} \frac{1}{((k_1+k_2)^2 - m^2)\, k_1^2 k_2^2}$$

$$= -\Gamma(-1+2\epsilon) \int_0^\infty \mathrm{d}x_1 \delta(1-x_1) \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \, \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{1-2\epsilon} ,$$

with Symanzik polynomials

$$\mathcal{U} = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U} .$$

- **can't expand in ϵ:**

$$T_{111}(m^2, 4-2\epsilon) = -\left(m^2\right)^{1-2\epsilon} \frac{\Gamma(-1+2\epsilon)\Gamma(\epsilon)\Gamma(1-\epsilon)}{1-\epsilon}$$

  $\Gamma(\epsilon)$ signals subdivergence
- **Euclidean** integrals: all divergencies from integration **boundaries**
- notation here: restrict to one or several parameters approaching **zero** (not infinity)

Systematic recognition of subdivergencies

- follow [Panzer '14]
- consider subsets

$$\{x_1, x_2\}, \quad \{x_1, x_3\}, \quad \{x_2, x_3\}, \quad \{x_1\}, \quad \{x_2\}, \quad \{x_3\}$$

- for each subset J consider **scaling with λ**:

$$J \to \lambda J$$

for integrand $P \equiv \mathcal{U}^{-3+3\epsilon}\mathcal{F}^{1-2\epsilon}$:

$$P \to P_{J_\lambda} = \lambda^{\deg_J(P)}\tilde{P} \qquad \text{where } \lim_{\lambda\to 0}\tilde{P} = \mathcal{O}(\lambda^0)$$

and the integral measure

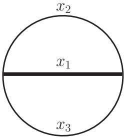
$$\prod_{i=1}^{3} \mathrm{d}x_i \to \lambda^{|J|}\prod_{i=1}^{3} \mathrm{d}x_i$$

and read off:

convergence index

$$\omega_J(P) = |J| + \deg_J(P)\,,$$

$$\lim_{\epsilon\to 0}\omega_J(P) \leq 0 \quad \Leftrightarrow \quad \text{presence of non-integrable subdivergence}$$

$T_{111}(m^2, 4-2\epsilon) =$ 

$$= \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \int \frac{\mathrm{d}^d k_2}{i\pi^{d/2}} \frac{1}{((k_1+k_2)^2 - m^2)\, k_1^2 k_2^2}$$
$$= -\Gamma(-1+2\epsilon) \int_0^\infty \mathrm{d}x_1 \delta(1-x_1) \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{1-2\epsilon} ,$$

with Symanzik polynomials

$$\mathcal{U} = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad \text{and} \quad \mathcal{F} = m^2 x_1 \mathcal{U} .$$

- **example:** for $J = \{x_2, x_3\}$:

$$|J| = 2$$
$$P|_{J\to\lambda J} = \lambda^{\epsilon-2} \left(m^2 x_1\right)^{1-2\epsilon} (x_1 x_2 + x_1 x_3 + \lambda x_2 x_3)^{\epsilon-2}$$

and thus $\omega_{\{x_2,x_3\}}(P) = 2 + (\epsilon - 2) = \epsilon$ and $\lim_{\epsilon\to 0} \omega_J(P) = 0$ signals subdivergence

Panzer's regularising shift

integrand can be regularized by dimension-shifts [Panzer '14]:

1. pick J for which $\lim_{\epsilon\to 0}\omega_J(P) \leq 0$
2. multiply by $1 = \int_0^\infty \mathrm{d}\lambda\ \delta(\lambda - x_J)$ with $x_J = \sum_{j\in J} x_j$
3. rescale $x_j \to \lambda x_j$ for all $j \in J$, account for delta functions, and perform partial integration

$$\int_0^\infty \mathrm{d}\lambda\ \lambda^{|J|-1} P_{J_\lambda} = \frac{1}{\omega_J(P)} \lambda^{|J|} P_{J_\lambda}\Big|_{\lambda=0}^{\infty} - \frac{1}{\omega_J(P)} \int_0^\infty \mathrm{d}\lambda\ \lambda^{\omega_J(P)} \frac{\partial}{\partial\lambda}\left(\lambda^{-\deg_J(P)} P_{J_\lambda}\right)$$

   where surface term vanishes

4. new integrand

$$P' = -\frac{1}{\omega_J(P)} \frac{\partial}{\partial\lambda}\left(\lambda^{-\deg_J(P)} P_{J_\lambda}\right)\Big|_{\lambda\to 1}.$$

   has improved convergence by design

5. iterate until no subdivergencies

for our **example** only one shift is needed:

$$T_{111}(m^2, 4-2\epsilon) = -\frac{2-\epsilon}{\epsilon}\Gamma(-1+2\epsilon) \int_0^\infty \mathrm{d}x_1 \delta(1-x_1) \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, x_2 x_3 \left(m^2 x_1\right)^{1-2\epsilon} \times$$
$$\times\, (x_1x_2 + x_1x_3 + x_2x_3)^{\epsilon-3}$$

Shortcomings of regularising shifts

real life problems:

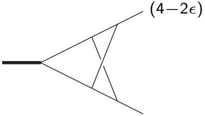
- proliferation of terms
- ambiguities
- spurious poles in ϵ

way out:

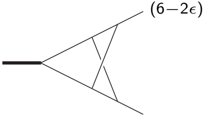
- consider full set of master integrals (basis)
- employ integration by parts (IBP) reductions

Our proposal: minimal dims & dots

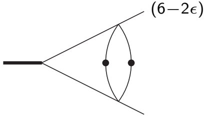
decompose wrt **quasi-finite basis**



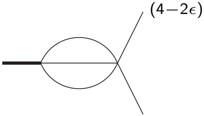
$$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$$



$$- \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2}$$



$$- \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3}$$



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basis consists of standard Feynman integrals, but

- in **shifted dimensions**
- with additional **dots** (propagators taken to higher powers)

Existence of quasi-finite basis

1. start with some basis B for topology and subtopologies
2. assume master b not quasi-finite and has integrand

$$P = \mathcal{U}^{\nu-(L+1)d/2}\mathcal{F}^{-\nu+Ld/2}\prod_{j=1}^{N} x_j^{\nu_j-1}, \qquad \text{where } \nu = \sum_{i=1}^{N}\nu_i$$

3. consider regularizating dimension shift:

$$P' = -\frac{1}{\omega_J(P)}\prod_{j=1}^{N} x_j^{\nu_j-1}\Bigg\{\Big(\nu - \tfrac{(L+1)d}{2}\Big)\mathcal{U}^{(\nu+L)-(L+1)(d+2)/2}\mathcal{F}^{-(\nu+L)+L(d+2)/2}\frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\Big|_{\lambda\to 1}$$
$$-\Big(\nu - \tfrac{Ld}{2}\Big)\mathcal{U}^{(\nu+L+1)-(L+1)(d+2)/2}\mathcal{F}^{-(\nu+L+1)+L(d+2)/2}\frac{\partial\widetilde{\mathcal{F}}}{\partial\lambda}\Big|_{\lambda\to 1}\Bigg\},$$

   with $\mathcal{U}_{J_\lambda} = \lambda^{\deg_J(\mathcal{U})}\widetilde{\mathcal{U}}$ and $\mathcal{F}_{J_\lambda} = \lambda^{\deg_J(\mathcal{F})}\widetilde{\mathcal{F}}$

4. picking any monomial from $\frac{\partial\widetilde{\mathcal{U}}}{\partial\lambda}\big|_{\lambda\to 1}$ or $\frac{\partial\widetilde{\mathcal{F}}}{\partial\lambda}\big|_{\lambda\to 1}$ gives

   dimension-shifted and **dotted** integral with **improved convergence** !

5. choose one term such that new integral b' is independent of $B \setminus b$
6. replace $b \to b'$ and iterate until B quasi-finite (algorithm terminates)

Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

Algorithm: construction of quasi-finite basis

- for each topology scan for quasi-finite integrals with dim-shifts and dots
- employ IBP + dimensional recurrence for actual basis change

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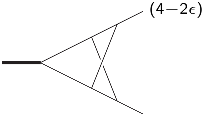
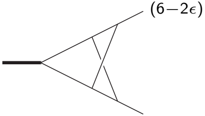
remarks:

- computationally expensive part shifted to IBP solver (`Fire`, `Reduze`, `LiteRed`)
- very efficient in practice
- easy to automate (implemented in dev. version of `Reduze 2`)
- any dim-shift good, e.g. shifts by [Tarasov '96], [R.N. Lee '10]
- see [Bern, Dixon, Kosower '93] for related one-loop pentagon example

Example 1: massless non-planar form factor family



$$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$$



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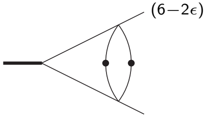

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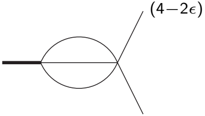

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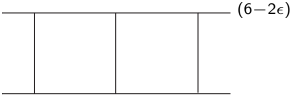
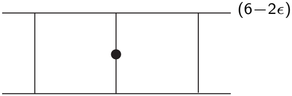
note:

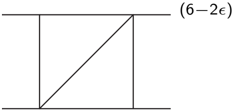
- old reg. shifts generated $\mathcal{O}(10\text{MB})$, here: 3 lines ! (more severe at higher loops)
- take $1/\epsilon$ from Γ prefactors into account
  - $1/\epsilon^4$ and $1/\epsilon^3$ poles originate from subtopologies
  - 6 line topology only from $1/\epsilon^2$ on

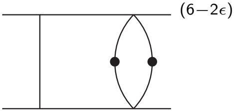
$\Rightarrow$ fewer difficult parameter integral evaluations !

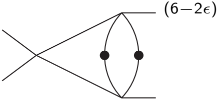
Example 2: massless planar double box family

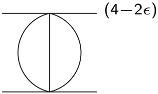
$b_1 =$ 

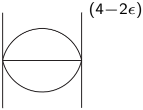
$b_2 =$ 

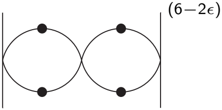
$b_3 =$ 

$b_4 =$ 

$b_5 =$ 

$b_6 =$ 

$b_7 =$ 

$b_8 =$ 

Numerical evaluations

advantages of quasi-finite basis:

- straight-forward to integrate numerically (in principle)
- no blow up in number of numerical integrations (speed, stability)
- no cancelation of spurious structures (stabitity)

first experiments with numerical evaluations

- naive straight-forward implementation works already quite well
- employ existing sector decomposition programs `Fiesta`, `SecDec` and `sector_decomposition`
- quasi-finite integrals much faster than naive master integrals
- reliable results also for notoriously problematic integrals

Conclusions & outlook

conclusions:

- presented simple and efficient method for **singularity resolution in multi-loop integrals**
- **analytical integrations:** quasi-finite integrals are Feynman integrals (dim-shifted, dotted)
- **numerical integrations:** faster and more stable evaluations

Conclusions & outlook

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outlook:

- massless QCD form factors at three-loops: first independent rederivation at heigher weights
- massless QCD form factors at four-loops: cusp anomalous dimension and more
- extend method to more general kinematics + phase space integrals
- truly finite basis